Marking Estimation in a Class of Time Labelled Petri Nets

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Abstract

This paper proposes an efficient marking estimation method for a subclass of *time labelled Petri nets* (TLPNs) in which each transition is associated with an infinite upper bound delay. The unobservable subnet of the considered subclass of TLPNs is backward-conflict-free, and all the output transitions of each conflict place are observable. The highlight of this method is that the markings set consistent with a given observation can be determined by a linear algebraic system based on the so-called *slow-bound* marking and *fast-bound marking* pairs. An algorithm to compute an online estimator is provided and an example is given. By this method the exhaustive construction of the full state space including the state class graph is avoided. This approach provides guidelines of sensor deployment in the design stage so that the online marking estimation problem can be efficiently solved.

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1 Introduction

Labelled Petri nets (LPNs) have been proposed as a fundamental model for *Discrete Event Systems* in a wide variety of applications and have been an asset to reduce the computational complexity involved in solving control problems. An LPN contains *unobservable* and *indistinguishable* transitions, since an external agent may not detect or distinguish some events. Hence in general it is not possible to determine the exact current marking from a given observation but only a set of possible markings called *consistent markings*. The *marking estimation* in LPNs plays an important role in Petri net theory since it is relevant to many problems, including *supervisory control* [1, 2], *observation* [3, 4, 5], *diagnosis* [6, 7], and *opacity* [8].

When time factor is considered, a *time labelled Petri nets* (TLPN) is an LPN such that each transition is associated with a timer, and an enabled transition can fire only if its timer belongs to a certain time window. The *state* of a TLPN is a pair that consists of the logical marking and the timer vector of transitions. In many practical problems, such as supervisory control, one is often interested in the *markings* rather than the *state* of TLPNs. A general scenario in the literature to estimate markings in TLPNs is to first enumerate all consistent states followed by extracting the marking information from them. A well-established framework to analyze the state space of a TLPN is the *state class graph* (SCG) [9], by which the state space of a TLPN is partitioned into a finite set of *classes*. Based on SCGs, a series of fruitful results on the state estimation in TLPNs have been achieved [10, 11, 12, 13, 14, 15, 16, 17]. In [10, 11] an SCG is treated as a *timed automaton* (TA), and hence the estimation techniques in TA models (e.g., [18, 19]) can be applied. In [12] several state classes of an SCG are merged into one node in the estimator in an explicit form. In [13] a structure called the *fault diagnosis graph* (FDG) is obtained by the reduction of the SCG. In [14] a structure called the *modified state class graph* (MSCG) is proposed by using a different notion of classification for fault diagnosis in TLPNs. Besides, in [15] a linear algebraic method is proposed to reconstruct the least/greatest sequence of unobservable transitions in timed Petri nets based on the online observation. The work in [16, 17] considers a different model called *P-Time Petri nets* (which are incomparable with TLPNs [20]). In [16, 17] the marking estimation is first done in the underlying untimed LPN to obtain a set of consistent marking candidates, and then a linear programming problem is formulated for each marking to check its *schedulability* so that time-spurious ones are removed.

The aforementioned SCG-based methods[1] provide a very general approach to perform the state and the marking estimation in TLPNs, but they have two drawbacks. First, the computation of an SCG is impeded by the *state-space explosion*, since it is well acknowledged that an SCG can be much larger than the reachability graph of the underlying untimed LPN [10, 12, 13, 14]. Although in some cases the state classes can be abstracted [21], they cannot be used for estimation since some timing information is not preserved. Second, at each on-line step a series of *integer linear programming problems* (ILPPs) have to be solved to single-out time-spurious state classes one-by-one, and in those ILPPs the number of inequalities continuously grows with the elapsed time [13, 14], which is computationally quite expensive. This motivates us to develop new methods to perform marking estimation in TLPNs without analyzing the full state space.

In this work we propose a marking estimation method for a subclass of TLPNs that satisfies the following structural assumptions: (1) each transition has a known firing window with a finite lower bound and an infinite upper bound; (2) the unobservable subnet is *backward-conflict-free* [22, 23]; (3) no transition in a structural conflict is unobservable, i.e., labeled by the empty string. The first assumption indicates that the firing of

[1] In the sequel, the term "SCG" refers to both the *state class graph* and its derivatives/variants.

each transition has a finite known delay but does not necessarily fire as soon as possible, while the remaining two assumptions are made on the observation structure. The observation structure of a plant depends on the deployment of sensors and can be modified by additional sensors [24, 25]. For example, in [26] and [27], by associating a cost to each transition, the control problem and the diagnosability enforcement problem in Petri nets are reformulated as optimization problems. Hence we propose a method to find an optimal sensor selection that satisfies these assumptions. We show that the marking estimation in such a model can be done in a very efficient way without computing the SCG which is the most burdensome part in state-space analysis in TLPNs. As far as we know, there is no method to perform the marking estimation in TLPNs without explicitly enumerating the full state space. Thus we believe that our approach also provides guidelines in the design stage for choosing a suitable set of sensors that will allow the plant operator to efficiently solve the marking estimation problem.

In this work we introduce the notions of *slow-bound time-transition-sequences* (slow-bound TSs)[2] and *fast-bound time-transition-sequences* (fast-bound TSs) with respect to an observation. In the aforementioned subclass of TLPNs the consistent marking set is the union of a set of *reachable hulls* of *slow-fast-marking-pairs* (SFM-pairs). We propose an algorithm to construct a marking estimator that keeps track of SFM-pairs and updates them by the observation, based on which the consistent marking set can be described by a linear algebraic system. The online computational load is much smaller than that of SCG-based methods, since the consistent marking set is compactly represented by the union of several *reachable hulls*, and no ILPPs such as those in [11, 12, 13, 14] need to be solved online, since this representation does not contain time-spurious markings.

This paper is organized in seven sections. The basics of Petri nets are recalled in Section II. Section III formulates the problem and introduces the used assumptions. Section IV presents several notions including slow- and fast-bound TSs that will be used later to establish our method. Section V introduces a series of useful results for computing the set of markings consistent with a given observation. In Section VI an algorithm is proposed for marking estimation in TLPN, and an illustrative example is presented. Conclusions are reached in Section VII.

2 Preliminaries

2.1 Petri Net

A Petri net is a four-tuple $N = (P, T, Pre, Post)$, where P is a set of m *places* represented by circles; T is a set of n *transitions* represented by bars; $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are respectively the *pre-* and *post-incidence functions* that specify the arcs in the net and are represented as matrices in $\mathbb{N}^{m\times n}$. The *incidence matrix* of a net is defined by $C = Post - Pre \in \mathbb{Z}^{m\times n}$.[3]

For a transition $t \in T$ we define the *set of its input places* as ${}^\bullet t = \{p \in P \mid Pre(p, t) > 0\}$ and the *set of its output places* as $t^\bullet = \{p \in P \mid Post(p, t) > 0\}$. The notions for ${}^\bullet p$ and $p^\bullet$ are analogously defined.

A *marking* is a function $M : P \to \mathbb{N}$ that assigns to each place of a Petri net a non-negative integer

[2] All the notions that appear in this section are formally defined in the rest of this paper.

[3] Here we use $\mathbb{N} = \{0, 1, \ldots, \}$ and $\mathbb{Z} = \{0, \pm 1, \ldots\}$ to denote the sets of natural numbers and integer numbers, respectively. We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ to denote the set of nonnegative real numbers, and $\mathbb{R}_0^{+\infty} = \mathbb{R}_0^+ \cup \{+\infty\}$.

number of tokens, graphically represented by black dots and can also be algebraically represented as an m-component vector. We denote by $M(p)$ the marking of place p. A *marked net* $\langle N, M_0 \rangle$ is a net N with an initial marking M_0.

A transition t is said to be *enabled* at M if $M \geq Pre(\cdot, t)$ and may fire reaching a new marking $M' = M_0 + C(\cdot, t)$. We write $M[\sigma\rangle$ to denote that the sequence of transitions σ is enabled at M, and we write $M[\sigma\rangle M'$ to denote that the firing of σ at M yields M'. We use $\mathbf{y}_\sigma$ to denote the *Parikh vector* of $\sigma \in T^*$, i.e., $y_\sigma(t) = k$ if transition t appears k times in σ.

We denote by $R(N, M_0)$ the set of all markings reachable from the initial one. A Petri net $\langle N, M_0 \rangle$ is said to be *bounded* if there exists an integer $k \in \mathbb{N}$ such that for all $M \in R(N, M_0)$, $M(p) \leq k$ holds for all $p \in P$.

Given a net $N = (P, T, Pre, Post)$ we say that $\hat{N} = (\hat{P}, \hat{T}, \hat{Pre}, \hat{Post})$ is a *subnet* of N if $\hat{P} \subseteq P$, $\hat{T} \subseteq T$ and $\hat{Pre}$ (resp., $\hat{Post}$) is the restriction of Pre (resp., $Post$) to $\hat{P} \times \hat{T}$. In particular, $\hat{N}$ is called the *$\hat{T}$-induced subnet* if $\hat{N} = (P, \hat{T}, \hat{Pre}, \hat{Post})$.

Given a net $N = (P, T, Pre, Post)$, a *cycle* is a sequence $x_1 x_2 \cdots x_k x_1$ where $x_i \in P \cup T$ and $x_i \in {}^\bullet x_{i+1}$, $i \in \{1, \ldots, k-1\}$, and $x_k \in {}^\bullet x_1$. A cycle is said to be *elementary* if it does not contain other cycles. The set of all elementary cycles in a Petri net is denoted as $\mathcal{O}$. A net is said to be *acyclic* if it does not contain any cycles, i.e., $\mathcal{O} = \emptyset$.

Proposition 2.1 *[28] Given a Petri net $N = (P, T, Pre, Post)$ that is acyclic and two markings M and M', if there exists $\mathbf{y} \in \mathbb{N}^n, \mathbf{y} \geq \mathbf{0}$ such that $M + C \cdot \mathbf{y} = M' \geq 0$, then there exists a sequence $\sigma \in T^*$ whose firing vector is $\mathbf{y}$ such that $M[\sigma\rangle M'$.*

2.2 Time Labelled Petri Nets

A *labelled Petri net* (LPN) is a 4-tuple $G = (N, M_0, E, \ell)$, where $\langle N, M_0 \rangle$ is a marked net, E is the *alphabet* (a set of labels), and $\ell : T \to E \cup \{\varepsilon\}$ is the *labeling function* that assigns to each transition $t \in T$ either a symbol from E or the empty word ε. Therefore, the set of transitions can be partitioned into two disjoint sets $T = T_o \cup T_{uo}$, where $T_o = \{t \in T \mid \ell(t) \in E\}$ is the set of *observable* transitions and $T_{uo} = T \setminus T_o = \{t \in T \mid \ell(t) = \varepsilon\}$ is the set of *unobservable* transitions. The labeling function can be extended to sequences $\ell : T^* \to E^*$, i.e., $\ell(\sigma t) = \ell(\sigma)\ell(t)$ with $\sigma \in T^*$ and $t \in T$ and $\ell(\lambda) = \varepsilon$.[4] We denote $w \in E^*$ the *word* that is observed when the sequence $\sigma \in T^*$ fires, i.e., $w = \ell(\sigma)$. We use $M_1[w\rangle M_2$ to denote that there exists a sequence $\sigma \in T^*$ such that $\ell(\sigma) = w$ and the firing of σ at M_1 yields M_2.

A *time labelled Petri net* (TLPN) is a 6-tuple $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ where (N, M_0, E, ℓ) is an LPN and $Q : T \to \mathbb{R}_0^+ \times \mathbb{R}_0^{+\infty}$ assigns each transition t a real[5] time window, i.e., $Q(t) = [l_t, u_t]$ where $l_t \leq u_t$. The *timer vector* $\Theta = [\theta_{t_1}, \ldots, \theta_{t_m}]^T \in \mathbb{R}_{0+}^m$ associates with each transition t a timer θ_t. The timer θ_t of transition t is initialized as zero and has a piecewise continuous evolution. When t is enabled, its timer increases with a unitary rate, i.e., it measures the passage of time. When t is not enabled, its timer remains constant. The timer θ_t is reset to zero whenever transition t fires or is disabled by the firing of another transition. Transition t can fire only if it is enabled and its timer belongs to its time window, i.e., $\theta_t \in [l_t, u_t]$.

[4] In this paper we use "λ" to denote an empty sequence, while we use "ε" to denote the *silent* label that is assigned to unobservable transitions.

[5] Most works dealing with TLPNs assume that the variable time takes rational values, to guarantee a finite state class graph. In our approach we do not need to compute such a graph and assume that time takes real values.

The LPN $G = (N, M_0, E, \ell)$ is said to be the underlying LPN of the TLPN G^T. We consider a *single-server semantics*, i.e., an enabled transition has a unique timer regardless of its enabling degree.

A *state* of a TLPN is presented as a pair (M, Θ) where M is a marking and Θ is a timer vector. We write "$(M_0, \Theta_0)[(\psi, \tau_{end})\rangle(M, \Theta)$" with $\psi = (t_1, \tau_1) \cdots (t_k, \tau_k)$ and $\tau_k \leq \tau_{end}$ to denote that: (1) the TLPN is initialized at (M_0, Θ_0) with the absolute time $\tau_0 = 0$; (2) each t_i fires at time τ_i; (3) after $\tau = \tau_k$ no more transition fires, and when $\tau = \tau_{end}$ the plant is at state (M, Θ).

Given a TLPN G^T, a *timed sequence* (TS) is a sequence $\psi = (t_{j_1}, \tau_1)(t_{j_2}, \tau_2) \cdots, (t_{j_k}, \tau_k)$, where t_{j_i} is the i-th transition occurring in the TS and $\tau_i \in \mathbb{R}_0^+$ is the time at which t_{j_i} is fired. We use τ to denote the *absolute* time determined by a global clock. The set of all TSs is denoted as Ψ.

An observation in a TLPN is a *timed observation* (TO) $\phi = (e_1, \tau_1)(e_2, \tau_2) \cdots (e_k, \tau_k)$ where each $e_i \in E, i = 1, \ldots, k$, is an observable event and $\tau_i \in \mathbb{R}_0^+$ denotes the time at which e_i is observed. The set of all TOs is denoted as Φ. The *time labeling operator* $P_o : \Psi \to \Phi$, which associates a timed sequence to its corresponding timed observation, is defined as:

- $P_o(\lambda) = \lambda$;
- $P_o((t, \tau_t)) = (e, \tau_t)$ if $\ell(t) = e$;
- $P_o((t, \tau_t)) = \varepsilon$ if $\ell(t) = \varepsilon$;
- $P_o(\psi(t, \tau_t)) = P_o(\psi)P_o((t, \tau_t))$.

We use $LOG(\psi) \in T^*$ to denote the logical firing sequence $t_{j_1} t_{j_2} \cdots t_{j_k}$ associated to a TS ψ, and we use $LOG(\phi) \in E^*$ to denote the logical observed word $e_1 e_2 \cdots e_{k'}$ associated to TO ϕ. Given a TS ψ, we denote $\mathbf{y}_\psi$ the Parikh vector of its logical firing sequence, i.e., $\mathbf{y}_{LOG(\psi)}$.

Definition 2.1 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ in which $T = T_o \cup T_{uo}$, the* unobservable sub-TLPN *of G^T is the TLPN $G^T_{uo} = (N_{uo}, M_{0,uo}, E, \ell, Q_{uo}, \Theta_{0,uo})$ where $(N_{uo}, M_{0,uo}, E, \ell)$ is the T_{uo}-induced subnet of G, and Q_{uo} and $\Theta_{0,uo}$ are Q and Θ_0 restricted to T_{uo}, respectively.* $\triangle$

The TLPNs considered in this paper satisfy the *divergent-free* property, i.e., for any (M, Θ) and any time τ there does not exist an infinite long TS ψ such that $(M, \Theta)[(\psi, \tau)\rangle$. This also means that the maximal number of transitions that can fire during a finite period of time must be finite.

3 Marking Estimation Problem Formulation in TLPNs

3.1 Estimation in TLPNs

In our setting, we assume that the initial marking M_0 is given and the initial timer vector is $\Theta_0 = \mathbf{0}$, i.e., the initial state $(M_0, \mathbf{0})$ is known. Let us now characterize the relationship between a given timed observation and the trajectories that may have produced it.

Definition 3.1 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$, a TS ψ is said to be* consistent *with a TO ϕ at time τ if $(M_0, \Theta_0)[(\psi, \tau)\rangle$ and $P_o(\psi) = \phi$.* $\triangle$

We write "$(M_0, \Theta_0)[(\phi, \tau_{end})\rangle(M, \Theta)$" to denote that there exists a consistent ψ such that $(M_0, \Theta_0)[(\psi, \tau_{end})\rangle(M, \Theta)$.

Definition 3.2 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$, a TO $\phi = (e_1, \tau_1) \cdots (e_k, \tau_k)$ and an absolute time $\tau > \tau_k$, the* consistent state set *(CSS) of (ϕ, τ) is:*

$$\mathcal{S}(\phi, \tau) = \{(M, \Theta) \mid (M_0, \Theta_0)[(\phi, \tau)\rangle(M, \Theta)\}.$$

The consistent marking set *(CMS) of (ϕ, τ) is:*

$$\mathcal{C}(\phi, \tau) = \{M \mid (\exists\Theta)(M, \Theta) \in \mathcal{S}(\phi, \tau)\}.$$

△

In plain words, given a TO ϕ and an absolute time τ, the consistent state set $\mathcal{S}(\phi, \tau)$ (resp., the consistent marking set $\mathcal{C}(\phi, \tau)$) contains all states (resp., markings) that can be reached from the initial state (M_0, Θ_0) (resp., the initial marking M_0) by firing some TS consistent with ψ. Hence there are two estimation problems in TLPNs stated as follows.

Problem 1 (State Estimation in TLPN) *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$, a TO $\phi = (e_1, \tau_1) \cdots (e_k, \tau_k)$ and an absolute time $\tau > \tau_k$, determine $\mathcal{S}(\phi, \tau)$.* △

Problem 2 (Marking Estimation in TLPN) *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$, a TO $\phi = (e_1, \tau_1) \cdots (e_k, \tau_k)$ and an absolute time $\tau > \tau_k$, determine $\mathcal{C}(\phi, \tau)$.* △

In the literature, the usual approach to compute the set of consistent markings $\mathcal{C}(\phi, \tau)$ is first to obtain the set of consistent states $\mathcal{S}(\phi, \tau)$ from the corresponding state class graph followed by extracting the marking information from it. However, as we have mentioned in Section I, to compute consistent states is computationally expensive since in practice [10, 12, 13, 14] the state class graph is much larger than the reachability set of the underlying untimed LPN, which is also illustrated by the example depicted at the end of Section VI. Moreover, there is no efficient method to single-out those state classes that are time-spurious [10, 12, 13, 14]. Since in many practical cases what we are really concerned with is the set of consistent markings $\mathcal{C}$, we propose a different scenario to estimate this set, which does not require to compute the consistent states.

3.2 Assumptions

The marking estimation approach to be presented in Section V is applicable to nets that satisfy the following assumptions.

Assumption 1 *For all transitions $t \in T$, $u_t = +\infty$ holds, i.e., $Q(t) = [l_t, +\infty)$.*

The infinite upper bound in Assumption 1 models cases where events can occur but cannot be forced. In other words, each operation t takes a minimal time l_t to occur, but may be indefinitely delayed, and such a delay cannot be pre-estimated. Note that when $u_t = +\infty$ for all transitions, the reachability space of G^T is identical to that of its underlying untimed net G. In addition, if a TLPN satisfies both $l_t = 0$ and $u_t = +\infty$ for all $t \in T$, then the marking estimation problem is reduced to the standard marking estimation in the underlying untimed net [29]. In such a case one can use other methods (such as *basis markings* [3]) that pertain to untimed nets.

Assumption 2 *The T_{uo}-induced subnet of G is backward-conflict-free, i.e., each place has at most one input transition.*

Assumptions 2 means that each event occurrence can be explained by a single causal path, which is widely used in the context of control [22, 23]. In addition, the backward-conflict-free assumption also plays a very important role in establishing our marking estimation method in TLPNs, as discussed at the end of next section.

To simplify the content of this paper, the following assumption is considered.

Assumption 3 *The T_{uo}-induced subnet of N is acyclic and does not contain sink transitions, i.e., for all $t \in T_{uo}, t^\bullet \neq \emptyset$.*

This assumption is purely technical and may be relaxed if needed. In fact, a net with sink transition t_s can be modified by adding a dummy place p_d with $Pre(p_d, t_s) = 0, Post(p_d, t_s) = 1$ and for all $t \neq t_s$, $Pre(p_d, t) = Post(p_d, t) = 0$, without affecting the behavior of the net. On the other hand, the acyclicity assumption can be lifted as shown in [30] by introducing the so-called *pseudo-observable* transitions.

Finally we consider the last assumption on the observation structure.

Assumption 4 *The underlying LPN G satisfies*

$$|p^\bullet| > 1 \quad \Rightarrow \quad p^\bullet \subseteq T_o,$$

i.e., if a place has more than one output transition, then all its output transitions are observable.

Assumption 4 indicates that no transitions in a structural conflict are unobservable, i.e., labeled by the empty string. Note that it does not require that transitions in a conflict should have different labels. Moreover, Assumptions 2 and 4 jointly imply that the topology of the unobservable subnet is a marked graph.

3.3 Sensor Selection

In a control system, the observation structure depends on the sensors that a plant has been equipped with, and as such it may often be suitably designed or modified if needed [24, 25, 26, 27]. In this subsection we propose a method to find an optimal sensor selection that satisfies the structural requirements in the assumptions mentioned above.

We define a *cost function* $c : T \to \mathbb{N} \cup \{+\infty\}$ that assigns to each transition t a nonnegative integer value, which represents the cost of making t observable (e.g., by deploying sensors accordingly). Transitions that cannot be made observable are assigned an infinite cost: the set of such transitions is $T_d = \{t \mid c(t) = +\infty\}$. The following proposition provides a method to obtain a sensor selection that satisfies Assumptions 2, 3, and 4.

Proposition 3.1 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ with no sink transitions and a cost function c, the following procedure determines a sensor-selection vector $\mathbf{z} \in \mathbb{N}^{|T|}$ satisfying Assumptions 2, 3, and 4 with a minimum cost.*

1. *Compute the set of all elementary cycles $\mathcal{O}$ in G^T;*

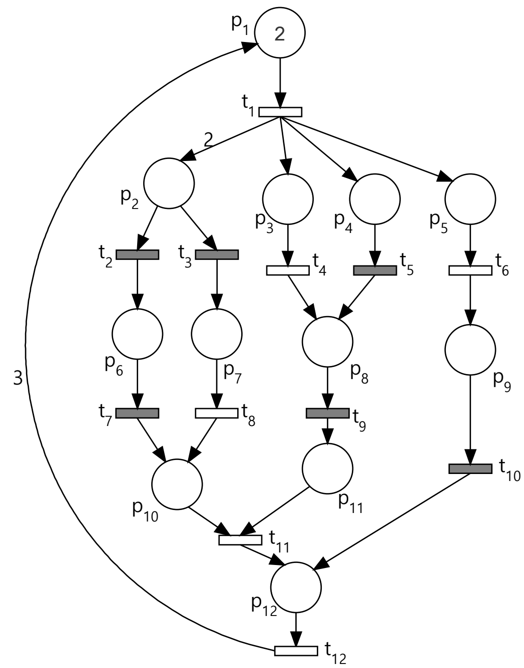

Figure 1: The Petri net used in Example 3.1.

2. Solve the following ILPP:

$$\begin{cases} \min & \sum_{t\in T\setminus T_d} c(t)\cdot z(t) \\ s.t. & \Sigma_{t\in {}^\bullet p} z(t) \geq |{}^\bullet p| - 1 (\forall p \in P) \\ & z(t) = 1, \forall t \in p^\bullet (\forall p \textit{ such that } |p^\bullet| \geq 2) \\ & \Sigma_{t\in O} z(t) \geq 1, \forall O \in \mathcal{O} \\ & z(t) = 0, \forall t \in T_d \\ & z(t) \in \{0,1\} \end{cases} \tag{1}$$

Proof: In the sensor-selection vector $\mathbf{z}$, $z(t) = 1$ means that a sensor is deployed for transition t to make it observable. In Eq. (1), the first condition implies that there does not exist a place with more than one unobservable input transitions, which enforces Assumption 2. The second condition means that for each conflict place all its output transitions are observable, which enforces Assumption 4. The third condition ensures that the unobservable subnet is acyclic, which enforces Assumption 3. The fourth condition means that all transitions that cannot be made observable are not selected. Finally, the objective function guarantees that the total cost of such a sensor deployment is minimum. □

Tarjan's algorithm [31] can be used to find all elementary cycles in a digraph. Although the number of elementary cycles may grow exponentially with the size of the net in the worse case, in practice the number of cycles is typically quite reasonable. Moreover, if the plant is too complex such that its elementary cycles cannot be enumerated, a near-optimal selection can be obtained heuristically.

Example 3.1 *Consider the net in Figure 1 with cost function* $c(t_i) = 5,3,2,6,5,2,7,5,4,3,5,7$ *for* $i = 1,\ldots,12$, *rspectively. We want to find a sensor deployment satisfying Assumptions 2, 3, and 4. There* *are five cycles in $\mathcal{O}$: $p_1t_1p_2t_2p_6t_7p_{10}t_{11}p_{12}t_{12}p_1$, $p_1t_1p_2t_3p_7t_8p_{10}t_{11}p_{12}t_{12}p_1$, $p_1t_1p_3t_4p_8t_9p_{11}t_{11}p_{12}t_{12}p_1$, $p_1t_1p_4t_5p_8t_9p_{11}t_{11}p_{12}t_{12}p_1$, and $p_1t_1p_5t_6p_9t_{10}p_{12}t_{12}p_1$. By solving ILPP* (1) *we have a sensor deployment strategy $T_o = \{t_2, t_3, t_5, t_7, t_9, t_{10}\}$ (grayed) with a minimal cost 20. Note that there is also another solution $T_o = \{t_2, t_3, t_5, t_6, t_7, t_{11}\}$ with the same cost.* □

If $T_d = \emptyset$, i.e., all transitions can be made observable, ILPP (1) always admits a solution since the constraint set is feasible ($\mathbf{z} = \mathbf{1}$ is a feasible solution). On the other hand, if T_d is not empty, a solution can be found if and only if the net induced by T_d satisfies Assumptions 2 and 3, and $T_d \cap p^\bullet = \emptyset$ holds for all p such that $|p^\bullet| \geq 2$.

4 Slow- and Fast-bound Markings in TLPNs

4.1 Notions

In this section we introduce the key notions of *slow-bound markings* and *fast-bound markings*, and derive some of their properties.

Definition 4.1 *A TS $\psi' = (t_{j'_1}, \tau'_1) \cdots (t_{j'_k}, \tau'_{k'})$ is said to be* faster *than a TS $\psi'' = (t_{j''_1}, \tau''_1) \cdots (t_{j''_k}, \tau''_{k''})$, denoted as $\psi' \prec \psi''$, if there exists a complete injective function[6] $\alpha : \{1, \ldots, k''\} \to \{1, \ldots, k'\}$ such that the following two conditions hold:*

- $t_{j'_{\alpha(i)}} = t_{j''_i}$ *and* $\tau'_{\alpha(i)} \leq \tau''_i$;
- *at least one of the following conditions is satisfied: (1) there exists $i \in \{1, \ldots, k''\}$ such that $\tau'_{\alpha(i)} < \tau''_i$ holds, or (2) $k' > k''$.*

△

In plain words, a TS ψ' is said to be faster than a TS ψ'' if ψ' fires each transition in ψ'' at a time no later than that in ψ'', and (1) at least one firing is earlier than the corresponding one in ψ'', or (2) ψ' fires some additional transitions with respect to ψ''. Note that $\psi' \prec \psi''$ does not require that in ψ' and ψ'' transitions should be in the same firing order. The following example illustrates Definition 4.1.

Example 4.1 *Consider the following three TS's: $\psi_1 = (t_1, 1)(t_1, 4)(t_2, 5)$, $\psi_2 = (t_1, 1)(t_2, 2)(t_1, 3)$, and $\psi_3 = (t_1, 1)(t_3, 2)(t_1, 4)(t_2, 5)$. By Definition 4.1, $\psi_2 \prec \psi_1$ and $\psi_3 \prec \psi_1$ hold, while ψ_2 and ψ_3 are incomparable.* △

Definition 4.2 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ and a TO ϕ, we define:*

- *A* fast-bound TS *of the TO ϕ is a TS $\psi^f = (t_{j_1}, \tau_1) \cdots (t_{j_k}, \tau_k)$ such that: (1) ψ^f is consistent with ϕ, and (2) there does not exist another consistent TS $\psi' \neq \psi^f$ such that $\psi' \prec \psi^f$. The state (M^f, Θ^f) reached by firing ψ^f from (M_0, Θ_0) is called the* fast-bound state *of ψ^f and the marking M^f_τ is called the* fast-bound marking *of ψ^f.*

[6] An injective function γ from set X to set Y is that for all $x_1, x_2 \in X$, $x_1 \neq x_2$ implies $\gamma(x_1) \neq \gamma(x_2)$. In other words, each element in X is uniquely mapped to a unique element in Y.

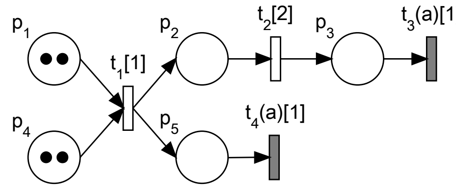

Figure 2: The time labelled Petri net used in Example 4.2.

- *A* slow-bound TS *of the TO ϕ is a TS $\psi^l = (e_1, \tau_1) \cdots (e_k, \tau_k)$ such that: (1) ψ^l is consistent with ϕ, and (2) there does not exist another consistent TS $\psi' \neq \psi^l$ such that $\psi^l \prec \psi'$. The state (M^l, Θ^l) reached by firing ψ^l from (M_0, Θ_0) is called the* slow-bound state *of ψ^l and the marking M^l_τ is called the* slow-bound marking *of ψ^l.*

- *A* slow-fast-sequence-pair *(SFS-pair) (ψ^l, ψ^f) of the TO ϕ is a pair of slow-bound TS ψ^l and fast-bound TS ψ^f such that $LOG(P_o(\psi^l)) = LOG(P_o(\psi^f))$. The corresponding* slow-fast-marking-pair *(SFM-pair) is (M^l, M^f) where M^l and M^f are the corresponding slow- and fast-bound markings of ψ^l and ψ^f, respectively.*

- *The set of all SFM-pairs is denoted as $SFM(\phi)$.*

$\triangle$

Given a TO ϕ, a fast-bound TS is a consistent TS ψ^f in which each unobservable transition fires as early as possible, and the slow-bound TS is a consistent TS ψ^l in which each unobservable transition fires as late as possible. A fast-bound (resp. slow-bound) state/marking is a state/marking reached by a fast-bound (resp. slow-bound) TS. A SFS-pair of TO ψ consists of a slow-bound TS and a fast-bound TS whose observable transitions are identical, i.e., for each event e in TO ϕ, both ψ^l and ψ^f agree that e is generated by firing the same transition t with label e.

Example 4.2 *Consider the TLPN in Figure 2 with an initial marking $M_0 = [2, 0, 0, 2, 0]^T$. For better readability, a transition t with label e and time interval $[l_t, +\infty)$ is denoted as $t(e)[l_t]$, while for $\ell(t) = \varepsilon$ its label is omitted, i.e., $t(\varepsilon)[l_t]$ is denoted as $t[l_t]$.*

As a first example, assume that time $\tau = 4$ is reached with a null observation. There exists a slow-bound TS $\psi^l = \lambda$ with the corresponding slow-bound marking $M^l = M_0$ and a fast-bound TS $\psi^f = (t_1, 1)(t_1, 2)(t_2, 3)$ with fast-bound marking $M^f = [0, 1, 1, 0, 2]$. Hence there is a unique SFM-pair (M^l, M^f) at $\tau = 4$ with $\psi^l_{\uparrow T_o} = \psi^f_{\uparrow T_o} = \lambda$.

As a second example, assume that after having observed event a at time $\tau = 4$ (i.e., $\phi = (a, 4)$), the absolute time reaches $\tau = 5$. In such a case, there exist two slow-bound TSs $\psi^l_1 = (t_1, 3)(t_4, 4)$ with $M^l_1 = [1, 1, 0, 1, 0]^T$ and $\psi^l_2 = (t_1, 1)(t_2, 3)(t_3, 4)$ with $M^l_2 = [1, 0, 0, 1, 1]^T$ and two fast-bound TSs $\psi^f_1 = (t_1, 1)(t_1, 2)(t_2, 3)(t_4, 4)(t_2, 5)$ with $M^f_1 = [0, 0, 2, 0, 1]^T$ and $\psi^f_2 = (t_1, 1)(t_1, 2)(t_2, 3)(t_3, 4)(t_2, 5)$ with $M^f_2 = [0, 0, 1, 0, 2]$. Hence in this case there are two SFM-pairs at $\tau = 5$: (M^l_1, M^f_1) with $\psi^l_{1\uparrow T_o} = \psi^f_{1\uparrow T_o} = t_4$ and (M^l_2, M^f_2) with $\psi^l_{2\uparrow T_o} = \psi^f_{2\uparrow T_o} = t_3$. $\triangle$

Example 4.2 shows that, given a TO and an absolute time, the SFS-pair is in general not unique. However, if the unobservable subnet satisfies Assumption 4, then each SFS-pair at τ uniquely determines an SFS-pair at $\tau + \Delta$, in case that the time period between τ and $\tau + \Delta$ is a *silence period*. An open period of time between absolute time τ and $\tau + \Delta$, denoted as $(\tau, \tau + \Delta)$, is called a *silence period* if no event is observed between absolute time τ and $\tau + \Delta$.

Proposition 4.1 *Given a TLPN $G^T = (N,\, M_0,\, E,\, \ell,\, Q,\, \Theta_0)$ satisfying Assumption 4, then for each fast-bound state $(M_\tau^f, \Theta_\tau^f)$ (resp. slow-bound state $(M_\tau^l, \Theta_\tau^l)$) at time τ, there exists a unique fast-bound state $(M_{\tau+\Delta}^f, \Theta_{\tau+\Delta}^f)$ (resp. unique slow-bound state $(M_{\tau+\Delta}^l, \Theta_{\tau+\Delta}^l)$) at time $\tau + \Delta$.*

Proof: For the fast-bound state, let us simulate the net G^T from state $(M_\tau^f, \Theta_\tau^f)$ at time τ. If a unique transition t is enabled and its timer reaches $\theta_t = l_t$, we let it fire immediately such that the new-yielded state is unique. Now consider the case that at some time instance two or more transitions $t_{j_1}, \ldots, t_{j_k}$ simultaneously satisfy $\theta_{j_i} = l_{j_i}$. since the unobservable subnet is conflict-free, each place has at most one output transition, and hence the firing of any t_{j_i} does not reset the timer of any other $t_{j_{i'}}$ with $i' \neq i$. This indicates that $t_{j_1}, \ldots, t_{j_k}$ can fire in an arbitrary order in an infinitesimal period of time. Hence the new-yielded state by firing all t_{j_i}'s is also unique. As a result, the fast-bound state $(M_{\tau+\Delta}^f, \Theta_{\tau+\Delta}^f)$ is uniquely determined at time $\tau + \Delta$. By a similar reasoning the slow-bound state $(M_{\tau+\Delta}^l, \Theta_{\tau+\Delta}^l)$ is also unique. $\square$

To simplify the notation, in the sequel we write "ψ_τ" and "M_τ" to denote a TS and marking at time τ, respectively. If a net satisfies $u_t = +\infty$ for all $t \in T$, the following property holds.

Proposition 4.2 *Given a TLPN $G^T = (N,\, M_0,\, E,\, \ell,\, Q,\, \Theta_0)$ satisfying Assumption 1, if M_τ^l is a slow-bound marking at time τ and $(\tau, \tau + \Delta)$ is a silent period, then M_τ^l is slow-bound marking at time $\tau + \Delta$, i.e.,*

$$M_{\tau+\Delta}^l = M_\tau^l \tag{2}$$

Proof: Suppose that $(M_\tau^l, \Theta_\tau^l)$ is a slow-bound state reached by a slow-bound TS ψ^l. Since $u_t = +\infty$ holds for all $t \in T$, clearly ψ^l is a slow-bound TS at $\tau + \Delta$. $\square$

Proposition 4.2 indicates that a slow-bound marking always remains unchanged during a silent period. On the other hand, a fast-bound state $(M_\tau^f, \Theta_\tau^f)$ for a silence period can be updated by simulating the net from state $(M_\tau^f, \Theta_\tau^f)$ by firing every unobservable transition as soon as possible (i.e., t fires immediately when $\theta_t = l_t$), which we call it the *asap-firing rule*. However, once an event is observed, then fast- and the slow-bound TS may not be unique. The computation of fast- and slow-bound states for such a case can be done by Theorem 5.3 in the next section.

We can now introduce the notion of *reachable hull* of two markings.

Definition 4.3 *Given a net N and two markings M_1, M_2, the* reachable hull *of a pair of markings (M_1, M_2), denoted as $R_h(M_1, M_2)$, is the set of markings that are both reachable from M_1 and co-reachable to M_2, i.e.:*

$$R_h(M_1, M_2) = \{M \in \mathbb{N}^m \mid (\exists \sigma', \sigma'' \in T^*) M_1[\sigma'\rangle M[\sigma''\rangle M_2\}.$$

$\triangle$

We use $R_{h,uo}(M_1, M_2)$ to denote the *unobservable reachable hull* of a pair of markings (M_1, M_2) in the unobservable subnet N_{uo}. If the unobservable subnet is acyclic, $R_{h,uo}(M_1, M_2)$ can be described by the following equation:

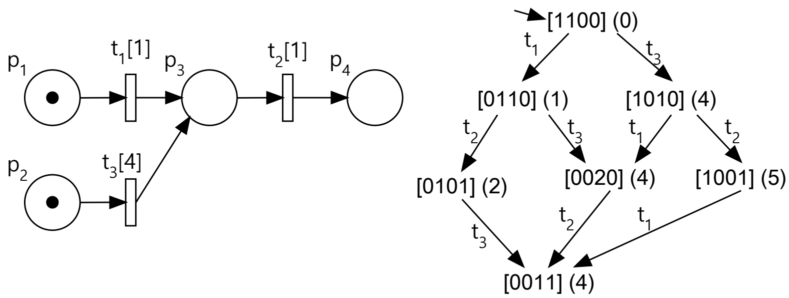

Figure 3: The counterexample used in Example 4.3.

$$R_{h,uo}(M_1, M_2) = \{M \in \mathbb{N}^m \mid (\exists \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{N}^n) \\ M = M_1 + C_{uo} \cdot \mathbf{y}_1, M_2 = M + C \cdot_{uo} \mathbf{y}_2\}. \tag{3}$$

According to Propositions 4.1 and 4.2, if the unobservable subnet is forward-conflict-free, during a silent period each SFM-pair (M^l_τ, M^f_τ) can be easily and uniquely determined from its precursor. This may lead one to conjecture that, if Assumptions 1 and 4 are satisfied, at time τ the unobservable reachable hull of (M^l_τ, M^f_τ) is the set of consistent markings. Unfortunately this conjecture is false.

Fact 1 *In a TLPN $G^T = (N,\ M_0,\ E,\ \ell,\ Q,\ \Theta_0)$ that satisfies Assumptions 1 and 4, after a silent period $(0, \tau)$ elapses the unobservable reachable hull of (M^l_τ, M^f_τ) may contain non-consistent markings, i.e., $R_{h,uo}(M^l_\tau, M^f_\tau) \setminus \mathcal{C}(\lambda, \tau) \neq \emptyset$.*

The statement in Fact 1 can be verified by Example 4.3.

Example 4.3 *Consider the TLPN in Figure 3 in which all transitions are unobservable. Clearly this TLPN satisfies Assumptions 1 and 4. On the right is a part of its logical reachability graph. Since the set of markings reachable in the underlying P/T net in this case coincides with the set of markings reachable in the time net, we denote each node of this graph as "$M(\tau)$" where τ denotes the minimal global time to reach marking M. By passing a silent period $(0, 4)$, the SFM-pair at $\tau = 4$ is $M^l = [1, 1, 0, 0]^T$ and $M^f = [0, 0, 1, 1]^T$, and the unobservable reachable hull consists of all seven markings in the graph. However, notice that marking $M = [1, 0, 0, 1]^T$ belongs to the reachable hull of $M^l = [1, 1, 0, 0]^T$ and $M^f = [0, 0, 1, 1]^T$, but is not reachable at time $\tau = 4$ but only at time $\tau \geq 5$.* $\square$

Example 4.3 shows the existence of time-spurious markings in $R_{h,uo}(M^l_\tau, M^f_\tau)$, i.e., an intermediate marking in the unobservable reachable hull of an SFM-pair at time τ is not necessarily reachable at time τ. The problem here is that for the fast-bound marking $M^f = [0, 0, 1, 1]^T$, the token in p_4 must be the one from p_1 and the token in p_3 must be the one from p_2, but this information is not maintained in Eq. (3), and there is no efficient method to distinguish the time-spurious markings like $[1, 0, 0, 1]^T$ from the unobservable reachable hull: to keep track of each individual token requires an exhaustive analysis. Another possible solution is to test the markings in $R_{h,uo}(M^l, M^f)$ by an ILPP that consists of a series of time-transition constraints. However, the scale of such a type of ILPP grows continually with the increase of the length of the firing sequences. On the other hand, the backward-conflict-free assumption, i.e., Assumption 2, eliminates such time-spurious solutions, which will be shown in the next section.

5 Computation of CMS in a Subclass of TLPNs

In this section we present a series of results to establish our online estimation algorithm. Before delving into the details we first provide a short roadmap of this section. First, when the plant is initialized (i.e., at $\tau = 0$ with empty observation), we show that the consistent marking set $\mathcal{C}(\lambda, 0)$ is the unobservable reachable hull of an SFM-pair (Section V-A). Second, if at time τ the consistent marking set can be described by a set of SFM-pairs, then after a silent period $(\tau, \tau + \Delta)$ has elapsed (Section V-B) or an event e has been observed (Section V-C), the new consistent marking set can also be described by a new set of SFM-pairs that can be determined from the former, respectively. Combining all these results provides us an iterative way to compute the consistent marking set without computing the consistent states.

5.1 Initialization of CMS

First, we propose a theorem showing that the initial consistent marking set is the unobservable reachable hull of an SFM-pair. To prove the theorem we first give the following lemma that will be used here and in Section V-B. The proof of this lemma can be found in the Appendix.

Lemma 5.1 *Consider a Petri net $G = (N, M_0, E, \ell)$ that is backward-conflict-free, acyclic, and does not contain sink transitions. Assume $M_1[\sigma\rangle M_2$ and let $M \in R_h(M_1, M_2)$ be a marking in the convex hull of M_1 and M_2. Then $\mathbf{y}_\sigma$ the Parikh vector associated to σ is the unique firing vector such that $M_1 + C \cdot \mathbf{y}_\sigma = M_2$, and for all $\sigma', \sigma'' \in T^*$ such that $M_1[\sigma'\rangle M[\sigma''\rangle M_2$, it holds $\mathbf{y}_{\sigma'}, \mathbf{y}_{\sigma''} \leq \mathbf{y}_\sigma$.*

Proof: See Appendix. □

Theorem 5.1 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, the initial set of consistent markings is*

$$\mathcal{C}(\lambda, 0) = R_{h,uo}(M_0^l, M_0^f)$$

where (M_0^l, M_0^f) is the SFM-pair corresponding to the empty observation λ at time $\tau = 0$, i.e.: (1) the slow-bound marking is $M_0^l = M_0$, (2) the fast-bound marking M_0^f is the unique marking reached from M_0 by firing a fast-bound TS composed only of transitions with $l_t = 0$.

Proof: The set $\mathcal{C}(\lambda, 0)$ corresponds to an empty TO $\phi = \lambda$ at a global time $\tau = 0$. Since by Assumption 1 and Proposition 4.2, the slow-bound marking M^l is M_0 (i.e., no transition has to fire at $\tau = 0$), the set $\mathcal{C}(\lambda, 0)$ coincides with the set of markings reachable in the untimed LPN with initial marking M_0 induced by the set of unobservable transitions with $l_t = 0$. Moreover, since the net is divergent-free, similar to the reasoning in Proposition 4.1, the fast-bound marking M_0^f obtained by the asap-firing-rule is unique. Thus, it is sufficient to prove this theorem by proving that in the untimed LPN induced by $T_{uo,0} = \{t \in T_{uo} \mid l_t = 0\}$, $\mathcal{C}(\lambda) = R_{h,uo,0}(M_0^l, M_0^f)$ holds.

Since in untimed acyclic nets, $\mathcal{C}(\lambda) = \{M \mid \exists \mathbf{y} : M = M_0 + C \cdot \mathbf{y}\}$ holds, and hence $\mathcal{C}(\lambda) \supseteq R_{h,uo,0}(M_0^l, M_0^f)$ is true. On the other hand, by Lemma 5.1 for all M reachable from M_0^l by firing σ', $\mathbf{y}_{\sigma'} \leq \mathbf{y}_\sigma$ holds. This indicates that M_0^f is reachable from M, and hence $\mathcal{C}(\lambda) \subseteq R_{h,uo,0}(M_0^l, M_0^f)$ holds. □

Theorem 5.1 shows that when the plant TLPN is initialized at M_0, in zero time and with null observation the set of consistent markings is the reachable hull of an SFM-pair.

5.2 Updating the CMS after a Silent Period

Now we prove that if the set of consistent markings is the reachable hull $R_{h,uo}(M^l_\tau, M^f_\tau)$ of an SFM-pair (M^l_τ, M^f_τ) at time τ, then after a silent period $(\tau, \tau+\Delta)$, the new set of consistent markings is the reachable hull $R_{h,uo}(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$ of the SFM-pair (M^l_τ, M^f_τ) at time $\tau+\Delta$.

Theorem 5.2 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, let $\mathcal{C}(\phi, \tau) = R_{h,uo}(M^l_\tau, M^f_\tau)$ where (M^l_τ, M^f_τ) is an SFM-pair at time τ. If $(\tau, \tau+\Delta)$ is a silent period, then the consistent marking set at the end of this period is*

$$\mathcal{C}(\phi, \tau+\Delta) = R_{h,uo}(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$$

where $(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$ is the new SFM-pair at time $\tau+\Delta$ derived from (M^l_τ, M^f_τ).

Proof: We prove both $\supseteq$ and $\subseteq$.

($\supseteq$) Let ψ^f be the fast-bound TS such that

$$(M_0, \Theta_0)[(\psi^f, \tau+\Delta)\rangle(M^f_{\tau+\Delta}, \Theta^f_{\tau+\Delta}),$$

i.e., $M^f_{\tau+\Delta}$ is reached at time $\tau+\Delta$ by firing a fast-bound TS $\psi^f = (t_{j_1}, \tau_1)(t_{j_2}, \tau_2)\cdots(t_{j_k}, \tau_k)$. Since the net satisfies Assumptions 2 and 3, by Lemma 5.1, there exists a unique Parikh vector $\mathbf{y}_\sigma$ such that $M^f_{\tau+\Delta} = M^l_{\tau+\Delta} + C_{uo}\cdot\mathbf{y}_\sigma$, and for any marking $M \in R_{h,uo}(M^l_0, M^f_0)$ such that $M^l_{\tau+\Delta}[\sigma'\rangle M[\sigma''\rangle M^f_{\tau+\Delta}$, it holds $\mathbf{y}_{\sigma'} \leq \mathbf{y}_\sigma$. We will prove that for all such markings M and corresponding vector $\mathbf{y}_{\sigma'}$ there exists a TS ψ' whose Parikh vector is $\mathbf{y}_{\sigma'}$ such that $(M_0, \Theta_0)[(\psi', \tau+\Delta)\rangle$, which implies $M \in \mathcal{C}(\phi, \tau+\Delta)$.

Without loss of generality, we assume that the transitions in the unobservable subnet are numbered from upstream to downstream, i.e., (transition t_j is in the upstream of transition $t_{j'}$) $\Rightarrow$ $(j < j')$. Now from the fast-bound TS ψ^f we construct a new TS ψ' by the following procedure:

1. let $\psi' = \psi^f$;
2. if $\mathbf{y}_{\psi'} = \mathbf{y}_{\sigma'}$ then End;
3. find the maximal subscript $\bar{\jmath}$ such that $\bar{\jmath} = \max\{j \mid y_{\psi'}(t_j) > y_{\sigma'}(t_j)\}$;
4. in ψ' find a pair $(t_{j_{\bar{\imath}}}, \tau_{\bar{\imath}})$ such that $\bar{\imath} = \max\{i \mid t_{j_{\bar{\imath}}} = t_{\bar{\jmath}}\}$, (i.e., the last $(t_{\bar{\jmath}}, \cdot)$ that appears in ψ');
5. delete $(t_{j_{\bar{\imath}}}, \tau_{\bar{\imath}})$ from ψ' and goto Step 2;

This procedure terminates in a finite number of steps determining a TS ψ' whose Parikh vector is $\mathbf{y}_{\sigma'}$.

Now we prove that at each iteration of the above procedure $(M_0, \Theta_0)[(\psi', \tau+\Delta)\rangle$ holds by contradiction. Suppose that during an iteration the new intermediate TS ψ' determined in Step 5 is not firable. This indicates that by deleting $(t_{j_{\bar{\imath}}}, \tau_{\bar{\imath}})$, some (t, τ) in ψ' with $\tau > \tau_{\bar{\imath}}$ cannot fire. Let (t_{j_k}, τ_k) be such a pair with the smallest index k (note that $k > \bar{\imath}$). By deleting $(t_{j_{\bar{\imath}}}, \tau_{\bar{\imath}})$ at least one of the input places of t_{j_k}, say p, does not receive sufficient tokens in advance so that t_{j_k} cannot fire at τ_k. Since by Assumption 2 place p can only receive tokens from the firing of $t_{j_{\bar{\imath}}}$, it implies that one of the future firings of transition $t = t_{j_k}$ is in fact permanently disabled, i.e., $M^l_\tau(p) + C_{uo}(p, \cdot)\cdot\mathbf{y}' < 0$ and $M^l_\tau + C_{uo}\cdot\mathbf{y}' \not\geq \mathbf{0}$, where $\mathbf{y}'$ is the Parikh of ψ'. Since the transitions in the unobservable subnet are numbered from upstream to downstream, and by Step 4 $(t_{j_{\bar{\imath}}}, \tau_{\bar{\imath}})$ is the pair with the largest index $j_{\bar{\imath}}$, all other (t_{j_i}, τ_i) with $j_i \geq j_{\bar{\imath}}$ deleted in further iterations are not in the

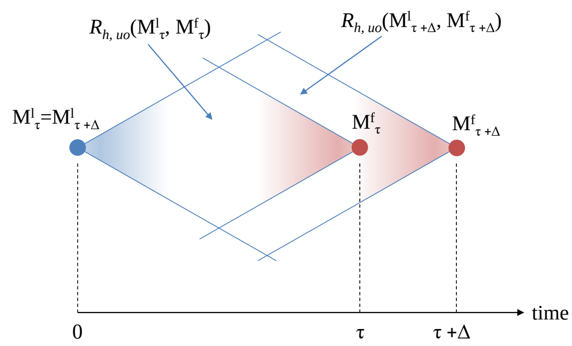

Figure 4: The evolution of the reachable hull $R_{h,uo}(M^l_\tau, M^f_\tau)$ to $R_{h,uo}(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$ from time τ to $\tau+\Delta$.

downstream of $t_{j_{\bar{i}}}$ nor t_{j_k}, leading to $p \notin {}^\bullet t_{j_i}$. This will eventually lead to the fact that $M^l_\tau + C_{uo} \cdot \mathbf{y}_{\sigma'} \ngeq \mathbf{0}$ necessarily holds, which contradicts the fact that marking M is logically reachable from $M^l_{\tau+\Delta}$. Hence $M \in \mathcal{C}(\phi, \tau+\Delta)$ necessarily holds.

($\subseteq$) We prove that for any marking $M \notin R_{h,uo}(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$, there does not exist a TS ψ' such that $LOG(\psi') = \phi$ and $(M_0, \Theta_0)[(\psi', \tau+\Delta)\rangle(M, \Theta)$. By contradiction, suppose that there exists such a TS ψ' whose firing yields a consistent marking M that does not belong to $R_{h,uo}(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$. Clearly $\mathbf{y}_{\psi'} \nleq \mathbf{y}$ holds, where $\mathbf{y}$ satisfies $M^f_{\tau+\Delta} = M^l_{\tau+\Delta} + C_{uo} \cdot \mathbf{y}$. It indicates that some transition, say t, in the unobservable subnet fires some more times in ψ' than that in the fast-bound TS $\psi^f_{\tau+\Delta}$ that yields $M^f_{\tau+\Delta}$. However, since by Assumption 4 the firing of one transition neither disables other transitions nor affects their timers, it indicates that at marking $M^f_{\tau+\Delta}$ transition t can fire some times more. This contradicts the fact that $M^f_{\tau+\Delta}$ is the marking reached by firing a fast-bound TS. □

From Theorem 5.2, if the set of consistent markings is the reachable hull of the fast- and slow-bound markings in N_{uo} at time τ, then after a period of time Δ without any observation, the new consistent marking is also the reachable hull of the new fast- and slow-bound markings. Thus we have the following corollary.

Corollary 5.1 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, if $\mathcal{C}(\phi, \tau) = \bigcup_i R_{h,uo}(M^l_{\tau,i}, M^f_{\tau,i})$ where each pair $(M^l_{\tau,i}, M^f_{\tau,i})$ is an SFM-pair at time τ, then after $\Delta\tau$ without any observation, the consistent marking set at the end of this period is*

$$\mathcal{C}(\phi, \tau+\Delta) = \bigcup_i R_{h,uo}(M^l_{\tau+\Delta,i}, M^f_{\tau+\Delta,i})$$

where each pair $(M^l_{\tau+\Delta,i}, M^f_{\tau+\Delta,i})$ is the new SFM-pair at time $\tau+\Delta$ derived from $(M^l_{\tau,i}, M^f_{\tau,i})$.

Example 5.1 *Consider the TLPN in Figure 2. Initially $M^l_0 = M^f_0 = M_0$ holds at time $\tau = 0$, which indicates that $R_{h,uo}(M^l_0, M^f_0) = \{M_0\}$, i.e., the only possible marking in $\mathcal{C}(\lambda, 0)$ is M_0. Suppose that the time elapses to $\tau = 3$ and no event is observed. At $\tau = 3$, the slow-bound marking $M^l_3 = M^l_0 = M_0$, while the fast-bound marking $M^f_3 = [0, 1, 1, 0, 2]^T$ by the fast-bound TS $(t_1, 1)(t_1, 2)(t_2, 3)$, i.e., $(M_0, 0)[(t_1, 1)(t_1, 2)(t_2, 3)\rangle(M^f_3, 3)$. By Theorem 5.2, the consistent marking set $\mathcal{C}(\lambda, 3) = R_{h,uo}(M^l_3, M^f_3) = R_{h,uo}(M_0, M^f_3)$ that consists of five markings, i.e.,:*

$$R_{h,uo}(M_0, M^f_3) = \{[2,0,0,2,0]^T, [1,1,0,1,1]^T, [0,2,0,0,2]^T, [1,0,1,1,1]^T, [0,1,1,0,2]^T\}.$$

$\triangle$

Figure 4 illustrates the evolution of $\mathcal{C}(\phi,\tau)$ when time passes. Suppose that the set of consistent markings at time τ is the reachable hull of an SFM-pair (M^l_τ, M^f_τ). When passing a silent time period $(\tau,\tau+\Delta)$, M^f_τ evolves to $M^f_{\tau+\Delta}$ while M^l_τ remains unchanged (i.e., $M^l_{\tau+\Delta}=M^l_\tau$), and the new set of consistent markings is the reachable hull of the new SFM-pair $(M^l_{\tau+\Delta}, M^f_{\tau+\Delta})$.

Theorem 5.2 characterizes the evolution of the consistent marking set when a silent period of time passes. Before we proceed, we briefly introduce the notion of *minimal explanation vector* [7] in labelled Petri nets. Given a marking M and an observable transition $t\in T_o$, a *minimal explanation* of t at M is an unobservable sequence $\sigma\in T^*_{uo}$ such that $M[\sigma\rangle[t\rangle$, and there does not exist another sequence $\sigma'\in T^*_{uo}, \sigma'\neq\sigma$ such that $M[\sigma'\rangle[t\rangle$ and $\mathbf{y}_{\sigma'}\lneq\mathbf{y}_\sigma$. For a minimal explanation σ, $\mathbf{y}_\sigma$ is called a *minimal explanation vector*. Since the net considered in this paper satisfies Assumption 2, by a result in [3], the minimal explanation vector at a marking M to enable an observable transition t, if exists, is unique and will be denoted as $\mathbf{y}_{(M,t)}$ in the sequel.

5.3 Updating the CMS after an Event is Observed

Suppose that the current absolute time is τ and the set of consistent markings is a reachable hull of SFM-pair, i.e., $\mathcal{C}(\phi,\tau)=R_{h,uo}(M^l_\tau, M^f_\tau)$. Now let us consider the case that at this moment an event $e\in E_o$ is observed. We show that the new set of consistent markings $\mathcal{C}(\phi(e,\tau),\tau)$ is the union of several reachable hulls, each of which is associated to the firing of a transition whose label is e. To prove this result, we first propose the following proposition showing that if an observable transition $t_o\in T_o$ can fire at the fast-bound state (M^f_τ,Θ^f_τ), then for any marking M belonging to the reachable hull $R_{h,uo}(M^l_\tau, M^f_\tau)$ and is reached from M^l_τ by a logical firing vector equal to or greater than the minimal explanation vector of t, there exists a consistent state (M,Θ) such that t_o can fire at it.

Proposition 5.1 *Given a TLPN $G^T=(N,\ M_0,\ E,\ \ell,\ Q,\ \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, if $\mathcal{C}(\phi,\tau)=R_{h,uo}(M^l_\tau, M^f_\tau)$, and $(M_0,\Theta_0)[\psi^f\rangle(M^f_\tau,\Theta^f_\tau)[(t_o,\tau)\rangle$ where $t_o\in T_o$, then for any $M\in R_{h,uo}(\bar{M}^l_\tau, M^f_\tau)$ where*

$$\bar{M}^l_\tau = M^l_\tau + C_{uo}\cdot\mathbf{y}_{(M^l_\tau,t)},$$

there exists a TS ψ such that $P_o(\psi)=P_o(\psi^f)$ and $(M_0,\Theta_0)[(\psi,\tau)\rangle(M,\Theta)[(t_o,\tau)\rangle$.

Proof: This proof is analogous to the proof of Theorem 5.1. Suppose that M^f_τ is reached by firing ψ^f. We use the procedure described in the proof of Theorem 5.1 to iteratively remove unobservable (t_{j_i},τ_i)'s from ψ^f by keeping $(M_0,0)[(\psi',\tau)\rangle(M',\tau)[(t_o,\tau)\rangle$, where ψ' is the modified TS in each iteration. Similar to the reasoning in Theorem 5.1, this procedure will eventually determine a new consistent TS ψ is obtained such that $(M_0,\Theta_0)[(\psi,\tau)\rangle(M,\Theta_\tau)[(t_o,\tau)\rangle$ holds. $\square$

Now we show that if the current consistent marking set is $R_{h,uo}(M^l_\tau, M^f_\tau)$, then by observing event e, the new set of consistent markings is the union of several reachable hulls, each of which is a new SFM-pair derived from (M^l_τ, M^f_τ) by firing some transition whose label is e.

Theorem 5.3 *Given a TLPN $G^T=(N,M_0,E,\ell,Q,\Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, if the consistent marking set $\mathcal{C}(\phi,\tau)=R_{h,uo}(M^l_\tau, M^f_\tau)$ at time τ, then $\mathcal{C}(\phi(e,\tau),\tau)$ which is the new consistent marking*

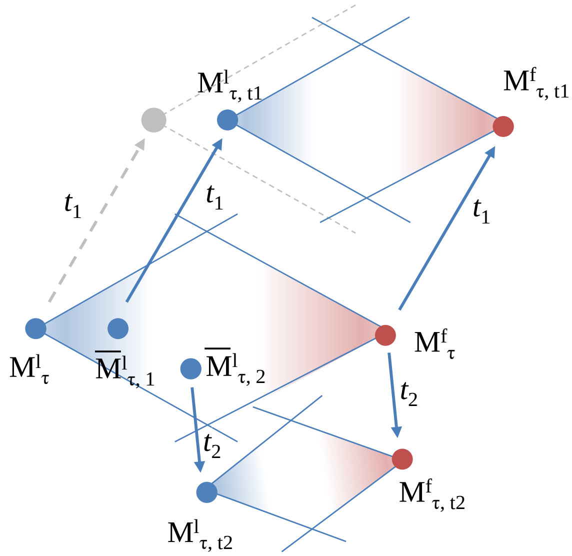

Figure 5: The update of $R_{h,uo}(M^l_\tau, M^f_\tau)$ by observing event e at time τ, where $\ell(t_1) = \ell(t_2) = e$.

set by observing e at time τ is:

$$\mathcal{C}(\phi(e,\tau),\tau) = \bigcup_{\ell(t_i)=e} R_{h,uo}(M^l_{\tau,t_i}, M^f_{\tau,t_i}) \tag{4}$$

where:

$$M^l_{\tau,t_i} = M^l_\tau + C_{uo} \cdot \mathbf{y}_{(M^l_\tau, t_i)} + C(\cdot, t_i) \tag{5}$$

and

$$M^f_{\tau,t_i} = M^f_\tau + C(\cdot, t_i). \tag{6}$$

Proof: First we claim that if the event e is generated by firing an arbitrary transition t_i such that $\ell(t_i) = e$, then $R_{h,uo}(M^l_{\tau,t_i}, M^f_{\tau,t_i})$ is a subset of $\mathcal{C}(\phi(e,\tau),\tau)$ due to the following reason. If t_i can fire at the fast-bound state $(M^f_\tau, \Theta^f_\tau)$, then by Proposition 5.1 for any marking $M \in R_{h,uo}(\bar{M}^l_\tau, M^f_\tau)$ there exists a corresponding state at which transition t_i is enabled. Hence the markings that are logically reachable from $R_{h,uo}(M^l_\tau, M^f_\tau)$ by firing t_i is $R_{h,uo}(\bar{M}^l_\tau + C(\cdot, t_o), M^f_\tau + C(\cdot, t_o))$. Hence by firing t_i the new set of consistent markings is $R_{h,uo}(M^l_{\tau,i}, M^f_{\tau,i})$. Therefore, the statement is true by taking all transitions t_i with $\ell(t_i) = e$ into account. $\square$

Example 5.2 *Let us consider again the TLPN in Figure 2. Initially $\mathcal{C}(\lambda, 0) = \{M_0\}$. If until time $\tau = 4$ no event is observed, the fast-bound state is (M^f_4, Θ^f_4) where $M^f_4 = [0,1,1,0,2]^T$ and $M^l_4 = M_0$. The set of consistent markings is $\mathcal{C}(\lambda, 3) = R_{h,uo}(M^l_4, M^f_4) = R_{h,uo}(M_0, M^f_4)$.*

Suppose that we observe a at $\tau = 4$. At the fast-bound state (M^f_4, Θ^f_4), both t_3 and t_4 can fire to generate a. By Theorem 5.3, the new consistent marking set

$$\mathcal{C}((a,4),4) = \bigcup_{i\in\{3,4\}} R_{h,uo}(M^l_{4,t_i}, M^f_{4,t_i})$$

where $(M^l_{4,t_4}, M^f_{4,t_4}) = ([1,1,0,1,0]^T, [0,1,1,0,1]^T)$ and $(M^l_{4,t_3}, M^f_{4,t_3}) = ([1,0,0,1,1]^T, [0,1,0,0,2]^T)$. $\triangle$

Theorem 5.3 can be generalized to the case that the set of consistent markings is the union of reachable hulls of several SFM-pairs, which is stated by the following corollary.

Corollary 5.2 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, if its consistent marking set $\mathcal{C}(\phi, \tau) = \bigcup_{(M^l_\tau, M^f_\tau) \in SFM(\phi)} R_{h,uo}(M^l_\tau, M^f_\tau)$ at time τ, then the set of consistent markings by observing e at time τ, i.e., $\mathcal{C}(\phi(e,\tau), \tau)$, is:*

$$\mathcal{C}(\phi(e,\tau), \tau) = \bigcup_{(M^l_\tau, M^f_\tau) \in SFM(\phi)} \bigcup_{\ell(t_i)=e} R_{h,uo}(M^l_{\tau,t_i}, M^f_{\tau,t_i}) \tag{7}$$

where M^l_{τ,t_i} and M^f_{τ,t_i} are from Eqs. (5) *and* (6)*, respectively.*

Proof: Straightforwardly from Theorem 5.3. □

Figure 5 illustrates the evolution of $\mathcal{C}(\phi, \tau)$ to $\mathcal{C}(\phi(e,\tau), \tau)$ by observing event e. Suppose that the consistent marking set before observing e is $\mathcal{C}(\phi, \tau) = R_{h,uo}(M^l_\tau, M^f_\tau)$, and there are two transitions t_1 and t_2 whose labels are both e, firable at the fast-bound state $(M^f_\tau, \Theta^f_\tau)$. Since t_1 can fire at the fast-bound state $(M^f_\tau, \Theta^f_\tau)$, by Theorem 5.3 the consistent markings by firing t_1 (which is a subset of $\mathcal{C}(\phi(e,\tau), \tau)$) are those in $(M^l_{\tau,t_1}, M^f_{\tau,t_1})$, and the consistent markings by firing t_2 (which is another subset of $\mathcal{C}(\phi(e,\tau), \tau)$) are those in $(M^l_{\tau,t_2}, M^f_{\tau,t_2})$. Hence the new consistent marking set $\mathcal{C}(\phi(e,\tau), \tau)$ is the union of $(M^l_{\tau,t_1}, M^f_{\tau,t_1})$ and $(M^l_{\tau,t_2}, M^f_{\tau,t_2})$.

5.4 The Final Result

Finally, by combining all results above, we reach the following conclusion.

Theorem 5.4 *Given a TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4, for a TO ϕ at the absolute time τ, the set of consistent markings is:*

$$\mathcal{C}(\phi, \tau) = \bigcup_{(M^l_\tau, M^f_\tau) \in SFM(\phi)} R_{h,uo}(M^l_\tau, M^f_\tau). \tag{8}$$

Proof: The statement holds by first applying Theorems 5.1 and 5.2 followed by repeatedly applying Corollary 5.1, Theorem 5.3, and Corollary 5.2 for each (e_i, τ_i) in ϕ. □

Theorem 5.4 provides us an iterative way to compute the consistent marking set, i.e., the consistent marking set is not presented in an explicit way but is described by a linear algebraic system by SFM-pairs that can be maintained online.

We conclude this section by observing that in our method the set of consistent markings $\mathcal{C}$ is represented by the fast- and slow-bound markings, which are in fact vertices of $\mathcal{C}$. On the other hand, in an untimed continuous Petri net the set of consistent markings can also be represented as a set of vertices of *convex polyhedrons* [32, 33]. We believe that by adding temporal constraints and similar assumptions, the approaches in [32, 33] may also be applied to the marking estimation in time continuous Petri nets, which will be explored in future work.

6 Online Marking Estimation Policy

By Theorem 5.4 the consistent marking set can be described by a linear algebraic system via SFM-pairs. In this section we propose an online marking estimation algorithm, i.e., a *marking estimator*, for a TLPN that satisfies the assumptions.

6.1 The Algorithm

Algorithm 1 Online Marking Estimation

Input: A TLPN $G^T = (N, M_0, E, \ell, Q, \Theta_0)$ satisfying Assumptions 1, 2, 3, and 4

Initialize $\mathcal{X} = \{(M_0^l, M_0^f, \Theta_0)\}$ by Theorem 5.1;
while time elapses Δ **do**
  Let $\tau = \tau + \Delta$;
  for all $(M^l, M^f, \Theta^f) \in \mathcal{X}$, **do**
    Update (M^f, Θ^f) by the *asap-firing rule*;
  end for
  if an event e is observed; **then**
    for all $(M^l, M^f, \Theta^f) \in \mathcal{X}$, **do**
      Remove (M^l, M^f, Θ^f) from $\mathcal{X}$;
      for all $t \in T_e, (M^f, \Theta^f)[(t, \tau)\rangle$, **do**
        Let $M_{new}^l = M^l + C_{uo} \cdot \mathbf{y}_{(M^l,t)} + C_{uo}(\cdot, t)$;
        Let $(M_{new}^f, \Theta_{new}^f)$ be the state $(M^f, \Theta^f)[(t, \tau)\rangle(M_{new}^f, \Theta_{new}^f)$;
        Let $\mathcal{X} = \mathcal{X} \cup \{(M_{new}^l, M_{new}^f, \Theta_{new}^f\}$;
      end for
    end for
  end if
end while

We briefly describe the main step of our algorithm that is presented as Algorithm 1. The *estimator* keeps track of a set of triples (M^l, M^f, Θ^f) that are called the *estimator states*. In an estimator state (M^l, M^f, Θ^f), M^l is a slow-bound marking and (M^f, Θ^f) is a fast-bound state. In the beginning, the estimator state is first initialized as that in Theorem 5.1. During the online stage, if no event is observed, the slow-bound marking M^l remains unchanged while the fast-bound state (M^f, Θ^f) is simulated by applying the *asap-firing rule*. Whenever an event is observed, each estimator state (M^l, M^f, Θ^f) in $\mathcal{X}$ is removed from $\mathcal{X}$ and then be checked one-by-one: for each transition t_o that is labelled with e and is firable at the fast-bound state (M^f, τ) (i.e, $(M^f, \tau)[t_o\rangle$), a new estimator state $(M_{new}^l, M_{new}^f, \Theta_{new}^f)$ is computed by Eqs. (5) and (6) and put into $\mathcal{X}$. Precisely speaking, M_{new}^f is the marking reached from M^f by firing t_o, while M_{new}^l is the marking reached from M^l by firing a minimal explanation (whose Parikh vector is $\mathbf{y}_{(M_\tau^l, t_o)}$) followed by t_o. Note that if at an estimator state (M^l, M^f, Θ^f) no transition labeled e is firable at (M^f, Θ^f), then this estimator state is abandoned. At any moment, the current consistent marking set is the union of reachable hulls of all those pairs (M^l, M^f).

Before presenting an example, we make some comments on Algorithm 1. Once an event e is observed, a current estimator state (M^l, M^f, Θ^f) is split to one or more new estimator states $(M_{new}^l, M_{new}^f, \Theta_{new}^f)$'s where each M_{new}^l is updated from M^l by Eq. (5). This implies that the slow-bound markings are in fact the so-called *basis markings* [7, 34] of the underlying LPN. As a result, if the LPN has finite number of basis markings, they can be computed offine so that the online computational load can be further reduced. The

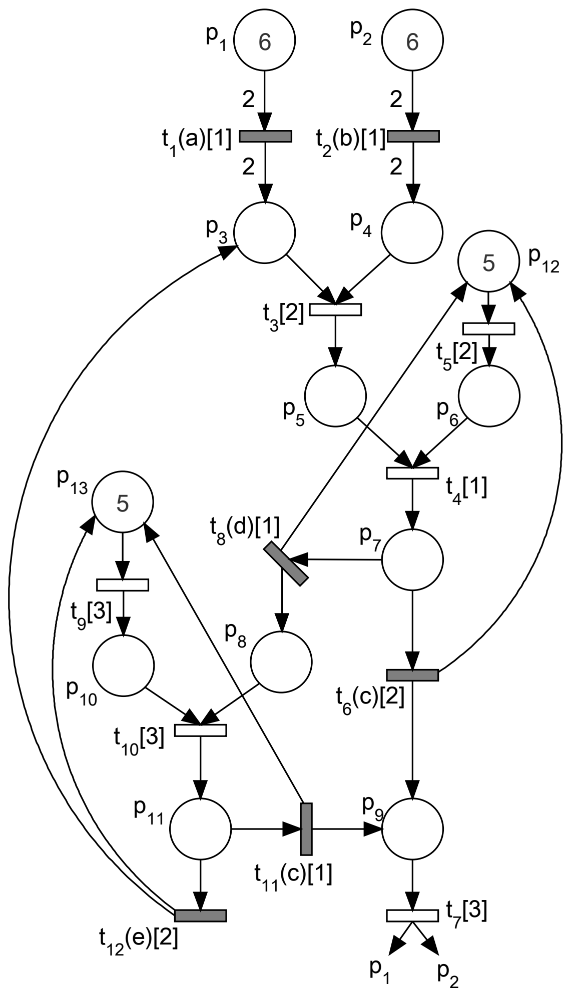

Figure 6: The TLPN for the example in Section VI-B.

number of basis markings is in general much smaller than the size of the associated reachability graph (in the worst case the same) of the LPN. On the other hand, SCGs that can also be computed offline are in general much larger than the reachability space. This indicates that our approach have an advantage since only a much smaller structure is computed during the offline stage. Moreover, it does not require that the underlying LPN be bounded, which the SCG and MSCG based methods require.

6.2 An Example

Consider the TLPN in Figure 6 which models an automatic manufacturing system. Two types of parts from two storages (p_1 and p_2) enter the workflow (t_1 and t_2, respectively) and are then assembled (t_3). The semi-finished product is then assembled (t_4) with another part from a third storage (p_6) to obtain a raw final product (p_7). Such a raw product may either be tested for quality control (t_8), or can be sent (t_6) to the product stock (p_9). A part to be tested (p_8) is assembled (t_{10}) with a test part from (p_{10}). A product that passes the test is put (t_{11}) into the stock (p_9), otherwise it is disassembled (t_{12}) to return the two initial parts to their stocks, respectively (the test part (from p_6) is discarded). In this TLPN model, transitions $t_1, t_2, t_6, t_8, t_{11}$, and t_{12}

are observable, and transitions t_6 and t_{11} are indistinguishable since they generate the same signal "*in-stock*". In the following we consider two cases. For better readability, a transition t with label e and time interval $[l_t, +\infty)$ is denoted as $t(e)[l_t]$, while for $\ell(t) = \varepsilon$ its label is omitted, i.e., $t(\varepsilon)[l_t]$ is denoted as $t[l_t]$.

1) Place p_6 and p_{10} are bounded: Suppose that the numbers of tokens in p_6 and p_{10} are bounded by 5, i.e., the initial marking is $[6, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5]$ as shown in Figure 6. The TLPN in Figure 6 has 169335 state classes in the SCG and 38483 reachable markings in the underlying LPN. Therefore to estimate the current markings by SCG-based methods, a very large SCG must be computed offline. On the contrary, its underlying LPN has only 110 basis markings, which is about 2% of the logical reachability set and 0.3% of the SCG.

Let us consider the following TO at time $\tau = 10$:

$$\phi = (a, 1)(b, 1)(d, 5)(c, 9)(e, 10). \tag{9}$$

The detailed evolution of the set $\mathcal{X}$ is listed in Table 1. Initially, $\mathcal{X}$ contains only one estimator state $(M_0^l, M_0^f, \Theta) = (M_0, M_0, \Theta_0)$, which indicates that $\mathcal{C}(\lambda, 0) = \{M_0\}$.

At time $\tau = 1$ before we observe any events, (M^l, M^f) (Entry 1) remains unchanged and hence the consistent marking set is $\mathcal{C}(\lambda, 1) = \{M_0\}$. Now at $\tau = 1$ we observe a, and (M^l, M^f, Θ^f) is updated accordingly, where $M_{new}^l = M_{new}^f = [6, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5]$ (Entry 2) which is also the only consistent marking. Then we observe b in an infinitesimal period of time after a, and (M^l, M^f, Θ^f) is again updated to $M_{new}^l = M_{new}^f = [4, 4, 2, 2, 0, 0, 0, 0, 0, 0, 0, 5, 5]$ (Entry 3).

Entries 4 to 7 shows the evolution of the estimator state during the silent period of time $\tau \in (1, 5)$. In this silent period, the slow-bound marking remains unchanged while the fast-bound state evolves according to the *asap-firing rule*. At the end of this period, the set of consistent markings is $\mathcal{C}((a, 1)(b, 1), 5) = R_{h,uo}(M^l, M^f)$ shown in Entry 7, where $M^l = [4, 4, 2, 2, 0, 0, 0, 0, 0, 0, 0, 5, 5]$ and $M^f = [4, 4, 2, 2, 0, 0, 0, 0, 0, 0, 0, 5, 5]$. Now we observe event d at time $\tau = 5$. By Steps 7 to 15 in Algorithm 1, $\mathcal{X}$ is updated to Entry 8 with $M^l = [4, 4, 1, 1, 0, 0, 0, 1, 0, 0, 0, 5, 5]$ and $M^f = [4, 4, 0, 0, 1, 1, 0, 1, 0, 1, 0, 4, 4]$. Here $M_{new}^l = M^l + C_{uo} \cdot \mathbf{y}_{(M^l, t_8)} + C(\cdot, t_8)$ is obtained by following the transition relation from M_{b1} to M_{b2}, and $(M_{new}^f, \Theta_{new}^f)$ is obtained by firing t_8 at state (M^f, Θ^f).

Entries 9 to 12 shows the evolution of (M^l, M^f, Θ^f) in the silent period of time $\tau \in (5, 9)$. At the end of this silent period, we have $M^l = [4, 4, 1, 1, 0, 0, 0, 1, 0, 0, 0, 5, 5]$ and $M^f = [4, 4, 0, 0, 0, 2, 1, 0, 0, 2, 1, 2, 2]$ as shown in Entry 12. By observing event c, we notice that both t_6 and t_{11} with label c can fire at the fast-bound state in Entry 12. As a result, the current estimator state (M^l, M^f, Θ^f) in Entry 12 is split into two new states $(M_{new,1}^l, M_{new,1}^f, \Theta_{new,1}^f)$ and $(M_{new,2}^l, M_{new,2}^f, \Theta_{new,2}^f)$ (in Entry 13), each of which corresponds to the firing of t_6 and t_{11}, respectively. Hence we have

$$\mathcal{C}((a, 1)(b, 1)(d, 5)(c, 9), 9) = \bigcup_{i=1,2} R_{h,uo}(M_i^l, M_i^f)$$

where M_i^l's and M_i^f's are those in Entry 13.

Finally after another silent period, the time reaches $\tau = 10$. At the end of this silent period, we have $\mathcal{C}((a, 1)(b, 1)(d, 5)(c, 9), 10) = \bigcup_{i=1,2} R_{h,uo}(M_i^l, M_i^f)$, where both (M_i^l, M_i^f)'s in Entry 14 are derived from (M_i^l, M_i^f) in Entry 13. By observing event e, each estimator state $(M_i^l, M_i^f, \Theta_i^f)$ in $\mathcal{X}$ is checked. Since at (M_2^f, Θ_2^f) (the second line in Entry 14) t_{12} with label e cannot fire, it indicates that the new consistent markings cannot be a continuation from the the reachable hull of (M_2^l, M_2^f). As a result, the estimator state

Entry	τ	Observed TO	(M^l, M^f, Θ^f) in $\mathcal{X}$
0	0	λ	$([6,6,0,0,0,0,0,0,0,0,0,5,5], [6,6,0,0,0,0,0,0,0,0,0,5,5],$ $[0,0,0,0,0,0,0,0,0,0,0])$
1	1	λ	$([6,6,0,0,0,0,0,0,0,0,0,5,5], [6,6,0,0,0,0,0,0,0,0,0,5,5],$ $[1,1,0,0,1,0,0,0,1,0,0,0])$
2	1	$(a,1)$	$([4,6,2,0,0,0,0,0,0,0,0,5,5], [4,6,2,0,0,0,0,0,0,0,0,5,5],$ $[0,1,0,0,1,0,0,0,1,0,0,0])$
3	1	$(a,1)(b,1)$	$([4,4,2,2,0,0,0,0,0,0,0,5,5], [4,4,2,2,0,0,0,0,0,0,0,5,5],$ $[0,0,0,0,1,0,0,0,1,0,0,0])$
4	2	$(a,1)(b,1)$	$([4,4,2,2,0,0,0,0,0,0,0,5,5], [4,4,2,2,0,1,0,0,0,0,0,4,5],$ $[1,1,1,0,0,0,0,0,2,0,0,0])$
5	3	$(a,1)(b,1)$	$([4,4,2,2,0,0,0,0,0,0,0,5,5], [4,4,1,1,1,1,0,0,0,1,0,4,4],$ $[2,2,0,0,1,0,0,0,0,0,0,0])$
6	4	$(a,1)(b,1)$	$([4,4,2,2,0,0,0,0,0,0,0,5,5], [4,4,1,1,0,1,1,0,0,1,0,3,4],$ $[3,3,1,0,0,0,0,0,1,0,0,0])$
7	5	$(a,1)(b,1)$	$([4,4,2,2,0,0,0,0,0,0,0,5,5], [4,4,0,0,1,1,1,0,0,1,0,3,4],$ $[4,4,0,0,1,1,0,1,2,0,0,0])$
8	5	$(a,1)(b,1)(d,5)$	$([4,4,1,1,0,0,0,1,0,0,0,5,5], [4,4,0,0,1,1,0,1,0,1,0,4,4],$ $[4,4,0,0,1,0,0,0,2,0,0,0])$
9	6	$(a,1)(b,1)(d,5)$	$([4,4,1,1,0,0,0,1,0,0,0,5,5], [4,4,0,0,0,1,1,1,0,2,0,3,3],$ $[5,5,0,0,0,0,0,0,0,1,0,0])$
10	7	$(a,1)(b,1)(d,5)$	$([4,4,1,1,0,0,0,1,0,0,0,5,5], [4,4,0,0,0,1,1,1,0,2,0,3,3],$ $[6,6,0,0,1,1,0,1,1,2,0,0])$
11	8	$(a,1)(b,1)(d,5)$	$([4,4,1,1,0,0,0,1,0,0,0,5,5], [4,4,0,0,0,2,1,0,0,1,1,2,3],$ $[7,7,0,0,0,2,0,2,2,0,0,0])$
12	9	$(a,1)(b,1)(d,5)$	$([4,4,1,1,0,0,0,1,0,0,0,5,5], [4,4,0,0,0,2,1,0,0,2,1,2,2],$ $[8,8,0,0,1,3,0,3,0,0,1,1])$
13	9	$(a,1)(b,1)(d,5)(c,9)$	$([4,4,0,0,0,0,0,1,1,0,0,5,5], [4,4,0,0,0,2,0,0,1,2,1,3,2],$ $[8,8,0,0,1,0,0,0,0,0,1,1])$
		$(a,1)(b,1)(d,5)(c,9)$	$([4,4,1,1,0,0,0,0,1,0,0,5,5], [4,4,0,0,0,2,1,0,1,2,0,2,3]$ $[8,8,0,0,1,3,0,3,0,0,0,0])$
14	10	$(a,1)(b,1)(d,5)(c,9)$	$([4,4,0,0,0,0,0,1,1,0,0,5,5], [5,5,0,0,0,3,0,0,0,2,1,2,2],$ $[9,9,0,0,0,0,0,0,1,0,2,2])$
		$(a,1)(b,1)(d,5)(c,9)$	$([4,4,1,1,0,0,0,0,1,0,0,5,5], [5,5,0,0,0,3,1,0,0,2,0,1,3],$ $[9,9,0,0,0,4,0,4,1,0,0,0])$
15	10	$(a,1)(b,1)(d,5)(c,9)(e,10)$	$([4,4,1,1,0,0,0,0,1,0,0,5,5], [5,5,0,0,0,3,0,0,0,2,1,2,2],$ $[9,9,0,0,0,0,0,0,1,0,0,0])$

Table 1: The stepwise evolution of $\mathcal{X}$ for the example in Section VI-B.

$(M_2^l, M_2^f, \Theta_2^f)$ is abandoned. On the other hand, transition t_{12} can fire at (M_1^f, Θ_1^f) (the first line in Entry 14). Hence the updated $\mathcal{X}$ contains only one estimator state $(M_{new}^l, M_{new}^f, \Theta_{new}^f)$ derived from $(M_1^l, M_1^f, \Theta_1^f)$ in Entry 14. Therefore we finally have:

$$\mathcal{C}((a,1)(b,1)(d,5)(c,9)(e,10),10) = R_{h,uo}(M^l, M^f) \tag{10}$$

where M^l and M^f are those in Entry 15.

2) Place p_6 and p_{10} are unbounded: Suppose that the numbers of tokens in p_6 and p_{10} are not bounded, i.e., in Figure 6 places p_{12} and p_{13} as well as the related arcs are removed. Since the underlying LPN is now unbounded, its SCG is not finite and hence cannot be computed offline. Although an on-the-fly method that computes the set of consistent state classes online may still be used, such a process is not efficient since the number of consistent classes may grow with the length of the observed sequence. On the other hand, the underlying LPN has also 110 basis markings. For the same TO $\phi = (a,1)(b,1)(d,5)(c,9)(e,10)$ at $\tau = 10$, we have exact the same estimation result as illustrated in the bounded case. Note, however, that in general the number of estimator states (i.e., the number of SFM-pairs) may be infinite if the TLPN is unbounded.

To conclude this section, let us compare our approach with other techniques available in the literature. All SCG-based methods have to explicitly record all consistent state classes. For the schedulability-based method [17], a set of logical consistent markings is explicitly computed and for each of them an ILPP needs to be solved to remove the time-spurious markings. On the other hand, in our method only the tuples of slow-bound marking and the fast-bound state are recorded, which requires much less computational effort and less memory requirement.

Let us take Entry 7 in Table 1 as an example, i.e., we observe $(a,1)(b,1)$ at $\tau = 5$. The set of consistent markings $\mathcal{C}((a,1)(b,1),5) = R_{uo}^h(M^l, M^f)$ where $M^l = [4,4,2,2,0,0,0,0,0,0,0,5,5]$ and $M^f = [4,4,2,2,0,0,0,0,0,0,0,5,5]$, which consists of 26 markings. In the SCG there are 54 state classes consistent with $(a,1)(b,1)$ at $\tau = 5$. By applying the schedulability-based method in [16, 17] and analyzing the underlying untimed LPN, there are 192 consistent marking candidates that need to be tested one-by-one by solving an ILPP, and finally all 26 consistent markings are explicitly recorded. On the contrary, applying our method only one marking M^l and one state (M^f, Θ^f) is recorded.

7 Conclusion

In this paper we propose an online marking estimation method for a subclass of TLPNs. The set of consistent markings can be determined by a linear algebraic system based on the so-called slow-bound marking and fast-bound marking pairs, which can be efficiently computed online. The proposed method does not require the computation of the full state space, and hence the exhaustive construction of the full state space including the state class graph is avoided. This approach provides guidelines of sensor deployment in the design stage such that the online marking estimation problem can be efficiently solved. In future work we expect to relax the assumptions used in this paper, and to consider *stochastic time Petri nets* in which the delay of a transition is not fixed but follows a particular probability distribution.

References

[1] J. Moody and P. Antsaklis, “Petri net supervisors for DES with uncontrollable and unobservable transitions,” *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 462–476, 2000.

[2] J. Park and S. A. Reveliotis, “Liveness-enforcing supervision for resource allocation systems with uncontrollable behavior and forbidden states,” *IEEE Transactions on Robotics and Automation*, vol. 18, no. 2, pp. 234–240, 2002.

[3] D. Corona, A. Giua, and C. Seatzu, “Marking estimation of Petri nets with silent transitions,” *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1695–1699, 2007.

[4] M. Cabasino, C. N. Hadjicostis, and C. Seatzu, “Probabilistic marking estimation in labeled Petri nets,” *IEEE Transactions on Automatic Control*, vol. 60, pp. 528–533, 2015.

[5] Y. Tong, Z. W. Li, and A. Giua, “On the equivalence of observation structures for Petri net generators,” *IEEE Transactions on Automatic Control*, vol. 61, no. 9, pp. 2448–2462, 2016.

[6] A. Ramirez-Trevino, E. Ruiz-Beltran, I. Rivera-Rangel, and E. López-Mellado, “Online fault diagnosis of discrete event systems: A Petri net-based approach,” *IEEE Transactions on Automation Science and Engineering*, vol. 4, no. 1, pp. 31–39, 2007.

[7] M. Cabasino, A. Giua, and C. Seatzu, “Fault detection for discrete event systems using Petri nets with unobservable transitions,” *Automatica*, vol. 46, no. 9, pp. 1531–1539, 2010.

[8] Y. Tong, Z. W. Li, C. Seatzu, and A. Giua, “Verification of state-based opacity using Petri nets,” *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2823–2837, 2017.

[9] B. Berthomieu and M. Diaz, “Modeling and verification of time dependent systems using time Petri nets,” *IEEE Transactions on Software Engineering*, vol. 17, no. 3, pp. 259–273, 1991.

[10] D. Lime and O. H. Roux, “Model checking of time Petri nets using the state class timed automaton,” *Discrete Event Dynamic Systems*, vol. 16, no. 2, pp. 179–205, 2006.

[11] G. Gardey, O. H. Roux, and O. F. Roux, “State space computation and analysis of time Petri nets,” *Theory and Practice of Logic Programming*, vol. 6, no. 3, pp. 301–320, 2006.

[12] M. Ghazel, A. Toguyeni, and P. Yim, “State observer for DES under partial observation with time Petri nets,” *Discrete Event Dynamic Systems*, vol. 19, no. 2, pp. 137–165, 2009.

[13] X. Wang, C. Mahulea, and M. Silva, “Diagnosis of time Petri nets using fault diagnosis graph,” *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2321–2335, 2015.

[14] F. Basile, M. P. Cabasino, and C. Seatzu, “State estimation and fault diagnosis of labeled time Petri net systems with unobservable transitions,” *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 997–1009, 2015.

[15] P. Declerck and P. Bonhomme, “State estimation of timed labeled Petri nets with unobservable transitions,” *IEEE Transactions on Automation Science and Engineering*, vol. 11, no. 1, pp. 103–110, 2014.

[16] P. Bonhomme, "Marking estimation of P-time Petri nets with unobservable transitions," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 45, no. 3, pp. 508–518, 2015.

[17] P. Declerck, A. Chouchane, and P. Bonhomme, "A strategy for estimation in timed Petri nets," in *Proceedings of the 4th International Conference on Control Decision and Information Technologies*, Barcelona, Spain, 2017, pp. 489–494.

[18] F. Cassez, "Dynamic observers for fault diagnosis of timed systems," in *Proc. 49th IEEE Conference on Decision Control*, 2010, pp. 4359–4364.

[19] S. H. Zad, R. Kwong, and W. Wonham, "Fault diagnosis in discrete-event systems: Incorporating timing information," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 1010–1015, 2005.

[20] W. Khansa, J. P. Denat, and S. Collart-Dutilleul, "P-time Petri nets for manufacturing systems," in *Proceedings of the 3rd International Workshop on Discrete Event Systems*, 1996, pp. 94–102.

[21] R. Hadjidj and H. Bouchene, "Efficient reachability analysis for time Petri nets," *IEEE Transactions on Computers*, vol. 60, no. 8, pp. 1085–1099, 2011.

[22] F. Basile, C. Carbone, and P. Chiacchio, "Feedback control logic for backward conflict free choice nets," *IEEE Transactions on Automatic Control*, vol. 52, no. 3, pp. 387–400, 2007.

[23] Z. Y. Ma, Z. W. Li, and A. Giua, "Characterization of admissible marking sets in Petri nets with conflicts and synchronizations," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1329–1341, 2017.

[24] E. Dallal and S. Lafortune, "On most permissive observers in dynamic sensor activation problems," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 966–981, 2014.

[25] X. Yin and S. Lafortune, "A general approach for solving dynamic sensor activation problems for a class of properties," in *Proceedings of the 54th IEEE Conf. on Decision and Control*, Osaka, Japan, 2015, pp. 3610–3615.

[26] F. Basile, P. Chiacchio, and A. Giua, "An optimization approach to Petri net monitor design," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 306–311, 2007.

[27] M. P. Cabasino, S. Lafortune, and C. Seatzu, "Optimal sensor selection for ensuring diagnosability in labeled Petri nets," *Automatica*, vol. 49, no. 8, pp. 2373–2383, 2013.

[28] G. T. Murata, "Petri nets: Properties, analysis and applications," *Proceedings of the IEEE*, vol. 77, no. 4, pp. 541–580, 1989.

[29] B. Berthomieu, F. Peres, and F. Vernadat, *Time Petri Nets - Analysis Methods and Verification with TINA*. Wiley-Blackwell, 2010, ch. 1, pp. 19–49.

[30] Z. Y. Ma, Y. Tong, Z. W. Li, and A. Giua, "Marking estimation in labelled Petri nets by the representative marking graph," in *Proc. 20th IFAC World Congress*, 2017, pp. 11 175–11 181.

[31] R. Tarjan, "Enumeration of the elementary circuits of a directed graph," *SIAM Journal on Computing*, vol. 2, pp. 211–216, 1973.

[32] C. Mahulea, C. Seatzu, M. P. Cabasino, and M. Silva, “Fault diagnosis of discrete-event systems using continuous Petri nets,” *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, vol. 42, no. 4, pp. 970–984, 2012.

[33] C. Mahulea, J. Júlvez, C. R. Vázquez, and M. Silva, *Continuous Petri Nets: Observability and Diagnosis*. London: Springer London, 2013, pp. 387–406.

[34] Z. Y. Ma, Y. Tong, Z. W. Li, and A. Giua, “Basis marking representation of Petri net reachability spaces and its application to the reachability problem,” *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1078–1093, 2017.

Appendix: Proof of Lemma 5.1

To prove Lemma 5.1 we first introduce the notion of *terminal places*.

Definition A.1: Given a non-zero firing vector $\mathbf{y}$, a place p is called a *terminal place* of $\mathbf{y}$ if there exists a non-zero component $y(t)$ such that $p \in t^\bullet$ holds, and for any other non-zero components $y(t')$ where $t' \neq t$, $p \notin {}^\bullet t'$ holds. $\triangle$

In other words, a terminal place of $\mathbf{y}$ is the output place of some transition with a nonnegative component in $\mathbf{y}$ and is not an input place of any other transitions with nonnegative components in $\mathbf{y}$. Note that the definition of terminal places is based on the net structure, and there does not necessarily exist a firable sequence associated to $\mathbf{y}$. It is not difficult to understand that under the assumptions of Lemma 5.1, for any non-zero firing vector $\mathbf{y}$ the set of its terminal places is always non-empty, otherwise it implies the existence of cycles and/or sink transitions. Moreover, by the definition, $C(p,\cdot)\cdot\mathbf{y} > 0$ holds if p is a terminal place of $\mathbf{y}$. The proof of Lemma 5.1 is given as follows.

Proof of Lemma 5.1: We first prove that for any markings M_1 and M_2 such that $M_1[\sigma\rangle M_2$ there exists a unique firing vector $\mathbf{y}_\sigma$ such that $M_1 + C\cdot\mathbf{y}_\sigma = M_2$ by contradiction.

Suppose that there exist two different firing vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ such that $M_1 + C\cdot\mathbf{y}_1 = M_2$ and $M_1 + C\cdot\mathbf{y}_2 = M_2$, which indicates that $C\cdot\mathbf{y}_1 = C\cdot\mathbf{y}_2$. We can remove all commonly fired transitions from both sides to obtain two new firing vectors $\mathbf{y}_{min,1}$ and $\mathbf{y}_{min,2}$ satisfying $C\cdot\mathbf{y}_{min,1} = C\cdot\mathbf{y}_{min,2}$ (i.e., $\mathbf{y}_{min,i} = \mathbf{y}_i - \min\{\mathbf{y}_1,\mathbf{y}_2\}$ for $i = 1, 2$ where the $\min$ operator on vector is intended componentwise) and $\mathbf{y}_{min,1}^T\cdot\mathbf{y}_{min,2} = 0$. If $\mathbf{y}_{min,1} = \mathbf{0}$, it indicates that $\mathbf{y}_{min,2}$ is a T-invariant of the net. However, this cannot happen since for any place p that is its terminal place, $C(p,\cdot)\cdot\mathbf{y}_{min,2} > 0$ holds. This argument also holds for the case that $\mathbf{y}_{min,2} = \mathbf{0}$. Thus, both $\mathbf{y}_{min,1}$ and $\mathbf{y}_{min,2}$ must be non-zero. In such a case, if a terminal place p of $\mathbf{y}_{min,1}$ satisfies $C(p,\cdot)\cdot\mathbf{y}_{min,1} = k$, then $C(p,\cdot)\cdot\mathbf{y}_{min,2} = k$ must also hold. However, since $\mathbf{y}_{min,1}$ and $\mathbf{y}_{min,2}$ do not share any transitions, it indicates that place p can receive tokens by firing at least two different transitions. This contradicts the backward-conflict-free assumption. As a result, we conclude that the firing vector $\mathbf{y}$ such that $M_1 + C\cdot\mathbf{y} = M_2$ is unique.

Since $\mathbf{y}$ is unique, by the fact that $\mathbf{y}_{\sigma'} + \mathbf{y}_{\sigma''} = \mathbf{y}$ and $\mathbf{y}_{\sigma'}, \mathbf{y}_{\sigma''} \geq \mathbf{0}$, Lemma 5.1 holds. ■