Modeling hybrid systems by high-level Petri nets
Modélisation des systèmes hybrides par réseaux de Petri de haut-niveau

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ABSTRACT. We present a hybrid model based on a Petri net formalism that merges the concepts of high-level nets with continuous nets. The model can represent jumps in the state space and switches in the dynamics, both autonomous and controlled. Classical Petri net concepts, such as the firing vector and the incidence matrix, can be generalized to this model and used to derive the evolution equation. We show how this behaviour can be represented by an evolution graph. Two examples of controlled switch and autonomous jump are discussed in detail.

RÉSUMÉ. Cet article présente un modèle hybride fondé sur un formalisme qui allie les concepts de réseaux de Petri de haut-niveau et de réseaux continus. Le modèle peut représenter des discontinuités dans l'espace d'état et des commutations de modes de fonctionnement, qu'ils soient autonomes ou contrôlés. Plusieurs notions classiques concernant les réseaux de Petri, tels que le vecteur de franchissement et la matrice d'incidence, peuvent être généralisées à ce modèle. Ces notions sont employées pour définir l'équation d'évolution et construire le graphe d'évolution correspondant. On présente un exemple de commutateur contrôlé et un exemple de système avec discontinuités.

KEY WORDS: hybrid systems, Petri nets, continuous nets, high–level nets.

MOTS CLÉS: systèmes hybrides, réseaux de Petri, réseaux continus, réseaux de haut–niveau.

1 Introduction

The control of hybrid systems, *i.e.*, systems with both continuous-time and discrete-event dynamics, is a domain of increasing importance and several hybrid models [GRO 93, BRO 93] have been presented in the literature. A comprehensive survey of different models and of the relationships among them can be found in [BRA 94].

Petri nets (PN) [MUR 89, REI 85] have originally been introduced to describe and analyze discrete event systems. Recently, much effort has been devoted to apply these models to hybrid systems. A recent survey of the relevant literature can be found in [DAV 97].

Most of the hybrid PN models are properly speaking fluid models, *i.e.*, the marking of the continuous places is a *nonnegative real* number following the original approach of David and Alla [DAV 92]. These models have also been extended to a stochastic framework by several authors [HOR 96, WOL 98, BAL 98]. However, to describe more general hybrid models, whose continuous state variables may also take negative values, extended net models have been considered. As an example, Differential Petri Nets by Demongodin and Koussoulas [DEM 98] are nets with *real* markings. In other approaches the Petri net formalism has been combined with Differential Algebraic Equations to model arbitrary continuous evolution [CHA 98, VAR 98].

In this paper, following [GIU 96], we describe a more general hybrid model, called *high-level hybrid Petri net* that merges the concepts of high-level Petri nets with continuous nets.

High-level nets are characterized by the use of structured individual tokens. *Colors*, *i.e.*, firing domains of the transitions and marking domains of the places, have been used in the literature to give a compact representation of systems. We use this feature in the discrete part of the nets. However, we also use vectors of real numbers to represent the continuous state space of the net. The use of real numbers as colors allows us to model arbitrary jumps in the state space, as discussed later on. In the other hybrid nets cited above, only a restricted type of state space jumps can be modeled, in the sense that the magnitude of the discontinuity is constant and does not depend on the state from which the jump occurs.

The continuous dynamics is ruled by the firing of continuous transitions, following the approach of [DAV 92]. However, we need to use marking dependent firing velocities to represent the ordinary differential equations describing the continuous evolution of systems. We also associate to a continuous transition a vector of firing velocities rather than a single velocity.

We add an explicit notion of time and take into account the presence of external inputs, continuous and discrete. Each discrete transition has associated a firing delay that may depend on the state of the system and on the external discrete inputs. The firing velocity of a continuous transition may depend on continuous external inputs.

Our model provides a simple graphical representation of hybrid systems and takes advantage of the modular structure of Petri nets in giving a compact description of systems composed of interacting subsystems, both continuous-time and discrete-event.

We define the *incidence matrix* of a high-level hybrid Petri net and use it to derive a fundamental equation that describes the evolution of such a net in terms of the firing of transitions, both continuous and discrete. This suggests that other Petri nets analysis and control techniques may be generalized within this framework as well.

In this paper we show that this modeling formalism is rich enough to encompass large classes of hybrid systems and can easily represent many of the features of the abstract model given in [BRA 94], that was shown to be extremely general. In particular we consider elementary net structures that can be used to represent jumps in the state space and switches in the dynamics, both autonomous and controlled.

We also show how it is possible to study the system's dynamical behaviour by constructing the evolution graph of the net. This graph describes the evolution (in terms of discrete event occurrences and phases associated with defined continuous dynamics) of the net with time.

We would like to give some motivation for the use of real numbers as colors. A dual semantic is associated with the arcs of a Petri net.

- Firstly, the Pre arcs from places to transitions specify the logical conditions that must be fulfilled to enable a given transition. A transition t is enabled and may fire only if the marking M is greater or equal to $Pre(\cdot, t)$. Note that the set of markings that enable a given transition is — in the notation of [VAL 85] — a *right-closed set*, *i.e.*, if M enables t, then any $M' \geq M$ also enables t.

- Secondly, the incidence matrix specifies the *constant* marking variation produced by the firing of a given transition t. Thus if $M'[t\rangle M''$, then $\Delta M = M'' - M' = Post(\cdot, t) - Pre(\cdot, t) = C(\cdot, t)$.

This double semantic is essential to the definition of place-transition nets and should be kept when extending the basic PN model to hybrid nets.

Let us now consider the counterpart of these semantics on an hybrid system.

A switch is a discrete event that corresponds to a change of dynamics of an hybrid system. An autonomous switch is defined by assigning in the continuous state space domain a switching boundary that is not necessarily right-closed. If one wants to represent a switch with a transition firing, it is necessary to specify for *each* state vector whether the transition may fire. Thus we need to use a different firing color for each state vector value from which the switch may occur.

A jump is a discrete event that corresponds to a discontinuity in the continuous–time state from $x(\tau^-)$ to $x(\tau^+)$. A jump represents a physical event whose state variation $\Delta x = x(\tau^+) - x(\tau^-)$ may depend on the values of

$x(\tau^-)$. As an example, in a later section we present the net model of a bouncing ball whose impact with the ground causes the abrupt change of velocity from $x(\tau^-) = -v$ to $x(\tau^+) = v$, *i.e.*, $\Delta x = -2x(\tau^-)$. To represent this event with a single transition, regardless of the value of $x(\tau^-)$, we need again to use a different firing color for each state vector value from which the jump may occur.

The paper is structured as follows. Section 2 presents the formal definition of high-level hybrid Petri nets and the rules governing their evolution. In Section 3 several basic structures that represent elementary hybrid behaviours are presented. Section 4 and Section 5 present two examples of dynamical hybrid system modeling.

2 High-level hybrid Petri nets

In this section we give a formal definition of the hybrid model we propose.

2.1 *Multisets and multirelations*

Let D be a set. A *multiset* (resp., *non negative multiset*) over D is a mapping $\alpha : D \to \mathbb{Z}$ ($\alpha : D \to \mathbb{N}$) and may be represented as $\alpha = \sum_{d \in D} \alpha(d) \otimes d$ where the sum is limited to the elements such that $\alpha(d) \neq 0$. Let $\mathcal{S}(D)$ denote the set of all non negative multisets over D. The multiset ε is the empty multiset such that for all $d \in D, \varepsilon(d) = 0$.

Given two multisets $\alpha, \beta \in \mathcal{S}(D)$ and a number $a \in \mathbb{N}$:

- The sum of α and β is denoted as $\gamma = \alpha + \beta$ and is defined as $\forall d \in D$: $\gamma(d) = \alpha(d) + \beta(d)$.
- The difference of α and β is denoted as $\gamma = \alpha - \beta$ and is defined as $\forall d \in D : \gamma(d) = \alpha(d) - \beta(d)$. Note that the difference of two non negative multisets may be negative.
- The product of α and a is denoted as $\gamma = a\ \alpha$ and is defined as $\forall d \in D$: $\gamma(d) = a\ \alpha(d)$.
- We write $\alpha \leq \beta$ iff $\forall d \in D : \alpha(d) \leq \beta(d)$.

Let $f : D \to \mathbb{R}^k$ be a function and $\alpha \in \mathcal{S}(D)$ a multiset over D. We define $f(\alpha) = \sum_{d \in D} \alpha(d) f(d) \in \mathbb{R}^k$, *i.e.*, $f(\alpha)$ is the linear combination with coefficients $\alpha(d)$ of the vectors $f(d)$.

Let D, D' be sets. A *non negative multirelation* over (D, D') is a non negative multiset $\rho \in \mathcal{S}(D \times D')$. Let $\mathcal{R}(D, D')$ denote the set of all non negative multirelations over (D, D'). The multirelation ϕ is the empty multirelation such that for all $d \in D, d' \in D', \phi(d, d') = 0$.

For $\rho \in \mathcal{R}(D, D')$ and $d \in D$, let $\rho[d]$ be the multiset over D' defined as $\forall d' \in D' : \rho[d](d') = \rho(d, d')$, *i.e.*, $\rho[d] = \sum_{d' \in D'} \rho(d, d') \otimes d'$.

For $\rho \in \mathcal{R}(D, D')$ and $\alpha \in \mathcal{S}(D)$, let $\rho[\alpha]$ be the multiset over D' defined as $\forall d' \in D' : \rho[\alpha](d') = \sum_{d \in D} \alpha(d)\rho(d, d')$, *i.e.*, $\rho[\alpha] = \sum_{d' \in D'} \left(\sum_{d \in D} \alpha(d)\rho(d, d')\right) \otimes d'$.

2.2 *Structure and marking*

A *High-Level Hybrid Petri Net* (HLHPN) is a 6-tuple $G = (P, T, F, \mathcal{I}, \delta, \nu)$ where:

- $P = P_D \cup P_C$ is a disjoint union of discrete places (represented by circles) and continuous places (represented by squares). Each place $p \in P$ has associated a marking domain $D(p)$. In particular if $p \in P_D$ then $D(p) = \{c_1, \ldots, c_k\}$ is a set of discrete colors, if $p \in P_C$ then $D(p) = \mathbb{R}^{c(p)}$, where $c(p) \in \mathbb{N}$ is the dimension of the continuous place. We define $m_D = |P_D|$, $m_C = |P_C|$, and $m = m_D + m_C$.
- $T = T_D \cup T_C$ is a disjoint union of discrete transitions (represented by bars) and continuous transitions (represented by boxes). Each transition $t \in T_C$ has associated a dimension $c(t) \in \mathbb{N}$. We define $n_D = |T_D|$, $n_C = |T_C|$, and $n = n_D + n_C$.
- $F \subseteq (P \times T) \cup (T \times P)$ is a relation specifying the arcs from places to transitions and vice versa. We write $F_{XY} = F \cap ((P_X \times T_Y) \cup (T_Y \times P_X))$ for $X, Y \in \{D, C\}$.

  For all $x \in P \cup T$ we write ${}^\bullet x = \{y \mid (y, x) \in F\}$ and $x^\bullet = \{y \mid (x, y) \in F\}$. For all transitions t, let $P^t = {}^\bullet t \cup t^\bullet$.

  Given a transition $t \in T_D$, let $P^t = \{p_1, \cdots, p_r\}$. We associate to t a *firing domain* $D(t) \subseteq D(p_1) \times \cdots \times D(p_r)$.

  Given a transition $t \in T_C$, let $P^t \cap P_D = \{p_1, \cdots, p_r\}$. We associate to t a *firing domain* $D(t) \subseteq D(p_1) \times \cdots \times D(p_r)$.
- $\mathcal{I}$ is an *inscription* that assigns to each arc in F a weight. There will be different kind of inscriptions depending on the kind of arcs. Let $a = (t, p)$ or $a = (p, t)$.
  - If $a \in F_{DD} \cup F_{DC}$ then $\mathcal{I}_a$ is a multirelation $\rho_a \in \mathcal{R}(D(t), D(p))$.
  - If $a \in F_{CD}$ then $\mathcal{I}_a$ is a vector function $f_a : D(t) \to \mathbb{R}^{c(p)}$.
  - If $a \in F_{CC}$ then $\mathcal{I}_a$ is a matrix $A_a \in \mathbb{R}^{c(p) \times c(t)}$.

  We will extend the inscription $\mathcal{I}$ to all elements in $(P \times T) \cup (T \times P)$ by assuming that if $a \notin F$, then $\mathcal{I}_a$ is the appropriate null element, *i.e.*, the null multirelation, the null function, or a zero matrix.
- The mapping δ associates to each discrete transition a time delay that is a function of the marking of the net and of the external discrete inputs. We denote by $\delta_t : \mathcal{M} \times \mathcal{U}_D \to \mathbb{R}^+ \cup \{0\}$ the component associated to transition t. Here $\mathcal{M}$ is the set of all possible markings, and $\mathcal{U}_D$ is the set of all possible discrete inputs, as defined in the following.

- The mapping ν associates to each continuous transition a firing velocity vector that is a function of the marking of the net and of the external continuous inputs. We denote by $\nu_t : \mathcal{M} \times \mathcal{U}_C \to \mathbb{R}^{c(t)}$ the component associated to transition t. Here $\mathcal{U}_C$ is the set of all possible continuous inputs of the net as defined in the following.

The *marking* of a net is a function of time $M(\tau)$ that associates to each place p a value $M_p(\tau)$. For all $p \in P_D, M_p(\tau) \in \mathcal{S}(D(p))$, while for all $p \in P_C, M_p(\tau) \in D(p)$. The set of all possible markings of a net is denoted $\mathcal{M}$. Note that while a discrete place may be empty, because its marking may be the empty multiset $\varepsilon \in \mathcal{S}(D(p))$, a continuous place is always marked by a vector belonging to $D(p) = \mathbb{R}^{c(p)}$.

We assume there are k_D discrete input signals $u_{Di}(\tau) : \mathbb{R} \to \{0, 1\}$. Thus a *discrete input* is a vector $u_D(\tau) = [u_{D1}(\tau) \cdots u_{Dk_D}(\tau)]^T$. The set of all possible discrete inputs of a net is denoted $\mathcal{U}_D$.

We assume there are k_C continuous input signals $u_{Ci}(\tau) : \mathbb{R} \to \mathbb{R}$. Thus a *continuous input* is a vector $u_C(\tau) = [u_{C1}(\tau) \cdots u_{Ck_C}(\tau)]^T$. The set of all possible continuous inputs of a net is denoted $\mathcal{U}_C$.

A *well formed* HLHPN, is a net whose arc relation F and inscription $\mathcal{I}$ satisfy the following conditions.

1. For all $p \in P_D$ and for all $t \in T_C$, $[(p,t) \in F \iff (t,p) \in F]$ and $\rho_{(p,t)} = \rho_{(t,p)}$.
2. For all $t \in T_C$, ${}^\bullet t \cap P_C = \emptyset$.

As in [DAV 97], the first condition implies that the firing of a continuous transition t does not change the marking of a discrete place p, even if there may be arcs between p and t because the enabling of t is conditioned by the marking of p.

The second condition implies that there can only be arcs from continuous transitions to continuous places, but not vice versa. This, however, is not a limitation because: (a) the state of a continuous place does not enable/disable a continuous transition; (b) the arc inscription or the firing velocity may take negative values, so that when $(t,p) \in F$ the marking of the continuous place p may decrease as the continuous transition t fires, even if $(p,t) \notin F$.

A hybrid system $\langle G, M(0) \rangle$ is a well formed net with an initial marking $M(0)$. In the following we will always assume that the nets are well formed.

2.3 *Enabling and firing*

2.3.1 *Discrete transitions*

A transition $t \in T_D$ is *enabled* with respect to (wrt) a firing element $d \in D(t)$ at a marking $M(\tau)$ if:

$$\begin{cases} \forall p \in {}^\bullet t \cap P_D, & M_p(\tau) \geq \rho_{(p,t)}[d]; \\ \forall p \in {}^\bullet t \cap P_C, & M_p(\tau) = f_{(p,t)}(d). \end{cases}$$

This means that a discrete transition t is enabled wrt a firing color $d \in D(t)$ if each input discrete place p has at least as many tokens as specified by the arc inscription $\rho_{(p,t)}[d]$, and if each input continuous place p contains the token $f_{(p,t)}(d)$.

An enabled discrete transition t fires only after it has remained enabled for a period of time equal to its firing delay δ_t (we call this condition firability). Its firing changes impulsively the marking of the net.

A transition $t \in T_D$ is *firable* wrt a firing element $d \in D(t)$ at a marking $M(\tau)$ and given an external discrete input $u_D(\tau)$ if:

$$\begin{cases} (\forall \tau' \in [\tau - \delta_t(M(\tau), u_D(\tau))\ ; \tau])\ \ (\exists d' \in D(t))\ :\ t \text{ was enabled at } \tau' \text{ wrt } d' \\ \text{AND} \\ t \text{ is enabled at } \tau \text{ wrt } d \end{cases}$$

In particular if at time τ only one discrete transition $t \in T_D$ is firable wrt a single element d, then that transition *must* fire and the marking of the net will change as follows:

$$\begin{cases} \forall p \in P_D, & M_p(\tau^+) = & M_p(\tau^-) + \rho_{(t,p)}[d] - \rho_{(p,t)}[d]; \\ \\ \forall p \in P_C, & M_p(\tau^+) = & M_p(\tau^-) + f_{(t,p)}(d) - f_{(p,t)}(d). \end{cases} \qquad [1]$$

It may be the case that more than one discrete transition is firable at time τ (or equivalently, one transition is firable wrt more than one element d). In this case we assume that between τ^- and τ^+ the net behaves as an untimed net in which the conflicts are resolved selecting with predefined rules (*e.g.*, priorities) one transition among all those firable. If after the firing of the selected transition other transitions are still firable, the procedure is repeated. We will assume that the evolution is such that only a finite number of transitions will fire between τ^- and τ^+. In this case, the evolution of the marking will be given by:

$$\begin{cases} \forall p \in P_D, & M_p(\tau^+) & = M_p(\tau^-) + \sum\{\rho_{(t,p)}[d] - \rho_{(p,t)}[d]\}; \\ \\ \forall p \in P_C, & M_p(\tau^+) & = M_p(\tau^-) + \sum\{f_{(t,p)}(d) - f_{(p,t)}(d)\}; \end{cases}$$

where the sum is taken over all the different transition firings. It may be possible to relax the hypothesis that only a finite number of transitions fire between τ^- and τ^+, as in [DRA 94].

As we have seen, the firability of a transition t depends on the value of δ_t. Some particular cases that may be worth discussing are the following:

- Autonomous constant delay transition (denoted by a label $\{r\}$):

$$\delta_t(M, u_D) = r = \text{const.}$$

- Controlled constant delay transition, *i.e.*, the effect of the control is only that of making the transition firable (denoted by a label $\{r\}_{u_D \in U_D}$)

$$\delta_t(M, u) = \begin{cases} r = \text{const} & \text{if } u_D \in U_D; \\ \infty & \text{if } u_D \notin U_D. \end{cases}$$

2.3.2 *Continuous transitions*

A transition $t \in T_C$ is *enabled* wrt a firing element $d \in D(t)$ at a marking $M(\tau)$ if:

$$\forall p \in {}^{\bullet}t \cap P_D, \quad M_p(\tau) \geq \rho_{(p,t)}[d].$$

This means that a continuous transition t is enabled wrt a firing color $d \in D(t)$ if each input discrete place p has at least as many tokens as specified by the arc inscription $\rho_{(p,t)}[d]$. The marking of continuous places does not affect the enabling of a continuous transition.

An enabled transition $t \in T_C$ fires continuously with velocity ν_t. Its effect is that of changing the marking of its output continuous places, while by the well-formedness condition 1 it does not modify the marking of any discrete place. The evolution of the marking of a place $p \in P_C$ due to the continuous firing of transition $t \in T_C$ is given by the differential equation:

$$\frac{dM_p(\tau)}{d\tau} = \sum_{t \in {}^{\bullet}p \cap T_C} A_{(t,p)} \cdot \nu_t(M(\tau), u_C(\tau)) \qquad [2]$$

Note that the firing of a continuous transition t does not depend on the particular element $d \in D(t)$ wrt whom the transition is enabled, but may depend on the marking M and on the external continuous input u_C. We denote the dependence of ν_t on external input u_C by the label $\{\nu_t\}_{u_C}$.

2.4 *Evolution and fundamental equation*

To keep track of the transition firings, we define a *firing vector* $\sigma(\tau)$. The component of σ associated to transition t is denoted σ_t and is defined as follows.

- For all $t \in T_D, \sigma_t(\tau) \in \mathcal{S}(D(t))$ is a multiset that expresses how many times and wrt which elements $d \in D(t)$ transition t has fired in the time interval $[0, \tau]$.

- For all $t \in T_C, \sigma_t(\tau) \in \mathbb{R}^{c(t)\times 1}$ is defined as

$$\sigma_t(\tau) = \int_0^\tau v_t(\tau')d\tau'$$

where $v_t(\tau)$ expresses the firing speed of transition t at time τ, *i.e.*,

$$v_t(\tau) = \begin{cases} \nu_t(M(\tau), u_C(\tau)) & \text{if } t \text{ is enabled at } M(\tau); \\ \{0\}^{c(t)\times 1} & \text{otherwise.} \end{cases}$$

We define the *incidence matrix* of a HLHPN as the matrix C of dimension $m \times n$ and such that:

$$C(p,t) = \begin{cases} \rho_{(t,p)} - \rho_{(p,t)} & \text{if } p \in P_D \text{ and } t \in T_D; \\ f_{(t,p)} - f_{(p,t)} & \text{if } p \in P_C \text{ and } t \in T_D; \\ \phi & \text{if } p \in P_D \text{ and } t \in T_C; \\ A_{(t,p)} & \text{if } p \in P_C \text{ and } t \in T_C. \end{cases}$$

where ϕ is the empty multirelation.

Let us now define the "$\circ$" operator (it is similar to the matrix product with substitution defined in [MUR 89]) as follows:

- $\forall \rho \in \mathcal{R}(D, D'), \forall \alpha \in \mathcal{S}(D), \rho \circ \alpha = \rho[\alpha]$ (the evaluation of ρ in α);
- $\forall f : D \to \mathbb{R}^k, \forall \alpha \in \mathcal{S}(D), f \circ \alpha = f(\alpha)$ (the evaluation of f in α);
- $\forall A \in \mathbb{R}^{h,k}, \forall B \in \mathbb{R}^{k,j}, A \circ B = A \cdot B$ (the matrix product).

Then from [[1]] and [[2]] it follows that

$$\begin{cases} \forall p \in P_D, \quad M_p(\tau) = M_p(0) & + \sum_{t \in T_D} \{\rho_{(t,p)}[\sigma_t(\tau)] - \rho_{(p,t)}[\sigma_t(\tau)]\} \\ \forall p \in P_C, \quad M_p(\tau) = M_p(0) & + \sum_{t \in T_D} \{f_{(t,p)}(\sigma_t(\tau)) - f_{(p,t)}(\sigma_t(\tau))\} \\ & + \sum_{t \in T_C} A_{(t,p)} \cdot \sigma_t(\tau) \end{cases}$$

and writing it in matrix form we obtain the fundamental equation:

$$M(\tau) = M(0) + C \circ \sigma(\tau)$$

There are some particular cases that may be worth considering.

- If $P_C = \emptyset$ and $T_C = \emptyset$, the net reduces to a timed colored net with possible external discrete inputs.
- If $P_C = \emptyset$, $T_C = \emptyset$ and for all $p, D(p) = \{c_1\} = \{\bullet\}$ (*i.e.*, the place domain is a singleton), we have the classical timed place/transition net with colorless tokens and possible external discrete inputs.

2.5 *Evolution graph*

The evolution, wrt time, of a hybrid system modeled by HLHPN can be concisely represented in a diagram composed by a sequence of phases (shown as boxes) and transitions (shown as bars) connected by arrows, called *Evolution Graph*.

Each bar represents a discrete transition and is labeled by its occurrence time instant τ_i. The firing vector and the marking of the net before the transition firing — $\sigma(\tau_i^-)$ and $M(\tau_i^-)$ — and after the transition firing — $\sigma(\tau_i^+)$ and $M(\tau_i^+)$ — are shown on the right hand side (RHS) of the bar. Only the initial firing vector and the initial value of the marking of the net are associated to the initial bar marked with $\tau = 0$.

The delay δ associated to the discrete transition is shown on the left hand side (LHS) of the bar. In the case of controlled transition, the value of the forcing discrete input is shown on the LHS of the bar as well.

The continuous behaviour between two discrete transitions (phase) is represented by a box containing the firing velocity vectors of the enabled continuous transitions $\nu_{t_i}(M(\tau), u_C(\tau))$, $(t_i \in T_C)$. On the RHS of each box the values of the discrete inputs u_D, of the firing vector $\sigma(\tau)$, and of the marking of the net $M(\tau)$ are shown.

The arrows connecting bars and boxes represent the logical sequence of continuous phases and discrete events. It is worth noting that simultaneous discrete transitions firings (multiple firings at the same time instant τ) can be well represented by the proposed evolution graph by a sequence of bars labeled $\tau^{(1)}$, $\tau^{(2)}$, etc.

3 Basic hybrid structures

A hybrid behaviour occurs when a system is characterized by interacting continuous and discrete event subsystems.

Usually continuous and discrete event systems are modeled by differential equations and by automata or Petri nets respectively. HLHPN are an extension of Petri net structures, and, therefore, can represent discrete event systems as a particular case.

The aim of this section is to show how HLHPN can model purely continuous systems and basic hybrid behaviours such as switches and jumps [BRA 94].

3.1 *Continuous evolution*

The model of continuous subsystems we consider is represented by a differential equation of the form

$$\begin{array}{l} \dot{x}_C(\tau) = \varphi[x_C(\tau); u_C(\tau); \tau] \\ \forall\, x_D(\tau)\ ;\ u_D(\tau) \end{array} \qquad [3]$$

where $x = [x_C^T; x_D^T]^T \in X$ is the whole system state vector, x_C and x_D being its continuous and discrete components respectively, $u = [u_C^T; u_D^T]^T \in U$ is the control state vector, u_C and u_D being its continuous and discrete components respectively, $\varphi[\cdot]$ is a vector function, and τ is the independent variable time.

In figure 3.1 the continuous subsystem dynamics is represented by means of a HLHPN. The continuous transition t_C fires with firing velocity ν_{t_C} depending on the marking x_C of the continuous place p_C, and on the continuous input u_C. The firing of t_C modifies the marking of p_C by the arc $a = (t_C, p_C)$ with inscription $\mathcal{I}_a$. The vector function φ is therefore defined as $\varphi = \mathcal{I}_a \cdot \nu_{t_C}$. As an example, $\mathcal{I}_a$ may be taken as the identity matrix of proper order, so that $\varphi = \nu_{t_C}$.

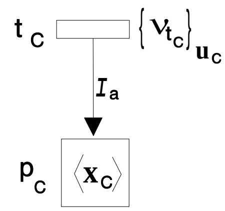

Figure 1. Continuous evolution

3.2 *Autonomous/Controlled switching*

In case the hybrid dynamics involves the switching among different vector functions $\varphi_i[x_C(\tau); u_C(\tau); \tau]$, depending both on the whole system state $x(\tau)$ and on the control $u(\tau)$, the continuous state dynamics is represented by the following

$$\begin{array}{l} \dot{x}_C(\tau) = \varphi_i[x_C(\tau); u_C(\tau); \tau] \\ (x; u) \in X_i \times U_i \end{array} \qquad [4]$$

where X_i and U_i are proper subset of the state and control space, respectively, such that $\bigcup_i X_i = X$ and $\bigcup_i U_i = U$.

If the switching between φ_i and φ_j, $(i \neq j)$, at a time instant τ_s, is due to the state evolution (*i.e.*, $x(\tau_s^-) \in X_i$, $x(\tau_s^+) \in X_j$, $x(\tau_s) \notin X_i \bigcap X_j$, and $u(\tau_s) \in U_i \bigcap U_j$) it is defined "autonomous". Analogously if it is due to the effect of the control input, both continuous or discrete, it is defined "controlled".

Switches can be easily represented with HLHPN. As an example, in figure 3.2 a controlled switching between the vector field $\varphi_1[x_C(t); u_C(t); t] = \mathcal{I}_{a_1} \cdot \nu_{t_{C_1}}$ and $\varphi_2[x_C(t); t] = \mathcal{I}_{a_2} \cdot \nu_{t_{C_2}}$ is represented. A discrete control u_D makes the discrete transition t_D firable after a delay δ. Its firing, moves the token from the discrete place p_{D_1}, which enables the continuous transition t_{C_1}, to p_{D_2}, which enables the continuous transition t_{C_2}.

Clearly if the firability of the discrete transition t_D does not depend on a discrete control, an autonomous switch will be represented.

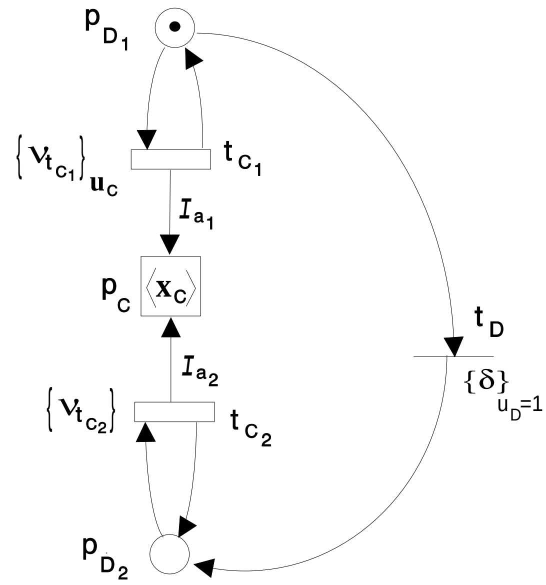

Figure 2. Controlled switching

3.3 *Autonomous/Controlled jump*

Consider a piecewise continuous state vector such that $x_C(\tau_h^-) \neq x_C(\tau_h^+)$, where τ_h $(h = 1, 2, \cdots)$ are the time instants at which a discontinuity occurs. At any τ_h the system state x_C change instantaneously and we can write

$$\dot{x}_C(\tau) = \varphi[x_C(\tau); u_C(\tau); t] + \mathbf{B}\delta\{\Gamma[x(\tau); u_D(\tau)]\} \qquad [5]$$

where $\mathbf{B}$ is a real matrix, and $\delta[\cdot]$ is a vector Dirac function whose elements δ_i are defined as follows

$$\delta_i(\Gamma_i) = \begin{cases} 0 & \Gamma_i \neq 0 \\ \infty & \Gamma_i = 0 \end{cases} \qquad [6]$$

Such a discontinuity can be represented by the enabling and firing of a discrete transition which, by means of proper arc inscriptions, changes the marking of the connected continuous places. In figure 3.3 an autonomous jump of the continuous state $x_C(\tau)$ from $x_C(\tau_h^-) = \mathcal{I}_{a_4}$ to $x_C(\tau_h^+) = \mathcal{I}_{a_5}$, due to the enabling and firing of the discrete transition t_D as soon as $x_C \in D(t_D)$, is represented. With reference to [[5]] and [[6]], we have $\mathbf{B}_{i,j(i\neq j)} = 0$, $\mathbf{B}_{i,i} = \mathcal{I}_{a_{5_i}} - \mathcal{I}_{a_{4_i}}$, $\Gamma_i = \Gamma\left[x_C; D(t_D)\right]$, $(i, j = 1, 2, \ldots, c(p_C))$.

The jump will be called "controlled" if the enabling and firing of the discrete transition depends on a discrete input.

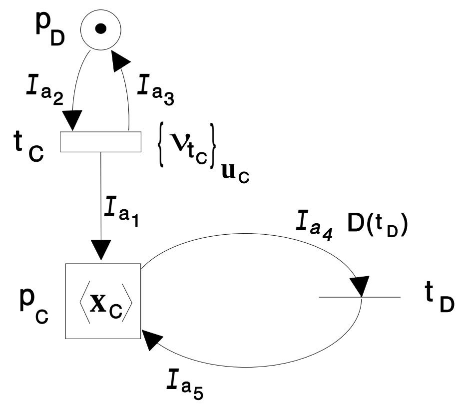

Figure 3. Autonomous jump

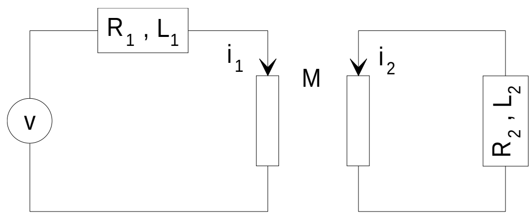

Figure 4. The electrical circuit

4 Example: an electrical circuit

Let us consider two RL circuits coupled or uncoupled by means of an external action that switches the mutual inductance from M to 0 and vice versa. One of the circuits is supplied by a voltage generator $v(\tau)$ (figure 4). This example shows how HLHPN can model switches in the dynamics.

Let $i_1(\tau)$ and $i_2(\tau)$ be the currents in the supplied and induced circuit respectively. The dynamics of the system is given by

$$\left\{\begin{array}{lll} \frac{di_1}{d\tau} & = g_1(i_1, i_2, v) & = -\frac{R_1 L_2 i_1}{L_1 L_2 - M^2} + \frac{M R_2 i_2}{L_1 L_2 - M^2} + \frac{L_2 v}{L_1 L_2 - M^2} \\ \frac{di_2}{d\tau} & = g_2(i_1, i_2, v) & = +\frac{M R_1 i_1}{L_1 L_2 - M^2} - \frac{L_1 R_2 i_2}{L_1 L_2 - M^2} - \frac{M v}{L_1 L_2 - M^2} \end{array}\right. \quad [7]$$

When the mutual inductance is switched from M to 0 the system evolves as two independent RL circuits according with the following relationships

$$\left\{\begin{array}{lll} \frac{di_1}{d\tau} & = G_1(i_1, v) & = -\frac{R_1}{L_1} i_1 + \frac{1}{L_1} v \\ \frac{di_2}{d\tau} & = G_2(i_2) & = -\frac{R_2}{L_2} i_2 \end{array}\right. \quad [8]$$

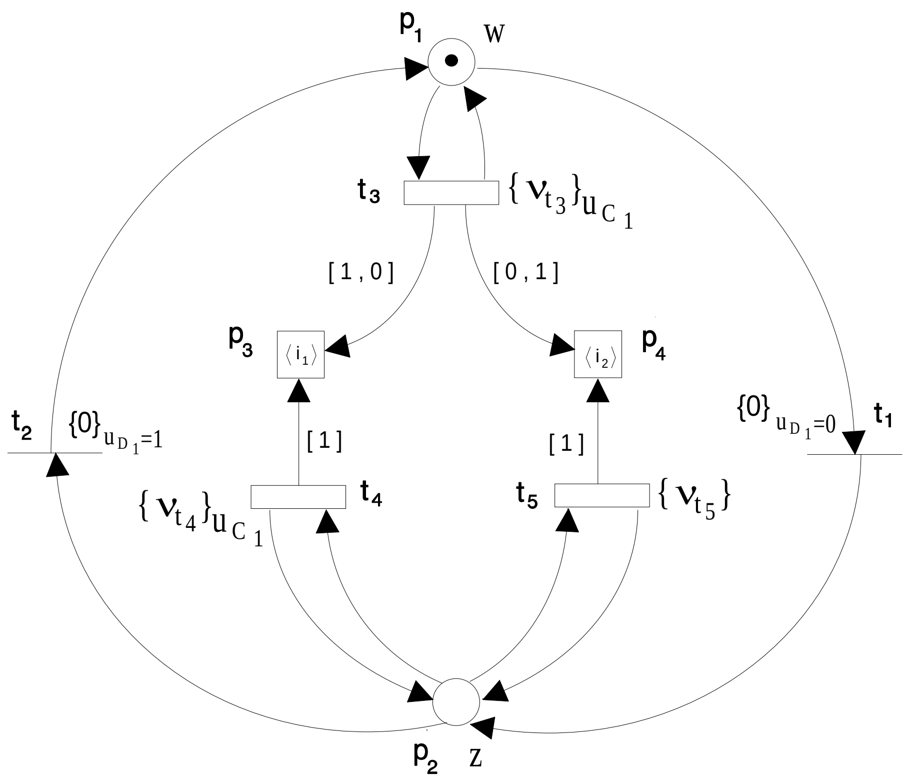

Figure 5. The electrical circuit net

In both [[7]] and [[8]] the voltage generator v is considered as an external continuous input.

It is possible to represent the dynamics of this system with the high-level hybrid Petri nets in figure 4.

- p_1 is a discrete place that is marked when the two circuits are coupled.
- p_2 is a discrete place that is marked when the two circuits are not coupled.
- p_3 is a 1-dimensional continuous place whose marking (denoted by the token $\langle i_1 \rangle$) represents the actual value of $i_1 \in \mathbb{R}$.
- p_4 is a 1-dimensional continuous place whose marking (denoted by the token $\langle i_2 \rangle$) represents the actual value of $i_2 \in \mathbb{R}$.
- u_{D1} is a discrete external input: when $u_{D1} = 0$ the two circuits are uncoupled while when $u_{D1} = 1$ they are coupled.

- u_{C1} is an external continuous input corresponding to the ideal voltage generator in figure 4, *i.e.* $u_{C1}(\tau) = v(\tau)$.

- t_1 is a discrete transition that, as soon as it is enabled, fires immediately, *i.e.*, its firing delay is $\{0\}$, when the discrete external input has a value $u_{D1} = 0$ (the circuits are decoupled).

- t_2 is a discrete transition that, as soon as it is enabled, fires immediately, *i.e.*, its firing delay is $\{0\}$, when the discrete external input has a value $u_{D1} = 1$ (the circuits are coupled).

- t_3 is a continuous transition that defines the coupled dynamics of the variables i_1 and i_2 by means of the "firing velocity" $\{\nu_{t_3}\} = \{[g_1(i_1, i_2, u_{C1}), g_2(i_1, i_2, u_{C1})]^T\}$; when t_3 is enabled, *i.e.*, when p_1 is marked, it fires continuously and changes the marking of the places p_3 and p_4 according to

$$\begin{aligned} i_1(\tau) &= i_1(\tau_i) + \int_{\tau_i}^{\tau} [1,0]\nu_{t_3}(\tau')d\tau' \\ i_2(\tau) &= i_2(\tau_i) + \int_{\tau_i}^{\tau} [0,1]\nu_{t_3}(\tau')d\tau' \end{aligned}$$

  while the firing of transition t_3 does not change the marking of the discrete place p_1.

- t_4 and t_5 are continuous transition that defines the uncoupled dynamics of the variables i_1 and i_2 (respectively) by means of the "firing velocity" $\{\nu_{t_4}\} = \{[G_1(i_1, u_{C1})]\}$ and $\{\nu_{t_5}\} = \{[G_2(i_2)]\}$; when t_4 and t_5 are enabled, *i.e.*, when p_2 is marked, they fire continuously and changes the marking of the place p_3 and p_3 (respectively) according to

$$\begin{aligned} i_1(\tau) = i_1(\tau_i) + \int_{\tau_i}^{\tau} [1]\nu_{t_4}(\tau')d\tau' \\ i_2(\tau) = i_2(\tau_i) + \int_{\tau_i}^{\tau} [1]\nu_{t_5}(\tau')d\tau' \end{aligned}$$

  while the firing of transition t_4 and t_5 does not change the marking of the discrete place p_2.

The complete structure of the HLHPN in figure 4 can be exactly defined as follows.

- $P_D = \{p_1, p_2\}$, $m_D = 2$, $P_C = \{p_3, p_4\}$, $m_C = 2$, $c(p_3) = c(p_4) = 1$, $D(p_1) = D(p_2) = \{\bullet\}$, $D(p_3) = D(p_4) = \mathbb{R}$;

- $M(p_1) = w$, $M(p_2) = z$, $M(p_3) = i_1$, $M(p_4) = i_2$;

- $T_D = \{t_1, t_2\}$, $n_D = 2$, $T_C = \{t_3, t_4, t_5\}$, $n_C = 3$, $c(t_3) = 2$, $c(t_4) = c(t_5) = 1$, $D(t_1) = D(t_2) = \{(w, z) | w \in D(p_1), z \in D(p_2)\}$, $D(t_3) = \{w | w \in D(p_1)\}$, $D(t_4) = D(t_5) = \{z | z \in D(p_2)\}$;

- $u_D = [u_{D1}] \in \{0, 1\}$, $u_C = [u_{C1}] \in \mathbb{R}$;

- $\delta_{t_1}(M, u_{D1}) = 0$ if $u_{D1} = 0$ else $\delta_{t_1}(M, u_{D1}) = \infty$,
  $\delta_{t_2}(M, u_{D1}) = 0$ if $u_{D1} = 1$ else $\delta_{t_2}(M, u_{D1}) = \infty$;

- $\nu_{t_3} = [g_1(i_1, i_2, u_{C1}), g_2(i_1, i_2, u_{C1})]^T$, $\nu_{t_4} = [G_1(i_1, u_{C1})]$, $\nu_{t_5} = [G_2(i_2)]$;
- $\mathcal{I}_{(p_1,t_1)}$: $\rho_{(p_1,t_1)}(d, d') = 1 \;\; \forall d \in D(t_1), d' \in D(p_1)$,
  $\mathcal{I}_{(p_1,t_3)}$: $\rho_{(p_1,t_3)}(d, d') = 1 \;\; \forall d \in D(t_3), d' \in D(p_1)$,
  $\mathcal{I}_{(t_1,p_2)}$: $\rho_{(t_1,p_2)}(d, d') = 1 \;\; \forall d \in D(t_1), d' \in D(p_2)$,
  $\mathcal{I}_{(p_2,t_2)}$: $\rho_{(p_2,t_2)}(d, d') = 1 \;\; \forall d \in D(t_2), d' \in D(p_2)$,
  $\mathcal{I}_{(p_2,t_4)}$: $\rho_{(p_2,t_4)}(d, d') = 1 \;\; \forall d \in D(t_4), d' \in D(p_2)$,
  $\mathcal{I}_{(p_2,t_5)}$: $\rho_{(p_2,t_5)}(d, d') = 1 \;\; \forall d \in D(t_5), d' \in D(p_2)$,
  $\mathcal{I}_{(t_2,p_1)}$: $\rho_{(t_2,p_1)}(d, d') = 1 \;\; \forall d \in D(t_2), d' \in D(p_1)$,
  $\mathcal{I}_{(t_3,p_1)} = \mathcal{I}_{(p_1,t_3)}$, $\mathcal{I}_{(t_4,p_2)} = \mathcal{I}_{(p_2,t_4)}$, $\mathcal{I}_{(t_5,p_2)} = \mathcal{I}_{(p_2,t_5)}$,
  $\mathcal{I}_{(t_3,p_3)}$:$A_{(t_3,p_3)} = [1, 0]$, $\mathcal{I}_{(t_3,p_4)}$:$A_{(t_3,p_4)} = [0, 1]$,
  $\mathcal{I}_{(t_4,p_3)}$:$A_{(t_4,p_3)} = [1]$, $\mathcal{I}_{(t_5,p_4)}$:$A_{(t_5,p_4)} = [1]$.

The incidence matrix is the following

$$C = \left[\begin{array}{cc|ccc} -1 \otimes \bullet & +1 \otimes \bullet & \phi & \phi & \phi \\ +1 \otimes \bullet & -1 \otimes \bullet & \phi & \phi & \phi \\ \hline [0] & [0] & [1,0] & [1] & [0] \\ [0] & [0] & [0,1] & [0] & [1] \end{array}\right]$$

The terms of type $1 \otimes \bullet$ in the upper-left part represent the constant multirelation such that $\rho[d] = 1 \otimes \bullet$, $\forall d$. The symbol ϕ in the upper-right part denotes the empty multirelation. The elements [0] in the lower-left part denote the zero 1-dimensional vector function, *i.e.*, the function $f(d) = 0 \; \forall d$. The elements in the lower-right part are constant matrices.

The evolution graph of this hybrid net is shown in figure 5. At the initial time instant $\tau = 0$ the two circuits are coupled, *i.e.*, place p_1 is marked and $u_{D1} = 1$, and no current is present, *i.e.*, $i_1 = i_2 = 0$. Due to the effect of the voltage generator, the system evolves according to [[7]], *i.e.*, $v_{t_3} = \nu_{t_3}$, $v_{t_4} = v_{t_5} = 0$. At an arbitrary time instant τ_1 an external action decouples the two circuits, *i.e.*, the discrete input changes its value from $u_{D1} = 1$ to $u_{D1} = 0$, then the discrete transition t_1 fires and the token passes from p_1 to p_2; from now on the system evolves as two distinct and independent circuits according to [[8]], *i.e.*, $v_{t_3} = 0$, $v_{t_4} = \nu_{t_4}$ and $v_{t_5} = \nu_{t_5}$, with initial marking $M(p_3) = i_1(\tau_1)$ and $M(p_4) = i_2(\tau_1)$.

5 Example: a bouncing ball

Consider the problem of modeling the motion of an elastic ball bouncing on an horizontal plane with infinite stiffness and subjected only to the gravitational force (figure 5).

This example shows how HLHPN can model jumps in the state space.

The continuous-time state vector of the system is $x_C = [x_1, x_2]^T$ where x_1 is the distance of the ball from the horizontal plane surface and x_2 is the velocity of the ball, assumed positive in the upwards direction. The horizontal plane

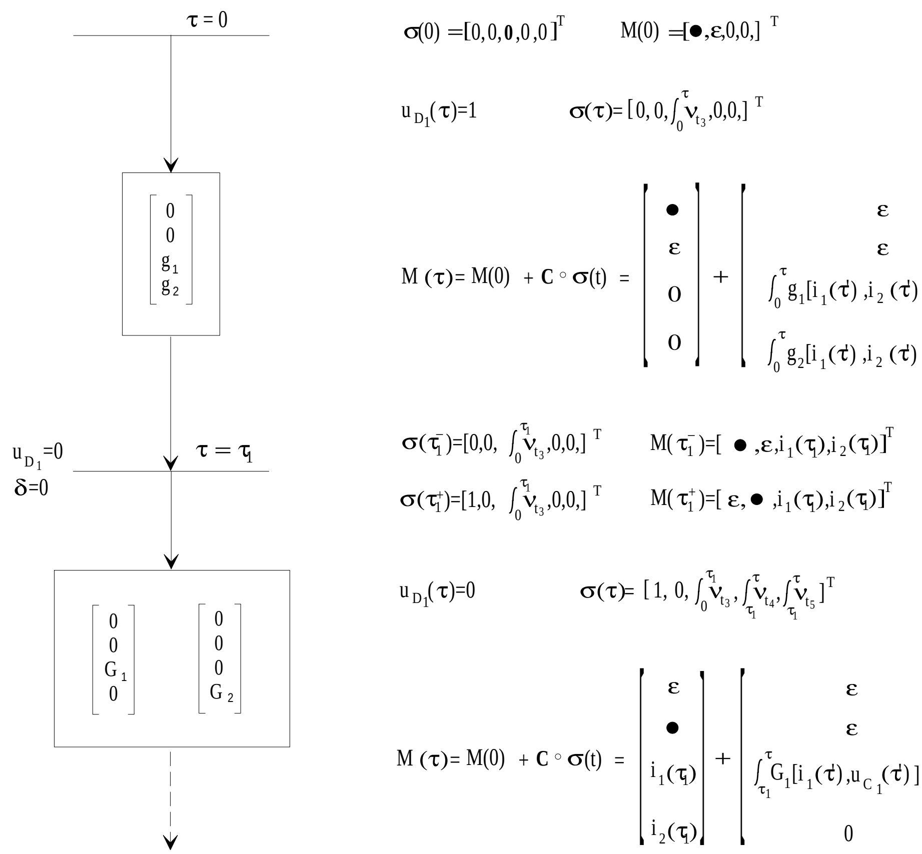

Figure 6. The electrical system net evolution graph

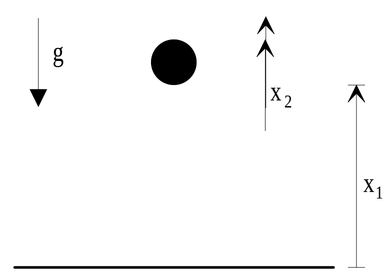

Figure 7. The bouncing ball

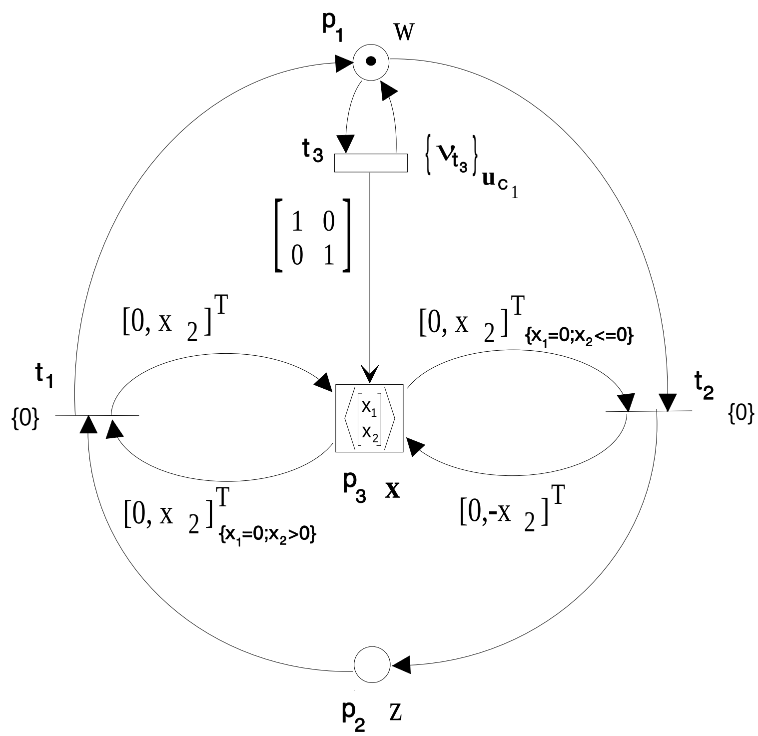

Figure 8. The bouncing ball hybrid net

constrains the state space to the half-plane in which $x_1 \geq 0$. The dynamics of the system is represented by the following

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g - 2\ x_2\ \delta(x_1) \end{cases} \qquad [9]$$

where $\delta(\cdot)$ is the Dirac function.

It is possible to represent the dynamics of this system by means of the HLHPN in figure 5. where:

- p_1 is a discrete place that is marked when the ball is on air.
- p_2 is a discrete place that is marked when the ball is on the plane.
- p_3 is a 2–dimensional continuous place whose marking (denoted by the token $\langle[x_1, x_2]^T\rangle$) represents the actual value of the system state vector $\mathbf{x} = [x_1, x_2]^T$.
- u_{C1} is an external continuous input corresponding to the constant gravity acceleration, *i.e.* $u_{C1}(\tau) = -g$.

- t_1 is a discrete transition that, as soon as it is enabled, fires immediately, *i.e.*, its firing delay is $\{0\}$, when the ball is on the plane and with a positive (upwards) velocity, *i.e.* $x_1 = 0$ and $x_2 > 0$ as shown in the pre arc between p_3 and t_1. This transition does not change the marking of the continuous place.

- t_2 is a discrete transition that, as soon as it is enabled, fires immediately, *i.e.*, its firing delay is $\{0\}$, when the ball reaches the plane surface with non positive velocity, *i.e.* $x_1 = 0$ and $x_2 \leq 0$ as shown in the pre arc between p_3 and t_2. This transition changes the marking of the continuous place p_3 from $\mathbf{x}$ to $-\mathbf{x}$ by removing the token $\langle [0, x_2]^T \rangle$ and adding the token $\langle [0, -x_2]^T \rangle$.

- t_3 is a continuous transition that defines the continuous dynamics of the state vector $\mathbf{x}$ by means of the "firing velocity" $\{\nu_{t_3}\} = \{[x_2, -g]^T\}$. When t_3 is enabled, *i.e.* p_1 is marked, it fires continuously and changes the marking of the place p_3 according with the following relationship

$$\mathbf{x}(\tau) = \mathbf{x}(\tau_i) + \int_{\tau_i}^{\tau} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \nu(\tau')d\tau'$$

  The transition t_3 does not change the marking of the discrete place p_1.

It is worth noting that when the ball reaches the surface plane with downwards velocity, say at time instant τ^*, the transitions t_2 and t_1 fire subsequently at the same time instant τ^*.

The complete structure of the HLHPN presented in figure 5 can be exactly defined as follows:

- $P_D = \{p_1, p_2\}$, $m_D = 2$, $P_C = \{p_3\}$, $m_C = 1$, $c(p_3) = 2$, $D(p_1) = D(p_2) = \{\bullet\}$, $D(p_3) = \mathbb{R}^2$;

- $M(p_1) = w$, $M(p_2) = z$, $M(p_3) = [x_1, x_2]^T$;

- $T_D = \{t_1, t_2\}$, $n_D = 2$, $T_C = \{t_3\}$, $n_C = 1$, $c(t_3) = 2$,
  $D(t_1) = \{(w, z, x_1, x_2) | w \in D(p_1), z \in D(p_2), x_1 = 0, x_2 > 0\}$,
  $D(t_2) = \{(w, z, x_1, x_2) | w \in D(p_1), z \in D(p_2), x_1 = 0, x_2 \leq 0\}$,
  $D(t_3) = \{w | w \in D(p_1)\}$;

- $u_C = [u_{C1}] = -g$;

- $\delta_{t_1}(M) = \delta_{t_2}(M) = 0$;

- $\nu_{t_3} = [x_2, -g]^T$;

- $\mathcal{I}_{(p_1,t_2)}$: $\rho_{(p_1,t_2)}(d, d') = 1 \ \ \forall d \in D(t_2), d' \in D(p_1)$,
  $\mathcal{I}_{(p_1,t_3)}$: $\rho_{(p_1,t_3)}(d, d') = 1 \ \ \forall d \in D(t_3), d' \in D(p_1)$,
  $\mathcal{I}_{(t_1,p_1)}$: $\rho_{(t_1,p_1)}(d, d') = 1 \ \ \forall d \in D(t_1), d' \in D(p_1)$,
  $\mathcal{I}_{(t_2,p_2)}$: $\rho_{(t_2,p_2)}(d, d') = 1 \ \ \forall d \in D(t_2), d' \in D(p_2)$,
  $\mathcal{I}_{(p_2,t_1)}$: $\rho_{(p_2,t_1)}(d, d') = 1 \ \ \forall d \in D(t_1), d' \in D(p_2)$,

$\mathcal{I}_{(p_3,t_2)}$: $f_{(p_3,t_2)}(d) = [0, x_2]^T \ \ \forall d \in D(t_2)$,
$\mathcal{I}_{(t_2,p_3)}$: $f_{(t_2,p_3)}(d) = [0, -x_2]^T \ \ \forall d \in D(t_2)$,
$\mathcal{I}_{(p_3,t_1)}$: $f_{(p_3,t_1)}(d) = [0, x_2]^T \ \ \forall d \in D(t_1)$,
$\mathcal{I}_{(t_3,p_1)} = \mathcal{I}_{(p_1,t_3)}$, $\mathcal{I}_{(t_1,p_3)} = \mathcal{I}_{(p_3,t_1)}$,
$\mathcal{I}_{(t_3,p_3)}$:$A_{(t_3,p_3)} = [[1,0]^T, [0,1]^T]$.

The incidence matrix is the following

$$C = \left[\begin{array}{cc|c} +1\otimes\bullet & -1\otimes\bullet & \phi \\ -1\otimes\bullet & +1\otimes\bullet & \phi \\ \hline \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -2x_2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}\right]$$

The terms in the upper part represent multirelations. The elements in the lower-left part denote vector functions, *i.e.*, $[0,0]^T$ represent the constant function $f(d) = [0,0]^T \forall d$ and $[0,-2x_2]^T$ represent the function $f(d) = [0,-2x_2]^T$ $\forall d = (w, z.x_1, x_2)$. The element in the lower-right part is a constant matrix.

The evolution graph of this hybrid net is shown in figure 5. At the initial time instant $\tau = 0$ the ball is at a distance x_{1_0} from the ground with no velocity, *i.e.*, $x_1 = x_{1_0}$, $x_2 = 0$, and the discrete place p_1 is marked. The ball falls down subjected to the gravity force according to [[9]]. At time instant $\tau_1 = (\frac{2x_{1_0}}{g})^{1/2}$ the ball reaches the ground with downwards velocity, *i.e.* $x_1 = 0$, $x_2 = -g\tau_1$. Then the discrete transition t_2 fires immediately, the token is moved from p_1 to p_2 and the marking of the continuous place p_3 changes from $\mathbf{x}(\tau_1^-)$ to $\mathbf{x}(\tau_1^+) = -\mathbf{x}(\tau_1^-)$. Transition t_1 fires immediately because $x_1 = 0$, $x_2 = g\tau_1 > 0$, p_2 is marked, and the system evolves as before subjected to the gravity force according to [[9]]. It is worth noting that during the infinitesimal time period in which the discrete place p_2 is marked, say from time instant τ_1^- to time instant τ_1^+ the continuous transition t_3 is not enabled, *i.e.* $v_{t_3} = 0$.

6 Conclusions

A hybrid model based on Petri nets that merges the concepts of high–level nets with continuous nets has been presented. A High Level Hybrid Petri Net can represent the dynamics of hybrid systems that can be characterized by jumps in the state space and switching in the dynamics, both autonomous and controlled by means of arbitrary external inputs. The evolution of the continuous dynamics, *i.e.* the rate of change of the marking of the continuous places, is influenced by the marking of the discrete places, that enable or disable the continuous transitions, and could be controlled by means of external continuous input.

The HLHPN herein presented constitutes a useful model that provides a simple graphical representation of hybrid systems and take advantage of the modular structure of Petri nets in giving a compact description of systems composed of interacting subsystems, both time-continuous and discrete-event.

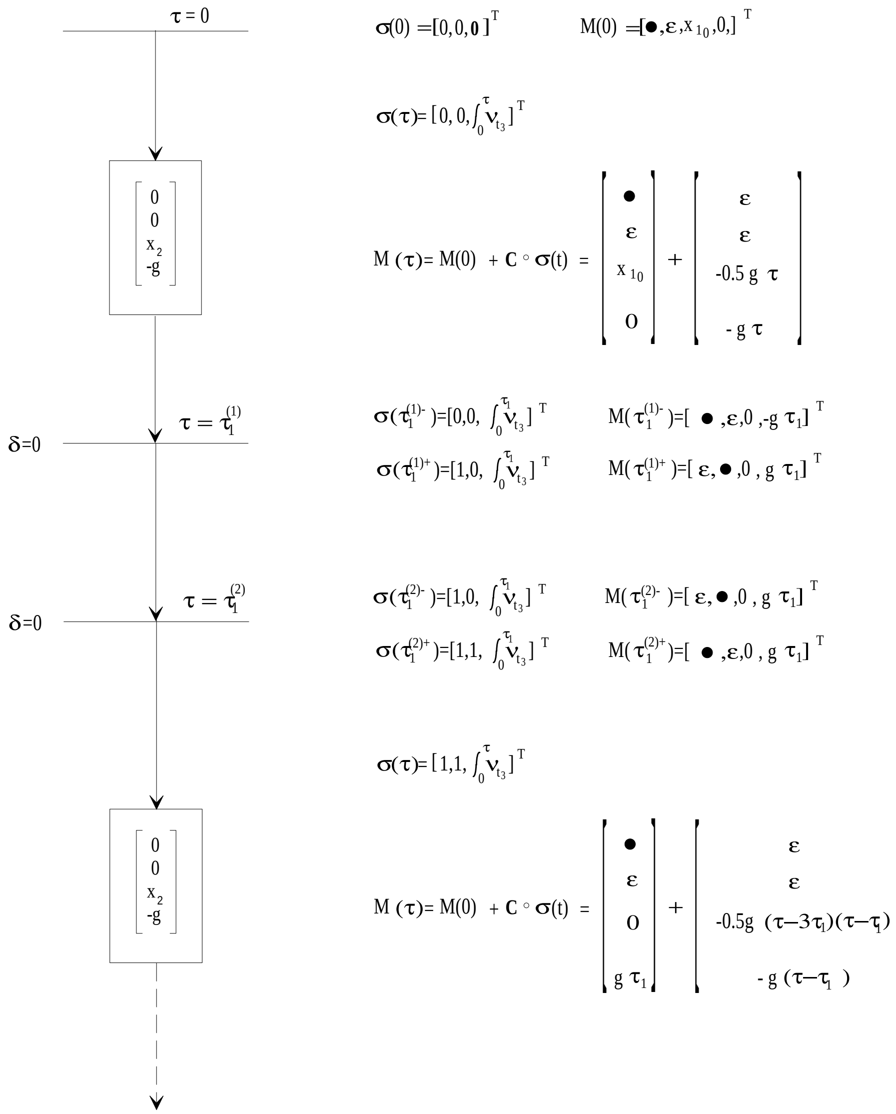

Figure 9. The bouncing ball evolution graph

The used formalism seems to be able to encompass large classes of hybrid systems and allows us to extend some of the standard structural concepts of Petri nets. This fact suggests that other Petri nets analysis and control techniques may be generalized within this framework as well. Further investigations on the analogies and correspondences between this model and the other presented in the literature could be profitable.

References

[BAI 91] LE BAIL J., ALLA H., DAVID R., "Hybrid Petri Nets," *Proc. 1st European Control Conference* (Grenoble, France), p. 1472– 1477, July 1991.

[BAL 98] BALDUZZI F., MENGA G., GIUA A., "Modelling Flexibile Manufacturing Systems with Hybrid Stochastic Petri Nets," *Proc. 4th Int. Work. on Discrete Event Systems* WODES98 (Cagliari, Italy), p. 432–438, August, 1998.

[BRA 94] BRANICKY M.S., BORKAR V.S., MITTER S.K., "A Unified Framework for Hybrid Control," *Proc. 33rd Conf. on Decision and Control* (Lake Buena Vista, Florida), p. 4228-4234, December, 1994.

[BRO 93] BROCKETT R.W., "Hybrid models for motion control systems," *Essays in Control*, H.L.Trentelman and J.C. Willems (Eds.), p. 29-53, Birkhauser, Boston, 1993.

[CHA 98] CHAMPAGNAT R., ESTEBAN Ph. , PINGAUD H., VALETTE R., "Modelling and Simulation of a Hybryd System Through Pr/Tr PN-DAE Model," *Proc. 3rd Int. Conf. on the Automation of Mixed Processes* ADMP98 (Reims, France), p. 131–137, March, 1998.

[DAV 92] DAVID R., ALLA H., "Autonomous and Timed Continuous Petri Nets," *Advances in Petri Nets 1993*, G. Rozenberg (Ed.), LNCS, vol. 674, p. 71–90, Springer-Verlag, 1992.

[DAV 97] DAVID R., "Modeling of Hybrid Systems Using Continuous and Hybrid Petri Nets," *Proc. Petri Nets and Performance Models* PNPM97 (St-Malo, France), p. 47–57, June, 1997.

[DEM 98] DEMONGODIN I., KOUSSOULAS N.T., "Modelling of Hybrid Control Systems Using Petri Nets," *Proc. 3rd Int. Conf. on the Automation of Mixed Processes* ADMP98 (Reims, France), p. 330–337, March, 1998.

[DRA 94] DRAKUNOV S., "Sliding Modes in Hybrid Systems - A Semigroup Approach," *Proc. 33rd Conf. on Decision and Control* (Lake Buena Vista, Florida), p. 4235-4240, December, 1994.

[GIU 96] GIUA A., USAI E. "High-Level Hybrid Petri Nets: a Definition," *Proc. 35th Conf. on Decision and Control* (Kobe, Japan), p. 148-150, December, 1996.

[GRO 93] GROSSMAN R., NERODE A., RAVN A., RISHEL H., (eds.), *Hybrid Systems,* LNCS, vol. 736, Springer-Verlag, 1993.

[HOR 96] HORTON G., KULKARNI V.G., NICOL D.M., TRIVEDI K.S., "Fluid Stochastic Petri Nets: Theory, Applications, and Solutions," *Technical Report 96-5*, ICASE, NASA Langley Research Center (Hampton, VA, USA), 1996.

[MUR 89] MURATA T., "Petri Nets: Properties, Analysis and Applications," *Proceedings IEEE*, vol. 77, n° 4, p. 541–580, April, 1989.

[REI 85] REISIG W., *Petri Nets: an Introduction*, Springer-Verlag, 1985.

[VAL 85] VALK R., JANTZEN M., "The Residue of Vector Sets with Applications to Decidability Problems in Petri Nets," *Acta Informatica*, n° 21, p. 643–674, Springer-Verlag, 1985.

[VAR 98] VALENTIN-ROUBINET C., "Modelling of Hybryd Systems: DAE Supervised by Petri Nets, The Example of a Gas Storage," *Proc. 3rd Int. Conf. on the Automation of Mixed Processes* ADMP98 (Reims, France), p. 142–149, March, 1998.

[WOL 98] WOLTER K., "Probabilistic and Deterministic Fluid Flow in Second Order FSPN's," *Proc. 3rd Int. Conf. on the Automation of Mixed Processes* ADMP98 (Reims, France), p. 324–329, March, 1998.