Finite-Time Consensus on the Median Value with Robustness Properties

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Abstract

In this paper we propose a novel continuous-time protocol that solves the consensus problem on the median value, i.e., it provides distributed agreement in networked multi-agent systems where the quantity of interest is the median value of the agents' initial values. In contrast to the average value, the median value is a statistical measure inherently robust to the presence of outliers, which is a significant robustness issue in large-scale sensor and multi-agent networks. The proposed protocol requires only binary information regarding the relative state differences among the neighboring agents and achieves consensus on the median value in finite time by exploiting a suitable ad-hoc discontinuous local interaction rule. In addition, we characterize certain resiliency properties of the proposed protocol against the presence of uncooperative agents which do not implement the underlying local interaction rule whereas they interact with their neighbors thus influencing the network. In particular, we prove that despite the persistent influence of (at most) a certain number of uncooperative agents, the cooperative agents achieve finite time consensus on a value lying inside the convex hull of the cooperative agents' initial conditions, provided that the special class of so-called "k-safe" network topology is considered. Capabilities of the proposed consensus protocol and its effectiveness are supported by numerical studies.

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I. Introduction

A networked multi-agent system consists of a set of dynamical systems interconnected by a communication network. One of the most popular research topics in networked multi-agent systems is the so-called "consensus" problem, i.e., the problem of designing a decentralized local interaction rule forcing the states of dynamical agents to converge to (or "agree upon") a common value (called "consensus value"). The reader is referred to [1] for a tutorial overview of cooperative consensus-based control. When the desired consensus value is the average of agents' initial states, the problem is denoted as "consensus on the average", which constitutes one of the most studied consensus problems.

In [2], [3], [4], [5] distributed protocols that guarantee consensus on the average were proposed to address the case of multi-agent systems with communication topology represented by time-varying, unbalanced and/or directed graphs. These approaches, based on linear local interaction rules, have in common the use of the so-called "companion" (or "storage") variables to preserve the average of the state variables across the (continuous or discrete) iterations of the algorithm.

The so-called *ratio* consensus algorithm was proposed in [6], [7]. Such an algorithm consists in a local interaction protocol that achieves consensus on the average by exploiting the ratio of two state variables that execute linear state updates, one initialized arbitrarily and the other initialized with a default value. A major feature of ratio consensus is that it achieves consensus on the average also in directed graphs while being additionally robust against packet drops.

In [8] a method was proposed to achieve consensus on the average exploiting the iterative and distributed scaling of a column-stochastic matrix. Additionally, in [9] the average consensus problem was solved by estimating with a distributed algorithm the left eigenvector of the stochastic matrix encoding the network topology, to subsequently be used for weighting the different final consensus values thus recovering the average of the initial agents' states.

Protocols yielding consensus on the average intrinsically suffer from a significant problem: in spite of the large-scale nature of the underlying multi-agent systems, the existence of even a single outlier agent (i.e., an agent whose initial value holds an abnormal value) may arbitrarily affect the emergent behavior of the network. In presence of one, or more, outliers the ultimate consensus value may largely differ from the average of the agents with nominal initial state.

This issue has been investigated in [10], [11], [12], [13] and [14] where the main idea to cope with such a problem is that of identifying in a decentralized way, and then removing from the network, the outlier agents, thus recovering the emergent network's behavior of interest. In [15] some fundamental limitations of the average consensus problem in unreliable networks are investigated, whereas in [16] a strategy to compute in a distributed fashion an arbitrary function over a network containing malicious nodes is presented. In [17], a method is proposed to achieve resilient consensus on the average by exploiting the knowledge of the maximum number of misbehaving agents and removing from the network those agents holding abnormally different state values with respect to their neighbors. In [18], an optimization-based method is proposed to attenuate the detrimental effects of the outliers on the distributed computation of the average of the initial values.

Although some appropriate ad-hoc adjustments to the local interaction protocols can certainly attenuate the sensitivity to the outliers, the average value of a set of variables is in fact a statistical measure inherently sensitive to the presence of outlier data [19].

Thus motivated, **the main contribution** of this paper is a consensus algorithm where the consensus value of interest is the median (in contrast to the average) of the initial agents' states. The median is a statistical measure which is significantly more robust to outliers as compared to the average, in that the existence of abnormal initial values is filtered out by the possibly large number of samples [20]. We consider the case of heterogenous tuning parameters, then we provide sufficient tuning inequalities involving such parameters. To properly tune the algorithm, an upper bound to the number n of agents is supposed to be available.

The primary target scenario of the present investigation is a network in which some of the sensors/agents feature abnormal initial values that corrupt significantly the corresponding average. Furthermore, another source of error consists in the so-called "uncooperative" agents, i.e., faulty agents which do not possess abnormal initial conditions, i.e., are not outlier agents, but do not update their own state according to the prescribed local interaction protocol while sending the state information to the corresponding neighbors, thereby disrupting the emergent network behaviour in absence of proper countermeasures. We consider oblivious uncooperative agents with arbitrary state trajectories, by assuming that these trajectories may intersect those of cooperative agents only at, possibly infinite, isolated instants of time. Oblivious uncooperative agents are not supposed to exploit maliciously the information they may gather from their neighbors. In this paper, we complement the analysis of the "nominal" scenario, where all agents are supposed to be cooperative, by presenting a characterization of certain robustness properties of the proposed consensus on the median value protocol against the presence of uncooperative agents.

Consensus on the median value can be usefully exploited to increase reliability in several applications of mobile multi-robot systems or sensor networks which make use of consensus algorithms for coordination and estimation purposes. Reliability of the entire network of robots/sensors is often questionable, and large-scale distributed protocols thus need to be fault-tolerant with respect to both outlier measurements and uncooperative agents.

In sensor networks which sample a random variable with symmetric probability distribution centered on the average value, the median value coincides with the average value, but it retains higher robustness properties with respect to outlier measurements violating the underlying probability distribution due to faulty sensors.

In mobile multi-robot systems, consensus protocols are used to achieve rendezvous on the network barycenter or leader following, among several other applications. A major weakness of robotic swarms interacting via average consensus algorithms is that in spite of the possibly large size of the network a single faulty robot can disrupt the global emergent behavior. The ability of the consensus on the median value protocol to both disregard displaced robots far apart from the group and uncooperative robots (e.g., due to faults such as broken wheels) allows to counteract such a weakness by profitably exploiting the redundancy of the network.

We point out that methods to achieve consensus on general functions, the so-called χ-consensus algorithms (see, e.g., [21], [22], [23]), cannot be applied to the present scenario since the median value is not a continuous function of the initial state of the network. Thus, the achievement of consensus on the median value requires an ad-hoc protocol, analysis and robustness characterization.

The approach presented in this paper is based on a discontinuous local interaction rule. We refer the reader to [24], [25], [26] for exhaustive tutorials on the analysis of discontinuous gradient flows and discontinuous dynamical systems by means of non-smooth Lyapunov theory.

While the main novelty of the present proposal is the achievement of consensus on the median value, the local interaction protocol proposed in this paper features the additional desirable property of finite-time convergence to the consensus value. Protocols that achieve finite-time converging consensus (on the average, or on different consensus values such as, for instance, the minimum or maximum value or the geometric mean) can be found, e.g., in [27], [28], [29], [21] and they work for undirected, directed and/or time-varying network topologies. In [30] and [31], the finite-time consensus on an arbitrary smooth function of the initial state is investigated, for agents modeled as single integrators and respectively second-order systems with unknown non-linear dynamics, by considering continuous non-smooth local interactions. In [32] a so-called binary protocol to achieve consensus is proposed, which basically exploits only the sign of the relative state differences between neighboring agents. In [33] the use of the sign function is suggested along with an appropriate discontinuous local interaction rule to achieve finite-time consensus in the case of switching network topologies with communication time-delays.

The present proposal differs from [32] in that instead of considering a leader agent and the task of making each agent converge towards the state of the leader, we introduce a specific ad-hoc term in the local interaction rule which depends on the agents' own initial state, thus being different for each agents. This term turns out to be instrumental in steering the states of all agents towards the median value of the corresponding initial values.

In our previous work [34], the disturbance rejection properties of a discontinuous local interaction rule based on the sum of the signs of the relative state differences were investigated in the case of undirected graphs with switching topology. In the present work the similar discontinuous term is present in the local interaction rule, but the introduction of the previously mentioned ad-hoc term, depending on the agent's initial conditions, drastically change the performance of the algorithm which, in contrast to [34], provides finite time consensus on the median value. In [35] the disturbance rejection properties of a discontinuous interaction rule based on the sign of the sum of relative state differences were investigated in directed graphs. Note that our previous work is mainly focused on reaching consensus in presence of disturbances: however, the consensus value remains unspecified and it is generally time-varying and possibly unbounded. A further difference of the present work with respect to the above mentioned papers consists in different and refined proof techniques, based on the non-smooth Lyapunov theory, which allow us to characterize the evolution of the consensus value also in presence of outliers and uncooperative agents.

A preliminary version of the proposed interaction protocol for the case of homogeneous tuning gains and with a limited characterization of its convergence properties was announced in [36].

The paper is structured as follows. In Section II some background material is introduced. In Section III the problem statement is given and the proposed protocol is described. Section IV investigates the finite-time convergence properties of the novel consensus on the median protocol. In Section V, robustness and resiliency of the proposed protocol are investigated and, particularly, it is shown that there exist appropriate conditions ensuring that uncooperative agents neither prevent the achievement of consensus between cooperative agents nor affect arbitrarily the consensus value. In Section VI, we present numerical simulations to corroborate the theoretical results. Finally, in Section VII concluding remarks and future perspectives of the present research are discussed.

II. Notation and preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected static graph, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes representing agents and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges representing information flow between the agents. Let $(i, j) \in \mathcal{E}$ be the edge joining the agents i and j. Let $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ be the set of neighbors of agent i, i.e., the set of agents that exchange information with the agent i.

A path in a graph $\mathcal{G}$ is a sequence of consecutive edges connecting two agents. A graph is said to be *connected* if there exists a path between any pair of nodes. A *cut* in graph $\mathcal{G}$ is a partition of its nodes into two sets which are joined by at least one edge. A *minimum cut* of a graph is a cut in which the two sets are joined by the minimum number of edges among all the possible cuts. A given graph may admit one or more minimum cuts. We now introduce the definition of k-connected graph.

***Definition** 2.1 (k-connected graph):* We denote a graph $\mathcal{G}$ as being "k-connected" if its minimum cuts partition the nodes of the graph in two sets joined by at least k edges. ■

Notation $\mathbf{1}_n$ stands for the n-dimensional vector with unit elements.

A. Preliminaries on non-smooth analysis

We recall some definitions and results that will be employed hereafter in the paper. We define the discontinuous "sign" function and the discontinuous and set-valued "SIGN" function as follows

$$\text{sign}(y) = \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ -1, & \text{if } y < 1, \end{cases} \qquad y \in \mathbb{R}, \tag{1}$$

$$\text{SIGN}\,(y) \in \begin{cases} 1 & \text{if} \quad y > 0, \\ [-1, 1] & \text{if} \quad y = 0, \\ -1 & \text{if} \quad y < 0. \end{cases} \qquad y \in \mathbb{R}. \tag{2}$$

Consider the (possibly discontinuous) dynamical system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{3}$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, is defined almost everywhere, i.e., it is defined for every $x \in \mathbb{R}^n \setminus W$, where W is a subset of $\mathbb{R}^n$ of measure zero. Furthermore, $f(x)$ is measurable in an open region $Q \subset \mathbb{R}^n$ and for all compact sets $D \subset Q$ there exists a constant A_D such that $\|f(x)\| \leq A_D$ almost everywhere in D.

If the differential equation (3) has discontinuous right-hand side, following [37] we understand the corresponding solution in the so-called *Filippov sense* as the solution of an appropriate differential inclusion, as explained in the next definition.

***Definition** 2.2 (Filippov solution):* A vector function $x(\cdot) \in \mathbb{R}^n$ is called a Filippov solution of (3) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and, for almost all $t \in [t_0, t_1]$, satisfies the differential inclusion

$$\dot{x} \in K(x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} co\left\{f\left(B(x, \delta) \setminus N, t\right)\right\}, \tag{4}$$

where $\bigcap_{\mu(N)=0}$ denotes the intersection over all sets N of Lebesgue measure zero, $co\{\cdot\}$ denotes the convex hull and $B(x, \delta)$ is a ball of radius δ centered at x. ■

If $f(x)$ is measurable and locally bounded then the set-valued map $K(x)$ is upper semicontinuous, compact, convex valued and locally bounded so that the differential inclusion (4) possesses a Filippov solution for each initial condition x_0.

The reader is referred to [25] for a comprehensive tutorial on the different alternative solution notions for discontinuous dynamical systems, and to [38] for a comprehensive theory of generalized gradients and their applications.

We recall the definition of the *Clarke's Generalized Gradient* [39].

***Definition** 2.3 (Clarke's Generalized Gradient):* Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient $\partial V(x)$ is defined as

$$\partial V(x) \triangleq co\left\{\lim_{i \to \infty} \nabla V(x_i) | x_i \to x, x_i \notin \Omega_V \cup N\right\},$$

where ∇V denotes the conventional gradient, $x_i \in \mathbb{R}^n$ represents a point of an infinite succession which converges to $x \in \mathbb{R}^n$ as i grows to infinity, Ω_V is a set of Lebesgue measure zero which contains all points where $\nabla V(x)$ does not exist, and N is an arbitrary set of measure zero. ■

The Clarke's generalized gradient coincides with the standard gradient at the points where the standard derivative of the scalar function exists. Further details and examples of computation can be found in [24], [25].

Next, we recall the definition of set-valued Lie derivative.

***Definition** 2.4:* Given a locally Lipschitz function $V(x)$, where $x \in \mathbb{R}^n$ is governed by the differential inclusion $\dot{x} \in K(x)$, the set-valued Lie derivative of $V(x)$ at x is

$$\widetilde{\mathcal{L}}V(x) = \{a \in \mathbb{R} | \exists v \in K(x) \text{ such that } \zeta \cdot v = a, \forall \zeta \in \partial V(x)\} \tag{5}$$

As mentioned in [21], the set-valued Lie derivative allows the study of the evolution of a Lyapunov function along the Filippov solutions of the system under study, according to the next theorem.

***Theorem** 2.5:* Evolution along Filippov solutions [39] *Let* $x(t) : [t_0, t_1] \to \mathbb{R}^n$ *be a Filippov solution of* (4). *Let* $V(x)$ *be a locally Lipschitz and regular function. Then* $\frac{d}{dt}\left(V(x(t))\right)$ *exists a.e. and* $\frac{d}{dt}\left(V(x(t))\right) \in \widetilde{\mathcal{L}}V(x(t))$ *a.e..* ■

Next we provide a generalization of an extended Lyapunov Theorem for non-smooth analysis previously presented, in different form, in the literature.

***Theorem** 2.6:* *Let $M = span(\mathbf{1}_n)$ be the subspace spanned by vector $\mathbf{1}_n$. Consider a scalar function $V(x) : \mathbb{R}^n \to \mathbb{R}$, with $V(x) = 0\ \forall x \in M$ and $V(x) > 0\ \ \forall x \notin M$. Let $x : \mathbb{R} \to \mathbb{R}^n$ and $V(x(t))$ be absolutely continuous on $[t_0, \infty)$ with $\frac{d}{dt}(V(x(t))) \leq -\epsilon < 0$ a.e. on $\{t | x(t) \notin M\}$. Then, $V(x(t))$ converges to 0 in finite time and $x(t)$ reaches the subspace M in finite time as well.*

Proof: See Appendix A. ■

The original version of Theorem 2.6, proven in [24], dealt with the conditions guaranteeing the finite-time convergence to the origin rather than to the consensus subspace M.

III. Consensus on the Median Value

We consider a network of n agents with single integrator dynamics, i.e.,

$$\dot{x}_i(t) = u_i(t), \qquad x_i(0) = z_i, \qquad i = 1, 2, \dots, n, \tag{6}$$

where $u_i(t) \in \mathbb{R}$ is the local control input, to subsequently be specified, and $x_i(t) \in \mathbb{R}$ is the state of the i-th agent. Define

$$z = [z_1, z_2, \dots, z_n], \tag{7}$$

and let the agents be labeled in ascending order according to the corresponding initial state in such a way that the inequalities

$$z_i \leq z_{i+1}, \quad i = 1, 2, ..., n-1, \tag{8}$$

hold.

***Remark** 3.1:* Although the agents' labels are ordered according to (8), we stress the fact that the agents ignore their respective labels and this ordering is only adopted to simplify the notation in the algorithm convergence analysis. We also point out that to implement the interaction protocol to be designed each agent needs to store the value of its own initial value z_i during its evolution. ■

The median value between the agents' initial values, stored in the vector z, is defined (see [20]) as

$$m(z) = \arg\min_{\ell \in \mathbb{R}} \sum_{i=1}^{n} |z_i - \ell|, \tag{9}$$

and it takes the explicit form given in the next definition.

***Definition** 3.2:* The median value $m(z)$ of vector z in (7), satisfying (8), takes the form

$$m(z) \in \begin{cases} \left\{ z_{\frac{n+1}{2}} \right\} & \text{if } n \text{ is odd,} \\ \left[z_{\frac{n}{2}}, z_{\frac{n}{2}+1} \right] & \text{if } n \text{ is even.} \end{cases} \tag{10}$$

■

Note that the median value $m(z)$ is uniquely defined only when the dimension n of vector z is odd, whereas the median belongs to the closed interval $\left[z_{\frac{n}{2}}, z_{\frac{n}{2}+1}\right]$ when n is even.

***Remark** 3.3:* In the remainder a slight abuse of notation will be taken. If n is even then the notations $c < m(z)$, $c > m(z)$ and $c = m(z)$, with $c \in \mathbb{R}$, will be adopted to denote, respectively, the relations $c < z_{\frac{n}{2}}$, $c > z_{\frac{n}{2}+1}$ and $c \in \left[z_{\frac{n}{2}}, z_{\frac{n}{2}+1}\right]$. ■

Next, we give a definition of finite-time consensus.

***Definition** 3.4:* The state variables $x_i(t) \in \mathbb{R}$, $i \in \mathcal{V}$, of a networked multi-agent system are said to reach a *finite-time consensus* if there exist $T > 0$ and $c(t) \in \mathbb{R}$ such that

$$x_i(t) = c(t), \quad \forall i \in \mathcal{V}, \quad \forall t \geq T, \tag{11}$$

where $c(t)$ is referred to as "consensus function". ■

Our objective is to design a decentralized consensus protocol such that the finite-time consensus condition (11) is achieved by the network (6) with the consensus function $c(t) = m(z)$. The protocol that addresses our objective is given by the following **local interaction rule**:

$$\boxed{u_i(t) = -\alpha_i \ \text{sign}\left(x_i(t) - z_i\right) - \sum_{j \in N_i} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right),} \tag{12}$$

where $\alpha_i \in \mathbb{R}^+$, $i \in \mathcal{V}$, and $\lambda_{ij} \in \mathbb{R}^+$, $(i,j) \in \mathcal{E}$, are tuning parameters. The resulting **collective dynamics** is thus

$$\begin{aligned} \dot{x}_i(t) &= -\alpha_i \ \text{sign}\left(x_i(t) - z_i\right) - \sum_{j \in N_i} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right), \\ x_i(0) &= z_i. \end{aligned} \tag{13}$$

governing the closed-loop behavior of the multi-agent network (6) under the proposed local interaction rule (12).

A Filippov solution to (13) exists for every initial condition since the corresponding right-hand side is uniformly bounded [25]. The emerging behavior of the collective network's dynamics (13) is analyzed and theoretically supported in the next section.

The following constants

$$\lambda_{max} = \max_{(i,j)\in\mathcal{E}} \lambda_{ij}, \quad \lambda_{min} = \min_{(i,j)\in\mathcal{E}} \lambda_{ij}, \tag{14}$$

$$\alpha_{max} = \max_{i\in\mathcal{V}} \alpha_i, \quad \alpha_{min} = \min_{i\in\mathcal{V}} \alpha_i. \tag{15}$$

are of relevance in the framework of the convergence and robustness analysis to subsequently be developed.

IV. Finite-time convergence properties

In this section we characterize the convergence properties of the collective dynamics (13). The finite-time convergence of the agents' states towards the median value of the initial conditions takes place in two consecutive steps. First, the consensus condition (11) is achieved in a finite time T_1, under certain inequalities involving the tuning parameters λ_{ij} and α_i of the proposed local interaction rule, and it is maintained indefinitely at any $t \geq T_1$. This is proven in Theorem 4.1. Then, we show in Theorem 4.3 that the time-varying consensus value $c(t)$ converges in finite time $T_2 > T_1$ to the median value $m(z)$.

***Theorem** 4.1: Consider the network dynamics* (6) *along with a k-connected undirected graph $\mathcal{G}$ describing the underlying communication topology, with $k \geq 1$. Let the local interaction rule* (12) *be implemented with tuning parameters α_i, λ_{ij} such that*

$$\lambda_{ij} = \lambda_{ji} > 0, \qquad \forall (i,j) \in \mathcal{E}, \tag{16}$$

$$0 < \alpha_{max} < \frac{2k\lambda_{min}}{n}. \tag{17}$$

Then, the consensus condition (11) *is achieved and the transient time T is such that*

$$T \leq T_1 = \frac{\max_{i\in\mathcal{V}} x_i(0) - \min_{i\in\mathcal{V}} x_i(0)}{\mu^2}, \quad \mu^2 = 2\left(\frac{2k\lambda_{min}}{n} - \alpha_{max}\right). \tag{18}$$

Proof: A complete proof is presented in the Appendix B. A short proof sketch, summarizing the main steps and lines of reasoning, is given hereinafter. First, we define the sets

$$I_{max}(t) = \{k \in \mathcal{V} : x_k = \max_{i\in\mathcal{V}} x_i(t)\},$$
$$I_{min}(t) = \{k \in \mathcal{V} : x_k = \min_{i\in\mathcal{V}} x_i(t)\},$$

and consider the next non-smooth Lyapunov candidate function

$$V_1(x(t)) = \frac{\sum_{i\in I_{max}(t)} x_i(t)}{|I_{max}(t)|} - \frac{\sum_{i\in I_{min}(t)} x_i(t)}{|I_{min}(t)|},$$

where $|I_{max}(t)|$ and $|I_{min}(t)|$ denote the cardinalities of the sets. By exploiting the property of absolute continuity of the state trajectories we argue that sets $I_{max}(t)$ and $I_{min}(t)$ may change cardinality only at isolated instants of time. Therefore, these time instants can be disregarded in the non-smooth Lyapunov analysis and $|I_{max}(t)|$, $|I_{min}(t)|$ can be treated as constants while evaluating the generalized gradient of $V_1(x(t))$.

Then, we exploit the definition of the generalized time-derivative $\frac{d}{dt}(V_1(x(t))$ to obtain the corresponding set-valued map. Particularly, we take advantage of the symmetry of interactions (16) to obtain the instrumental relations

$$\sum_{i\in I_{max}(t)} \left(\sum_{j\in N_i \bigcap I_{max}(t)} \lambda_{ij} \mathrm{sign}\left(x_i(t) - x_j(t)\right) \right) = 0,$$

$$\sum_{i\in I_{min}(t)} \left(\sum_{j\in N_i \bigcap I_{min}(t)} \lambda_{ij} \mathrm{sign}\left(x_i(t) - x_j(t)\right) \right) = 0,$$

on the basis of which, by assuming the graph $\mathcal{G}$ to be k-connected, the set-valued generalized time-derivative $\frac{d}{dt}(V_1(x(t))$ is shown to fulfill the estimate

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq -\mu^2,$$

where μ^2 is the strictly positive constant defined in (18). The above estimate straightforwardly yields the finite-time convergence of $V_1(x(t))$ to zero, according to Theorem 2.6, which in turns implies that the finite time consensus condition (11) is achieved. □

Remark *4.2:* We point out that the protocol tuning condition (17) depends upon the number n of nodes in the network, which is a global information. This information, however, needs not to be exactly known by the designer since any overestimate $n_{max} \geq n$ can be substituted for n in (17). On the other hand, online distributed algorithms capable of estimating the number of agents are available in the literature (see, e.g., [40]). ■

In the next theorem we prove that the time-varying consensus value $c(t)$, achieved due to the local interaction protocol (13), converges in finite time to the median value $m(z)$ of vector (7).

Theorem *4.3: Consider the network dynamics* (6) *along with a k-connected undirected graph $\mathcal{G}$ describing the underlying communication network, with $k \geq 1$, and the local interaction rule* (12) *with tuning parameters that satisfy conditions* (16) *and* (17)*. Let the consensus condition* (11) *be in force starting from the finite moment $t = T$ on. If the additional tuning inequality*

$$\frac{\alpha_{max} - \alpha_{min}}{\alpha_{min}} < \frac{1}{n}, \tag{19}$$

is met, then there exist $T_2 \geq T$ such that the consensus value $c(t)$ meets the relation

$$c(t) = m(z), \quad \forall t \geq T_2, \tag{20}$$

where $m(z)$ denotes the median value (10) *and*

$$T_2 \leq 2n \frac{|c(T) - m(z)|}{\alpha_{max}} + T. \tag{21}$$

Proof: A complete proof is presented in the Appendix C. A short proof sketch summarizing its main steps and rationale is given in the sequel.

The analysis starts at $t \geq T$, i.e., after that the consensus condition (11) is established. We consider the Lyapunov function candidate

$$V_2(x(t)) = |c(t) - m(z)|, \quad t \geq T.$$

Due to (11) all agents hold the same state value $c(t)$, thus the corresponding average state value is equal to $c(t)$, i.e.:

$$c(t) = \frac{\sum_{i \in \mathcal{V}} x_i(t)}{n}, \quad t \geq T.$$

By exploiting the symmetry of local interactions, formalized by (16), it can be shown that the consensus value $c(t)$ obeys the discontinuous differential equation

$$\dot{c}(t) = -\frac{\sum\limits_{i \in \mathcal{V}} \alpha_i \text{sign}\left(x_i(t) - z_i\right)}{n}. \tag{22}$$

We then define the sets

$$I_{up} = \{k \in \mathcal{V} : x_k < z_k\}, \quad I_{down} = \{k \in \mathcal{V} : x_k > z_k\},$$
$$I_{equal} = \{k \in \mathcal{V} : x_k = z_k\}.$$

which are disjoint and such that their union forms the set $\mathcal{V}$. By manipulating the set-valued map which defines the differential inclusion governing the Filippov solutions of the discontinuous differential equation (22), we show that the set valued Lie derivative of $V_2(x(t))$ is such that

$$\widetilde{\mathcal{L}}V_2(x(t)) \in -\frac{1}{n}\text{SIGN}(c(t) - m(z)) \left(\sum_{i \in I_{down}} \alpha_i - \sum_{i \in I_{up}} \alpha_i + \sum_{i \in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t)) \right).$$

Further manipulations, taking into consideration the definition of median value and that of sets I_{down}, I_{up} and I_{equal}, relation (19), and the fact that the network is at consensus for all $t \geq T$, yield that unless $c(t) = m(z)$ (i.e. $V_2(x(t))) = 0$) it holds

$$\frac{d}{dt}\left(V_2(x(t))\right) \leq -\frac{\alpha_{max}}{2n},$$

which is sufficient to assess the finite-time convergence of $c(t)$ to the median value $m(z)$ and it also allows for straightforwardly determining the upper bound (21) to the convergence time. □

***Remark** 4.4:* The results of Theorems 4.1 and 4.3 can be straightforwardly generalized to graphs with switching topology provided that the time-varying graph $\mathcal{G}(t)$ remains k-connected for almost every $t \geq 0$ and the topology changes take place at isolated time instants whose union forms a set of measure zero.

***Remark** 4.5:* The proposed strategy does not work for directed graph topologies. The reason is that the symmetry of local interactions, given by (16), is only achievable for undirected graphs, and such property is instrumental to get the consensus value dynamics (22). If property (16) is not in force one obtains a more complex form for $\dot{c}(t)$, also depending on the signs of the relative state differences, which does not yield convergence of $c(t)$ towards the median value. Thus, in the directed graph scenario a different protocol has to be devised to achieve consensus on the median value.

V. Robustness properties

In this section we explore the robustness properties of the consensus protocol (13) against uncooperative agents, i.e., agents that belong to the network but do not adjust their own state according to protocol (13). Agents may act uncooperatively due to a variety of reasons such as faults or sabotages.

A critical vulnerability of distributed averaging networks is that a single uncooperative agent may arbitrarily influence the emergent behavior of the network despite the large number of agents/sensors. The consensus protocol (13) overcomes the issue of outlier initial states/measurements due to the inherent robustness of the median value with respect outlier data. We now show that, in some circumstances, protocol (13) is also able to limit the effect of uncooperative agents.

We are in need of introducing some further notation. We classify the agents of the network in two categories: i. agents that are known to be cooperative; ii.) agents that may be uncooperative. Let $\mathcal{V}_{safe}$ be the subset of agents which are *known* to be cooperative, and let $\mathcal{V}_{unsafe}$ be the subset of agents which are not known for certainty to be cooperative. $\mathcal{V}_{unsafe}$ may thus contain both cooperative and uncooperative agents. Additionally, let $\mathcal{V}_c$ be the subset of cooperative agents and $\mathcal{V}_u$ be the subset of uncooperative agents. Uncooperative agents do not implement protocol (13), and follow arbitrary trajectories, i.e., $\dot{x}_i = u_i(t)$, with $u_i(t) : \mathbb{R}^+ \to \mathbb{R}$, while interacting with the rest of the network. We consider oblivious uncooperative agents which are not supposed to exploit maliciously the information they may gather from the state of the neighbors. As a result, it is sensible to assume that the trajectories of the uncooperative agents may cross the trajectories of other agents only at isolated instants of time.

From these definitions, it follows that nodes in the set $\mathcal{V}_{safe}$ are all cooperative, i.e., $\mathcal{V}_{safe} \subseteq \mathcal{V}_c$. Uncooperative agents belong all to the set $\mathcal{V}_{unsafe}$, i.e., $\mathcal{V}_u \subseteq \mathcal{V}_{unsafe}$. It also holds that $\mathcal{V}_{safe} \bigcup \mathcal{V}_{unsafe} = \mathcal{V}_c \bigcup \mathcal{V}_u = \mathcal{V}$.

Let $\mathcal{G}_{safe} = (\mathcal{V}_{safe}, \mathcal{E}_{safe})$ be the subgraph of $\mathcal{G}$ representing the interconnections between agents in the set $\mathcal{V}_{safe}$. We now introduce the notion of k*-safe* network which allows us to characterize a class of network topologies which, under the local interaction rule (13), holds certain robustness features against uncooperative agents.

***Definition** 5.1:* k*-safe* network. A connected network $\mathcal{G}$, whose nodes are partitioned into the sets $\mathcal{V}_{safe}$ and $\mathcal{V}_{unsafe}$ according to the above-defined notation, is said to be k*-safe* if the corresponding subgraph $\mathcal{G}_{safe} \subseteq \mathcal{G}$ is k-connected and all nodes in the set $\mathcal{V}_{unsafe}$ are connected only to nodes in the set $\mathcal{V}_{safe}$. ■

A simple way to achieve a k-safe network is to let nodes in $\mathcal{V}_{safe}$ form a rigid graph in k dimensions (a rigid planar graph when $k = 2$, see [41]) and let nodes in $\mathcal{V}_{unsafe}$ share edges only with nodes in $\mathcal{V}_{safe}$.

In Figure 1 it is shown an example of 2-safe network. A plausible scenario in which k-safe network topologies can occur is, e.g., when a set of expensive and cheap sensors are scattered in an area which they need to monitor. Cheap sensors are large in number but prone to faults, and they only interact with a smaller set of more expensive "reliable" sensors which must be sufficiently connected between each other so as to form a k-connected graph.

Let us define

$$\begin{aligned}
\lambda_{max}^{safe} &= \max_{(i,j)\in\mathcal{E}_{safe}} \lambda_{ij}, & \lambda_{min}^{safe} &= \min_{(i,j)\in\mathcal{E}_{safe}} \lambda_{ij}, \\
\lambda_{max}^{unsafe} &= \max_{(i,j)\in\mathcal{E}\setminus\mathcal{E}_{safe}} \lambda_{ij}, & \lambda_{min}^{unsafe} &= \min_{(i,j)\in\mathcal{E}\setminus\mathcal{E}_{safe}} \lambda_{ij}.
\end{aligned}$$

The following theorem proposes sufficient conditions for the finite time achievement of consensus by cooperative agents under the influence of uncooperative agents.

***Theorem** 5.2: Consider an undirected k-safe network $\mathcal{G}$ of single integrator agents* (6)*. Let $\bar{\delta}$ be the maximal number of uncooperative agents, whose arbitrary trajectories are supposed to cross the trajectories of the cooperative agents only at isolated instants of time. Let Δ_{max} be the maximum degree of nodes in the set V_{unsafe}. Let the local interaction rule* (12) *be*

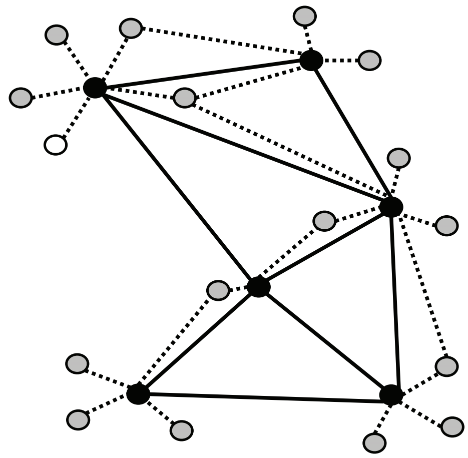

Fig. 1. Example of 2-safe network partitioned into safe nodes (black dots) and unsafe nodes (other nodes). The white node is uncooperative. The topology of $\mathcal{G}_{safe}$ is 2-connected and is highlighted with edges depicted as solid lines.

implemented for all $i \in \mathcal{V}_c$, with tuning parameters such that

$$\lambda_{min}^{safe} > \lambda_{max}^{unsafe}, \tag{23}$$

$$\lambda_{ij} = \lambda_{ji} > 0, \quad \forall (i,j) \in \mathcal{E}, \tag{24}$$

$$\alpha_{max} < 2\frac{k\lambda_{min}^{safe} - \Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}}, \tag{25}$$

$$\frac{\lambda_{min}^{safe}}{\lambda_{max}^{unsafe}} > \frac{\Delta_{max}\bar{\delta}}{k}. \tag{26}$$

Then, there exists a finite $T > 0$ such that

$$x_i(t) = c(t), \quad \forall i \in \mathcal{V}_c, \quad \forall t \geq T. \tag{27}$$

with

$$\begin{aligned} T \leq T_1 &= \frac{\max_{i\in\mathcal{V}_c} x_i(0) - \min_{i\in\mathcal{V}_c} x_i(0)}{\beta^2}, \\ \beta^2 &= 2\left(\frac{2(k\lambda_{min}^{safe} - \Delta_{max}\bar{\delta}\lambda_{max}^{unsafe})}{n-\bar{\delta}} - \alpha_{max}\right). \end{aligned} \tag{28}$$

Proof:

The complete proof is presented in Appendix D, whereas a short proof sketch in presented hereinafter.

The proof develops along the same lines followed in the proof of Theorem 4.1 by considering the subset of agents $\mathcal{V}_c \subseteq \mathcal{V}$ as opposed to set $\mathcal{V}$. For this purpose we first redefine sets I_{max} and I_{min} as

$$I_{max}^c(t) = \{k \in \mathcal{V}_c : x_k = \max_{i\in\mathcal{V}_c} x_i(t)\},$$
$$I_{min}^c(t) = \{k \in \mathcal{V}_c : x_k = \min_{i\in\mathcal{V}_c} x_i(t)\}.$$

We then consider the next function as non-smooth Lyapunov candidate

$$V^c(x(t)) = \frac{\sum_{i\in I_{max}^c} x_i(t)}{|I_{max}^c|} - \frac{\sum_{i\in I_{min}^c} x_i(t)}{|I_{min}^c|}.$$

As in the proof of Theorem 4.1, sets I_{max}^c and I_{min}^c may change cardinality only at isolated instants of time. By exploiting the fact that the network is k-safe and each uncooperative agent is connected only to cooperative agents which belong to set $\mathcal{V}_{safe}$, we show that with protocol parameters that satisfy the tuning constraints (23)-(26) one has that

$$\frac{d}{dt}\left(V^c(x(t))\right) \leq -\beta^2, \tag{29}$$

where $\beta^2 = 2\left(\frac{2(k\lambda_{min}^{safe} - \bar{\delta}\lambda_{max}^{unsafe})}{n-\bar{\delta}} - \alpha_{max}\right) > 0$, which proves the finite-time convergence of $V^c(x(t))$ to zero, according to Theorem 2.6, within a maximal transient time T_1 satisfying (28). □

In the next theorem, we show that despite the presence of uncooperative agents in the network, if consensus is reached then under appropriate conditions the consensus value cannot be arbitrarily affected by the uncooperative agents.

***Theorem** 5.3: Consider an undirected k-safe network $\mathcal{G}$ of agents* (6) *along with the local interaction rule* (12)*, implemented for all $i \in \mathcal{V}_c$, with tuning parameters such that*

$$\begin{array}{l} \lambda_{ij} = \lambda_{ji} > 0, \quad \forall (i,j) \in \mathcal{E}, \\ 2\frac{\Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}} < \alpha_{max}, \\ \frac{\alpha_{max}-\alpha_{min}}{\alpha_{min}} < \frac{1}{n}, \end{array} \tag{30}$$

where $\bar{\delta}$ is the maximal number of uncooperative agents. If

$$x_i(t) = c(t), \quad \forall i \in \mathcal{V}_c, \quad \forall t \geq T, \tag{31}$$

and if the trajectories of the uncooperative agents cross the trajectories of the cooperative agents only at isolated instants of time, then there exists a finite $\bar{T} > 0$ such that

$$c(t) \in [z^c_{min}, z^c_{max}], \qquad \forall t \geq T + \bar{T}. \tag{32}$$

where

$$z^c_{max} = \max_{i \in \mathcal{V}_c} z_i, \quad z^c_{min} = \min_{i \in \mathcal{V}_c} z_i, \tag{33}$$

and $\bar{T}$ satisfies

$$\begin{array}{l} \bar{T} \leq \dfrac{\left| c(T) - \frac{z^c_{max}+z^c_{min}}{2} \right| - \frac{z^c_{max}-z^c_{min}}{2}}{\beta^2}, \\ \beta^2 = \dfrac{\alpha_{max}}{2} - \lambda_{max}^{unsafe}\dfrac{\Delta_{max}\bar{\delta}}{n-\bar{\delta}}. \end{array} \tag{34}$$

Proof:
A complete proof is presented in Appendix E, the main steps of which are summarized in the following.
We consider the Lyapunov function

$$V_3(x(t)) = |c(t) - \bar{c}|, \quad \bar{c} = \frac{z^c_{max} + z^c_{min}}{2}, \quad t \geq T.$$

Taking into account that $|\mathcal{V}_c| = n - \delta$, where δ is the actual unknown number of uncooperative agents, we show that

$$c(t) = \frac{\sum_{i \in \mathcal{V}_c} x_i(t)}{n-\delta}.$$

Exploiting the symmetry of local interactions between the cooperative agents, formalized by the first relation of (30), we show that the time-varying consensus value $c(t)$ obeys the discontinuous differential equation

$$\dot{c}(t) = \begin{array}{l} -\dfrac{\displaystyle\sum_{i \in \mathcal{V}_c} \alpha_i \text{sign}\left(x_i(t) - z_i\right)}{n-\delta} \\ -\dfrac{\displaystyle\sum_{i \in \mathcal{V}_c} \sum_{j \in \mathcal{V}_u \bigcap N_i} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right)}{n-\delta}. \end{array}$$

Differently from the proof of Theorem 4.3, we define $I^c_{up} = \{k \in \mathcal{V}_c : x_k < z_k\}$, $I^c_{down} = \{k \in \mathcal{V}_c : x_k > z_k\}$, $I^c_{equal} = \{k \in \mathcal{V}_c : x_k = z_k\}$, which actually refer to the subset $\mathcal{V}_c$ of collaborative agents rather than to the full set $\mathcal{V}$ of agents.

We then assume that $V(x(t)) > \frac{z^c_{max}-z^c_{min}}{2}$, which happens when $c(t) > z^c_{max}$ or when $c(t) < z^c_{min}$. If $c(t) > z^c_{max}$ one has that $|I^c_{down}| = n - \delta$, whereas if $c(t) < z^c_{min}$ then $|I^c_{up}| = n - \delta$. In both cases, it holds that $|I^c_{down}| - |I^c_{up}| = n - \delta$ and $|I^c_{equal}| = 0$. By assumption, uncooperative agents are only connected with cooperative agents and their trajectories cross the trajectories of cooperative agents only at isolated instants of time, which can thus be disregarded. Furthermore, by definition $\delta \leq \bar{\delta}$. Then, by exploiting these arguments and with further manipulations, we show that if $\frac{\alpha_{max}-\alpha_{min}}{\alpha_{min}} < \frac{1}{n}$ then the set valued Lie derivative of $V_3(x(t))$ fulfills the following relation

$$\frac{d}{dt}\left(V_3(x(t))\right) \in \widetilde{\mathcal{L}}V_3(x) \leq -\frac{\alpha_{max}}{2} + \lambda_{max}^{unsafe}\frac{\Delta_{max}\bar{\delta}}{n-\bar{\delta}}, \tag{35}$$

which implies, due to the second relation of (30), that the domain $V^c(x) \leq \frac{z_{max}-z_{min}}{2}$ is finite-time attracting and invariant, with a transient time $T + \bar{T}$ fulfilling the estimation (34) thereby completing the proof. □

The tuning conditions considered in Theorem 5.3 are different from those of Theorem 5.2. In particular, if the tuning conditions of Theorem 5.2 are satisfied, then cooperative agents reach consensus. If the tuning conditions of both Theorems 5.2 and 5.3 are satisfied then cooperative agents achieve consensus and their state value can not be arbitrarily affected by uncooperative agents, i.e., the consensus value stays inside the convex hull of the cooperative agents' initial states. If the tuning conditions of Theorem 5.2 are not satisfied, then it is not guaranteed that cooperative agents achieve consensus but they may do as shown in the numerical simulations section. Therefore, it might be the case that only the tuning conditions of Theorem 5.3 are satisfied, and in this case the claimed results hold only in the event that cooperative agents achieve consensus.

In light of the results of Theorem 5.2 and Theorem 5.3, we now discuss the qualitative behavior of the proposed consensus protocol depending on the chosen values of parameters α_i and λ_{ij}, when the network topology is k-safe.

Due to relation $\frac{\alpha_{max}-\alpha_{min}}{\alpha_{min}} < \frac{1}{n}$, which is a requirement of both Theorems 4.2 and 5.3, parameters α_i should be designed with identical values or values that differ little, especially when the number n of nodes grows. If one selects them all identical, the former relation is identically satisfied. Provided that consensus is achieved within the set of cooperative agents, then if

$$\alpha_{max} > 2\frac{\Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}},$$

the uncooperative agents cannot arbitrarily influence the consensus value, which is constrained inside the convex hull of the cooperative agents' initial states. If

$$\alpha_{max} < 2\frac{k\lambda_{min}^{safe} - \Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}},$$

then all cooperative agents are guaranteed to achieve consensus.

Therefore, if the condition in inequality (19) and both previous conditions hold simultaneously, i.e., if

$$2\frac{\Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}} < \alpha_{max} < 2\frac{k\lambda_{min}^{safe} - \Delta_{max}\bar{\delta}\lambda_{max}^{unsafe}}{n-\bar{\delta}},$$

we can guarantee that the network achieves consensus with a value inside the convex hull of the cooperative agents initial states despite the influence of the $\delta \leq \bar{\delta}$ uncooperative agents. Such choice of parameters that satisfies both conditions exists only if

$$k\lambda_{min}^{safe} - 2\Delta_{max}\bar{\delta}\lambda_{max}^{unsafe} > 0,$$

therefore choosing a large λ_{min}^{safe} and a well connected network with $k \geq 1$ improves the robustness properties of the approach, i.e., a greater number of uncooperative agents can be disregarded.

VI. Simulations

In the first simulation we consider a network of 23 agents interacting by the graph in Figure 1 which is a 2-safe network with maximum degree of unsafe agents being equal to $\Delta_{max} = 3$. We consider the case in which three agents hold an outlier initial value. The initial network state is chosen at random with values in the range $[0, 10]$ while the initial values of outlier agents are chosen to be far off the average value and equal to $40, 43, 50$. The initial network state is thus

$$\begin{aligned} x(0) = z = \quad & [0.17, 1.14, 2.49, 2.87, 3.70, 4.40, 5.30, 5.71, \\ & 6.00, 6.97, 7.05, 7.78, 8.08, 8.65, 8.83, 8.89, \\ & 9.55, 9.69, 9.97, 9.99, 40.00, 43.00, 50.00]^T. \end{aligned}$$

The initial states' average is 11.31 whereas the median value $m(z)$ is 7.78. The median value disregarding the outlier agents belongs to the interval $[6.97, 7.05]$. In the first simulation we choose parameters that satisfy the conditions (16) and (30). We choose the tuning parameters uniformly at random within the ranges $\alpha_i \in [0.5, 0.51]$ for $i \in \mathcal{V}$, $\lambda_{ij} \in [1.5, 2]$ and $\lambda_{ij}^{safe} \in [10, 11]$ for $(i, j) \in \mathcal{E}$. This choice of parameters also satisfies condition (30) for $\delta = 1$ and $\Delta_{max} = 3$.

In Figure 2 it is shown the time evolution of the network states in a preliminary test where no uncooperative agents are considered. It is apparent that after a finite transient time consensus on a common value is achieved, and afterwards the time varying consensus value converges in finite time to the median value 7.78. Agents move at different speeds depending on the number of their neighbors. Agents with a single neighbor and tuning parameters λ_{ij} close to λ_{min} represent the worst case scenario with respect to the convergence speed. Figure 3, which is a zoom extracted from Figure 2, shows the transient evolution of the consensus value $c(t)$, highlighting its finite-time convergence towards the median value according to our theoretical findings.

In Figure 4 it is shown the network evolution during a second simulation run which considers the same tuning parameters as in the first simulation, which fulfill the tuning conditions (23) and (30) but it also includes one uncooperative agent (white node in Figure 1). Cooperative agents interact with the uncooperative agent while it is following its own independent trajectory. The trajectory of the uncooperative agent is represented by the bold line. It is evident that despite the presence of the uncooperative agent the network of cooperative agents converges to a consensus value inside the convex hull of the initial network state, according to (32)-(33). The steady state consensus value is 8.2.

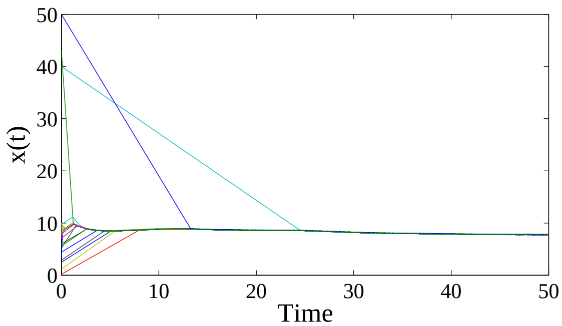

Fig. 2. Evolution of the network state $x(t)$ with tuning parameters $\alpha_i \in [0.5, 0.51]$, $\lambda_{ij} \in [1.5, 2]$ and $\lambda_{ij}^{safe} \in [10, 11]$ of the network in Figure 1 with three outliers and no uncooperative agents.

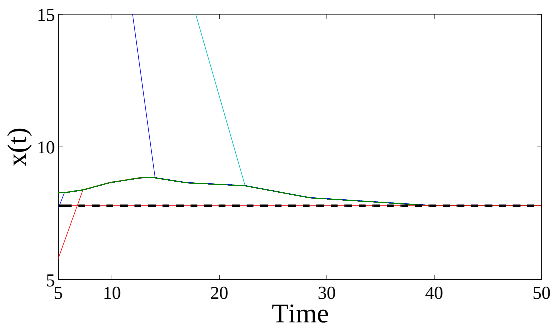

Fig. 3. Zoom of the evolution of the network state in Figure 2.

In Figure 5 we repeat the previous simulation with one uncooperative agent by choosing tuning parameters $\alpha_i = 1.5$ for all $i \in \mathcal{V}$, $\lambda_{ij} = 2$ and $\lambda_{ij}^{safe} = 2$ for all $(i,j) \in \mathcal{E}$, thus violating the tuning conditions (23) of Theorem 5.2. It can be seen that consensus is achieved by the cooperative agents and that the final consensus value is 7.788, thus well approximating the median value. Numerical simulations show that if parameters α_i are close to, but less than, parameters λ_{ij}, then the performance of the algorithm is improved but there exist network topologies and initial conditions for which it fails to converge to consensus.

In another simulation run, whose results are shown in the Table VI, we consider a scenario where the network topology is a connected Erdős-Rényi random graph $\mathcal{G}(p, n)$ with $n = 100$ nodes and probability of edge existence equal to $p = \frac{ln(n)}{n}$, just above the theoretical threshold which ensure almost sure connectivity in large networks [42]. We choose the tuning parameters $\alpha = 1$ and $\lambda = 2$, and have made 1000 simulation runs with random initial conditions and a number of uncooperative agents varying between 1 and 5. In Table VI the simulations results are summarized. We have measured the success rate of cooperative agents in the task of achieving consensus. Simulations show that for the given scenario, the protocol is always robust in the case of one uncooperative agents and starts to fail more frequently as the number of uncooperative agents grows. In this context failure to achieve consensus does not imply that all the network states tend to arbitrary value but only that some cooperative agents may not reach the consensus value. The number of cooperative agents which do not converge to the consensus value

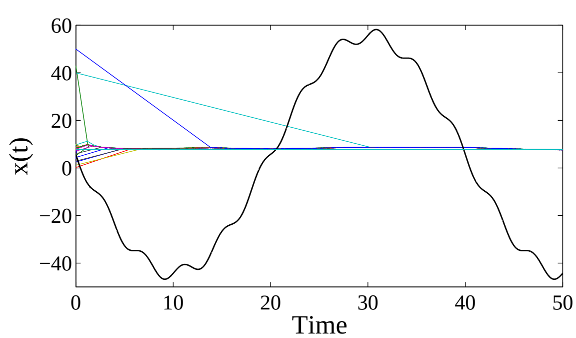

Fig. 4. Evolution of the network state $x(t)$ with tuning parameters as in Figure 2 with three outliers and one uncooperative agent with arbitrary trajectory (bold line).

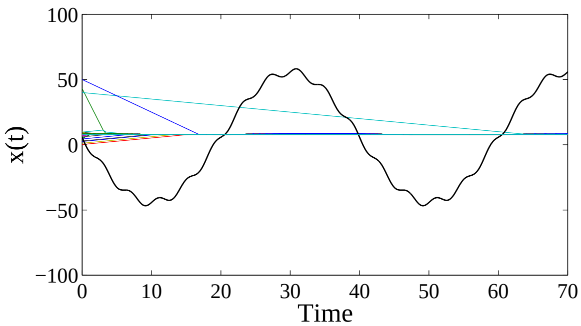

Fig. 5. Evolution of the network state $x(t)$ with tuning parameters $\alpha_i = 1.5$ for all $i \in \mathcal{V}$ and $\lambda_{ij} = 2$ for all $(i, j) \in \mathcal{E}$, three outliers and one uncooperative agent with arbitrary trajectory (bold line).

is surprisingly small. This indicates that in general, while it is not possible to guarantee robustness to uncooperative agents for arbitrary networks, it is possible to observe robustness, in a probabilistic sense, in large networks where the number of uncooperative agents is small with respect to the total number of agents. This feature seems to be more evident when the parameters λ_{ij} are close to the parameters α_i. In summary, very small values of the α_i's, as suggested by Theorem 5.2, can guarantee global consensus at the price of making the emergent behavior very sensitive to uncooperative agents. Instead, if parameters α_i's are close to, but less than, the λ_{ij}'s, global consensus between cooperative agents is not guaranteed anymore but sensitivity with respect to uncooperative agents is greatly reduced.

	Number of uncooperative agents				
	1	2	3	4	5
Percentage of simulations where all cooperative agents achieve consensus	100%	95%	83%	82%	81%
Maximum number of cooperative agents not at consensus	0	1	2	4	4

TABLE I
ROBUSTNESS WITH RESPECT TO UNCOOPERATIVE AGENTS ON ERDŐS - RÉNYI RANDOM GRAPHS $\mathcal{G}(p, n)$ WITH $n = 100$, $p = \frac{ln(n)}{n}$ AND PROTOCOL TUNING PARAMETERS $\alpha = 1$, $\lambda = 2$.

Finally, in Figure 6 we show the results of a large scale simulation considering a random network with 1000 agents and tuning parameters $\alpha_i = 1$ and $\lambda_{ij} = 2$. In this simulation we considered 5 uncooperative agents with state value equal to 100 for the whole simulation time. In Figure 6 it can be seen that most of the network, except one cooperative agent, reaches the consensus value of 5.40. The initial average value of the set of cooperative agents is 5.07 whereas the median value is 5.10. In general, numerical simulations of protocol (13) show that if parameter λ_{ij}'s are close to parameters α_i, while it is not guaranteed that all cooperative agents reach consensus on the median value, which happens or not depending on the current network topology, with high probability most of the cooperative agents converge to a value close to the median value of the network while being able to significantly disregard the existence of uncooperative agents which can not arbitrarily affect the network emerging behavior. We are interested in proving this conjecture in future works.

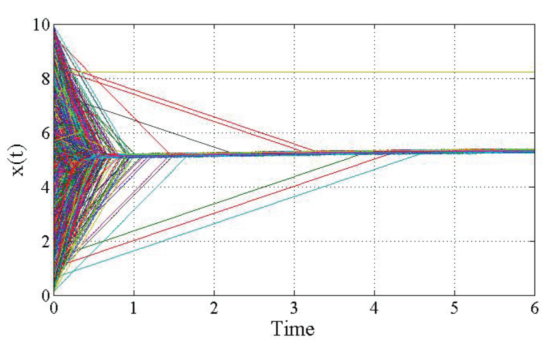

Fig. 6. Time evolution of a random network with 1000 nodes with 5 uncooperative agents.

VII. Conclusions and future work

In this paper we proposed a novel consensus protocol which achieves agreement with respect to the median value of the initial states of a network of continuous-time single integrators in finite time. The proposed protocol achieves distributed agreement towards an inherently robust statistical measure with respect to outlier states corresponding to large abnormal values due to measurement errors or faulty equipment. We characterized the finite-time convergence properties of the proposed protocol and some robustness properties with respect to uncooperative agents. In particular, we proved that there exist tuning conditions for the protocol parameters so that for a network with the so-called k-safe topology the achievement of consensus is robust to the influence of uncooperative agents and the consensus value lies inside the convex hull of the initial states of the cooperative agents. Future work will involve the characterization of the robustness properties with respect to noisy relative state measurements and the application of possibly modified versions of the protocol to concrete problems involving multi-robot networks. The convergence and robustness properties of the discrete-time versions of the proposed interaction protocol are also worth to be investigated in future research activities.

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APPENDIX

A. Proof of Theorem 2.6

This proof is a slight extension of the proof of Theorem 2 (Lypunov's Theorem Generalized) in [24]. We show that the similar result holds in the case one is interested in proving the convergence of $x(t)$ toward the consensus subspace $span(\mathbf{1}_n)$ rather than towards the origin.

We begin by proving by contradiction that there exists $T_0 \geq t_0$ such that $V(T_0) = 0$. If there exists no such T_0, then $x(t) \notin M$ $\forall t \in [t_0, \infty)$ and $\frac{d}{dt}(V(x(t))) \leq -\epsilon$ a.e. on $[t_0, \infty)$. Thereby, $\lim_{t\to\infty} V(t) = V(0) + \int_{t_0}^{\infty} \frac{d}{dt}(V(x(t)))\, dt = -\infty$, thus contradicting that $V(T_0) = 0$ never holds since $V(0) \geq 0$. Therefore, there exists such a $T_0 \geq t_0$. We now prove by contradiction that $V(t) = 0$ for $t \geq T_0$. Suppose that there exists $T_1 \geq T_0$ such that $V(T_1) > 0$, then $\int_{T_0}^{T_1} \frac{d}{dt}(V(x(t)))\, dt = V(T_1) > 0$. It follows that $\frac{d}{dt}(V(x(t))) > 0$ on a set of positive measure, thus contradicting that $\frac{d}{dt}(V(x(t))) \leq -\epsilon < 0$ a.e. on $[t_0, \infty)$. Therefore, there exists $T_0 \geq t_0$ such that $V(t) = 0$ $\forall t \geq T_0$. Furthermore, $V(t) = 0$ implies $x(t) \in M = span(\mathbf{1}_n)$. This concludes the proof. □

B. Proof of Theorem 4.1

Let

$$\begin{aligned} I_{max}(t) &= \{k \in \mathcal{V} : x_k = \max_{i \in \mathcal{V}} x_i(t)\}, \\ I_{min}(t) &= \{k \in \mathcal{V} : x_k = \min_{i \in \mathcal{V}} x_i(t)\}. \end{aligned} \tag{36}$$

In the sequel, the explicit dependence of $I_{max}(t)$ and $I_{min}(t)$ on t will be omitted to simplify the notation. Consider the non-smooth Lyapunov candidate function

$$V_1(x(t)) = \frac{\sum\limits_{i \in I_{max}} x_i(t)}{|I_{max}|} - \frac{\sum\limits_{i \in I_{min}} x_i(t)}{|I_{min}|}. \tag{37}$$

It is clear that $V_1(x(t)) \geq 0$ and $V_1(x(t)) = 0$ iff the network is at consensus. It is worth noting that by construction the next relations

$$\frac{\sum\limits_{i \in I_{max}} x_i(t)}{|I_{max}|} \equiv \max_{i \in \mathcal{V}} x_i(t), \quad \frac{\sum\limits_{i \in I_{min}} x_i}{|I_{min}|} \equiv \min_{i \in \mathcal{V}} x_i(t), \tag{38}$$

hold, which means that the alternative definition $V_1(x(t)) = \max_{i \in \mathcal{V}} x_i - \min_{i \in \mathcal{V}} x_i$ could in principle be adopted. However, definition (37) proves to be convenient to get a subsequent Lyapunov analysis yielding less conservative requirements on the tuning parameters of the local interaction protocol.

The cardinalities of sets I_{max} and I_{min} change over time. We first observe that the cardinality of the sets $I_{max}(t)$ and $I_{min}(t)$ is a piecewise constant function whose instants of discontinuity belong to a set of measure zero. We show this for $I_{max}(t)$, but a similar reasoning applies to $I_{min}(t)$. Since for all $i \in \mathcal{V}$ $\dot{x}_i$ is a sum of sign functions for every t, it is bounded. Therefore, $x_i(t)$ is locally Lipschitz. This implies that $x_i(t)$ is absolutely continuous. Consider any $j \notin I_{max}(t)$ then, since $x_j(t)$ is absolutely continuous with bounded derivative, it takes a finite time before $x_j(t)$ can become maximal. Therefore, increments of $|I_{max}(t)|$ occur at isolated instants of time. Now assume, by contradiction, that there exists an interval of time positive measure in which $|I_{max}(t)|$ is discontinuous. In this interval, $|I_{max}(t)|$ should take an infinite number of decrements which is impossible, given that the cardinality is an integer between 1 and n. This shows that the instants of discontinuity of $I_{max}(t)$ are isolated points. These time instants can be disregarded in the non-smooth Lyapunov analysis and $|I_{max}(t)|$, $|I_{min}(t)|$ can be treated as constants while evaluating the generalized gradient of $V_1(x(t))$.

Let us note that $V_1(x(t)) : \mathbb{R}^+ \to \mathbb{R}$ is absolutely continuous because it is the composition of the function $V_1(\cdot)$, which is locally Lipschitz continuous, and the absolutely continuous function $x(t)$. Therefore, $\frac{d}{dt}\left(V(x(t)\right)$ exists almost everywhere and there exists a set N of measure zero such that for all $t \in [0,\infty)\backslash N$, both $\dot{x}(t)$ and the generalized time derivative $\frac{d}{dt}\left(V_1(x(t)\right)$ exist.

Now, fix $t \in [0,\infty)\backslash N$. Following [43], owing on the local Lipschitz continuity of $V_1(x(t)$ one has that

$$\frac{d}{dt}\left(V_1(x(t)\right) = \lim_{h\to 0} \frac{V_1(x(t)+h\dot{x}(t)) - V_1(x(t))}{h}. \tag{39}$$

By substituting (37) into (39), it holds

$$\begin{aligned}\tfrac{d}{dt}\left(V_1(x(t)\right) = \quad & \lim_{h\to 0}\left\{\frac{\sum\limits_{i\in I_{max}} x_i(t)+h\dot{x}_i(t)}{h|I_{max}|} - \frac{\sum\limits_{i\in I_{min}} x_i + h\dot{x}_i(t)}{h|I_{min}|}\right. \\ & \left. -\frac{\sum\limits_{i\in I_{max}} x_i(t)}{h|I_{max}|} + \frac{\sum\limits_{i\in I_{min}} x_i(t)}{h|I_{min}|}\right\},\end{aligned}$$

which simplifies as follows

$$\frac{d}{dt}\left(V_1(x(t)\right) = \frac{\sum\limits_{i\in I_{max}} \dot{x}_i(t)}{|I_{max}|} - \frac{\sum\limits_{i\in I_{min}} \dot{x}_i(t)}{|I_{min}|}. \tag{40}$$

By taking into account the discontinuous collective dynamics (13), one straightforwardly derives

$$\begin{aligned}\sum_{i\in I_{max}} \dot{x}_i = \quad & \sum_{i\in I_{max}}\Bigg(-\alpha_i \ \mathrm{sign}\left(x_i(t)-z_i\right) \\ & -\sum_{j\in N_i \bigcap I_{max}} \lambda_{ij}\mathrm{sign}\left(x_i(t)-x_j(t)\right) \\ & -\sum_{j\in N_i\backslash I_{max}} \lambda_{ij}\mathrm{sign}\left(x_i(t)-x_j(t)\right)\Bigg).\end{aligned} \tag{41}$$

Due to the symmetry of interactions, formalized by (16), it holds

$$\sum_{i\in I_{max}}\left(\sum_{j\in N_i \bigcap I_{max}} \lambda_{ij}\mathrm{sign}\left(x_i(t)-x_j(t)\right)\right) = 0. \tag{42}$$

By virtue of (42), we can manipulate (41) into

$$\begin{aligned}\sum_{i\in I_{max}} \dot{x}_i = \quad & \sum_{i\in I_{max}}\Bigg(-\alpha_i \ \mathrm{sign}\left(x_i(t)-z_i\right) \\ & -\sum_{j\in N_i\backslash I_{max}} \lambda_{ij}\mathrm{sign}\left(x_i(t)-x_j(t)\right)\Bigg).\end{aligned} \tag{43}$$

Thus, one derives the following set-valued map associated to (43)

$$\begin{aligned}\sum_{i\in I_{max}} \dot{x}_i \in \quad & \sum_{i\in I_{max}}\Bigg(-\alpha_i \ \mathrm{SIGN}\left(x_i(t)-z_i\right) \\ & -\sum_{j\in N_i\backslash I_{max}} \lambda_{ij}\mathrm{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg).\end{aligned} \tag{44}$$

By making similar developments, one also derives that

$$\begin{aligned}\sum_{i\in I_{min}} \dot{x}_i \in \quad & \sum_{i\in I_{min}}\Bigg(-\alpha_i \ \mathrm{SIGN}\left(x_i(t)-z_i\right) \\ & -\sum_{j\in N_i\backslash I_{min}} \lambda_{ij}\mathrm{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg).\end{aligned} \tag{45}$$

Substituting (44) and (45) into (40), one obtains that $\frac{d}{dt}\left(V_1(x(t))\right)$ takes values in the following set-valued map

$$\begin{aligned}\frac{d}{dt}\left(V_1(x(t))\right) \in \quad & \frac{1}{|I_{max}|}\sum_{i\in I_{max}}\Bigg(-\alpha_i \ \mathrm{SIGN}\left(x_i(t)-z_i\right)\\ & -\sum_{j\in N_i\backslash I_{max}}\lambda_{ij}\mathrm{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg)\\ & -\frac{1}{|I_{min}|}\sum_{i\in I_{min}}\Bigg(-\alpha_i \ \mathrm{SIGN}\left(x_i(t)-z_i\right)\\ & -\sum_{j\in N_i\backslash I_{min}}\lambda_{ij}\mathrm{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg).\end{aligned} \tag{46}$$

By the definition of set I_{max}, any agent j in the set $N_i\backslash I_{max}$ is such that if the network is not at consensus then $x_i(t) > x_j(t)$ $\forall i \in I_{max}$, which implies that $\mathrm{SIGN}\left(x_i(t)-x_j(t)\right) = 1$ $\forall i \in I_{max}$. Similarly, one concludes that $\mathrm{SIGN}\left(x_i(t)-x_j(t)\right) = -1$ $\forall i \in I_{min}$.

Furthermore, since graph $\mathcal{G}$ is k-connected, if the network is not in the consensus state then there exist at least k edges connecting nodes in I_{max} with nodes with a different state value, and at least k edges connecting nodes in I_{min} with nodes with a different state value.

Therefore, the following estimate can be derived by (46)

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq \quad \alpha_{max} - \frac{k\lambda_{min}}{|I_{max}|} + \alpha_{max} - \frac{k\lambda_{min}}{|I_{min}|}. \tag{47}$$

Additionally, if the network is not in the consensus state then $|I_{max}| = p < n$ and $|I_{min}| \leq n-p$. Thus, (47) is further elaborated as follows

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq 2\alpha_{max} - \frac{nk\lambda_{min}}{p(n-p)}. \tag{48}$$

The upper bound is maximized taking $p = \frac{n}{2}$, therefore it yields

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq 2\alpha_{max} - \frac{4k\lambda_{min}}{n}. \tag{49}$$

In view of (16), by letting $\mu^2 = 2\left(\frac{2k\lambda_{min}}{n} - \alpha_{max}\right)$ one derives

$$\frac{d}{dt}\left(V_1(x(t))\right) \leq -\mu^2, \tag{50}$$

where, by (17), $\mu^2 > 0$, thereby proving the finite-time convergence of $V_1(x(t))$ to zero according to Theorem 2.6. To evaluate the convergence time, let us write down the inequality

$$\begin{aligned} V_1(x(t)) \quad & = V_1(x(0)) + \int_0^t \frac{d}{dt}\left(V_1(x(t))\right)dt \\ & \leq V_1(x(0))) - \int_0^t \mu^2 dt = V_1(x(0)) - \mu^2 t. \end{aligned} \tag{51}$$

By (51), it derives that the finite-time consensus condition (11) is achieved within a maximal transient time T_1 satisfying (18). Theorem 4.1 is proved. □

C. Proof of Theorem 4.3

Let us consider the Lyapunov function

$$V_2(x(t)) = |c(t) - m(z)|, \quad t \geq T. \tag{52}$$

i.e., our analysis begins after that the consensus condition (11) has been already achieved.

It is worth to note that due to (11) all agents hold the same state value $c(t)$, and therefore the average state value is given by $c(t)$, i.e.

$$c(t) = \frac{\sum_{i\in\mathcal{V}} x_i(t)}{n}, \quad t \geq T. \tag{53}$$

The generalized gradient of $V_2(x(t))$ takes the form

$$\partial V_2(x) = \mathrm{SIGN}(c(t) - m(z)). \tag{54}$$

By (53), the time varying consensus value $c(t)$ obeys the discontinuous differential equation

$$\begin{aligned}\dot{c}(t) &= \frac{\sum_{i\in\mathcal{V}} \dot{x}_i(t)}{n} = -\frac{\sum_{i\in\mathcal{V}} \alpha_i \text{sign}\left(x_i(t) - z_i\right)}{n} \\ &\quad -\frac{\sum_{i\in\mathcal{V}} \sum_{j\in N_i} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right)}{n}.\end{aligned} \tag{55}$$

Due to the symmetry of local interactions, specified by (16), one has that

$$\sum_{i\in\mathcal{V}} \left(\sum_{j\in N_i} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right) \right) = 0,$$

therefore (55) straightforwardly simplifies as

$$\dot{c}(t) = -\frac{\sum_{i\in\mathcal{V}} \alpha_i \text{sign}\left(x_i(t) - z_i\right)}{n}. \tag{56}$$

The Filippov solutions of (56) are governed by the differential inclusion

$$\dot{c}(t) \in -\frac{1}{n} \sum_{i\in\mathcal{V}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right). \tag{57}$$

Let

$$\begin{gathered} I_{up} = \{k \in \mathcal{V} : x_k < z_k\}, \quad I_{down} = \{k \in \mathcal{V} : x_k > z_k\}, \\ I_{equal} = \{k \in \mathcal{V} : x_k = z_k\}. \end{gathered} \tag{58}$$

Clearly, the sets I_{up}, I_{down} and I_{equal} are disjoint, and their union forms the set $\mathcal{V}$. We can thus rewrite (57) as follows

$$\begin{aligned}\dot{c}(t) \in \; & -\frac{1}{n} \left\{ \sum_{i\in I_{up}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right) \right. \\ & + \sum_{i\in I_{down}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right) \\ & \left. + \sum_{i\in I_{equal}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right) \right\}.\end{aligned} \tag{59}$$

By construction, the next relations hold

$$\text{SIGN}\left(x_i(t) - z_i\right) = -1 \qquad \forall i \in I_{up}, \tag{60}$$

$$\text{SIGN}\left(x_i(t) - z_i\right) = 1 \qquad \forall i \in I_{down}. \tag{61}$$

Therefore, by (60) and (61) one manipulates (59) as follows

$$\begin{aligned}\dot{c}(t) \in \; & -\frac{1}{n} \left(\sum_{i\in I_{down}} \alpha_i - \sum_{i\in I_{up}} \alpha_i \right. \\ & \left. + \sum_{i\in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t)) \right),\end{aligned} \tag{62}$$

and the set-valued Lie derivative of $V_2(x(t))$ correspondingly takes the form

$$\begin{aligned}\widetilde{\mathcal{L}}V_2(x(t)) = \; & -\frac{1}{n} \text{SIGN}(c(t) - m(z)) (\sum_{i\in I_{down}} \alpha_i - \sum_{i\in I_{up}} \alpha_i \\ & + \sum_{i\in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t))).\end{aligned} \tag{63}$$

Taking into account that

$$x_i(t) = c(t), \qquad \forall i \in \mathcal{V}, \quad \forall t \geq T, \tag{64}$$

by construction it holds

$$c(t) < z_i, \quad \forall i \in I_{up}, \tag{65}$$

$$c(t) > z_i, \quad \forall i \in I_{down}. \tag{66}$$

Having in mind the Definition 3.2 of the median value, the next implication holds

$$|I_{down}| = |I_{up}| \quad \Longrightarrow \quad c(t) = m(z). \tag{67}$$

Therefore, we concentrate on the case in which $|I_{down}| \neq |I_{up}|$. When $|I_{down}| \neq |I_{up}|$, two cases may occur

$$\text{Case 1}: ||I_{down}| - |I_{up}|| > |I_{equal}|, \tag{68}$$

$$\text{Case 2}: ||I_{down}| - |I_{up}|| \leq |I_{equal}|, \tag{69}$$

which will be treated separately.

Case 1. When relation (68) is in force, it holds

$$\sum_{i \in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t)) \in [-\alpha_{max}|I_{equal}|, \alpha_{max}|I_{equal}|]. \tag{70}$$

We now derive a lower bound to the next quantity

$$\left| \sum_{i \in I_{down}} \alpha_i - \sum_{i \in I_{up}} \alpha_i + \sum_{i \in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t)) \right|. \tag{71}$$

Without loss of generality we consider the case $|I_{down}| > |I_{up}|$ and thus inequality (68) becomes $|I_{down}| - |I_{up}| > |I_{equal}|$ (the same derivation can be carried out for the case $|I_{down}| < |I_{up}|$). Thus, we consider the value for each term α_i in (71) with the aim to find the minimal magnitude of (71) and thus yielding the next estimate

$$\begin{aligned} &\left| \sum_{i \in I_{down}} \alpha_i - \sum_{i \in I_{up}} \alpha_i + \sum_{i \in I_{equal}} \alpha_i \text{SIGN}(x_i(t) - z_i(t)) \right| \\ &\geq \alpha_{min}|I_{down}| - \alpha_{max}|I_{up}| - \alpha_{max}|I_{equal}|. \end{aligned} \tag{72}$$

Denote

$$k = |I_{up}| + |I_{equal}|. \tag{73}$$

By substituting eq. (73) into $|I_{down}| + |I_{up}| + |I_{equal}| = n$, it follows that

$$|I_{down}| = n - k. \tag{74}$$

Now substituting (73) into the previously derived condition $|I_{down}| > |I_{up}| + |I_{equal}|$, it holds that $|I_{down}| > k$. Since $|I_{down}|$ and k are integer numbers, this implies the next inequality

$$|I_{down}| \geq k + 1. \tag{75}$$

Thus, since the smaller lower bound of $|I_{down}|$ occurs for n odd, it holds

$$k \leq \frac{n-1}{2}. \tag{76}$$

In light of eq. (73) and eq. (74), the right hand side of eq. (72) can be rewritten as

$$\begin{aligned} &\alpha_{min}|I_{down}| - \alpha_{max}|I_{up}| - \alpha_{max}|I_{equal}| \\ &= \alpha_{min}(n-k) - \alpha_{max}k. \end{aligned} \tag{77}$$

Since the right hand side of eq. (77) is a decreasing function of k, its minimum subject to the constraint in eq. (76) is obtained when $k = \frac{n-1}{2}$. Thus, it holds

$$\begin{aligned} \alpha_{min}(n-k) - \alpha_{max}k \quad &\geq \alpha_{min}(n - \tfrac{n-1}{2}) - \alpha_{max}\tfrac{n-1}{2} \\ &= \alpha_{min}\tfrac{n+1}{2} - \alpha_{max}\tfrac{n-1}{2}. \end{aligned} \tag{78}$$

By rewriting the tuning inequality (19) in the equivalent form

$$\frac{\alpha_{min}}{\alpha_{max}} > \frac{n}{n+1}, \tag{79}$$

we further manipulate the right-hand side of eq. (78) as follows

$$\begin{aligned} \alpha_{min}\tfrac{n+1}{2} - \alpha_{max}\tfrac{n-1}{2} \quad &= \alpha_{max}\left(\tfrac{\alpha_{min}}{\alpha_{max}}\tfrac{n+1}{2} - \tfrac{n-1}{2}\right) \\ &> \alpha_{max}\left(\tfrac{n}{n+1}\tfrac{n+1}{2} - \tfrac{n-1}{2}\right) \\ &= \alpha_{max}\left(\tfrac{n}{2} - \tfrac{n-1}{2}\right) \\ &= \alpha_{max}\tfrac{1}{2}. \end{aligned} \tag{80}$$

By virtue of (80) one derives that

$$\left|\sum_{i\in I_{down}}\alpha_i - \sum_{i\in I_{up}}\alpha_i + \sum_{i\in I_{equal}}\alpha_i \mathrm{SIGN}(x_i(t)-z_i(t))\right| > \frac{\alpha_{max}}{2}. \tag{81}$$

Additionally, (68), (19) and (70) also imply that

$$\begin{aligned}&\mathrm{sign}\left(\sum_{i\in I_{down}}\alpha_i - \sum_{i\in I_{up}}\alpha_i + \sum_{i\in I_{equal}}\alpha_i \mathrm{SIGN}(x_i(t)-z_i(t))\right)\\ &= \mathrm{sign}\left(|I_{down}|-|I_{up}|\right).\end{aligned} \tag{82}$$

By exploiting the definition of the median value, and considering (64), (68) and (19) one derives the following implications

$$|I_{down}| < |I_{up}| \quad \Longleftrightarrow \quad c(t) < m(z), \tag{83}$$

$$|I_{down}| > |I_{up}| \quad \Longleftrightarrow \quad c(t) > m(z). \tag{84}$$

Relations (83) and (84) imply in turns that

$$\mathrm{sign}\left(|I_{down}|-|I_{up}|\right) = \mathrm{sign}\left(c(t)-m(z)\right). \tag{85}$$

Therefore, by (63), (81) and (85) it derives that

$$\max_{\xi\in\widetilde{\mathcal{L}}V_2(x(t))} \xi \leq -\frac{\alpha_{max}}{2n}. \tag{86}$$

Since $\frac{d}{dt}\left(V_2(x(t))\right) \in \widetilde{\mathcal{L}}V_2(x(t))$, according to eq. (86), unless $c(t)=m(z)$ (i.e. $V_2(x(t)))=0$) it holds

$$\frac{d}{dt}\left(V_2(x(t))\right) \leq -\frac{\alpha_{max}}{2n}. \tag{87}$$

Thus, in Case 1, the finite time achievement of condition $c(t)=m(z)$ is guaranteed by analogous developments as those made in the proof of Theorem 4.1, with a finite transient time satisfying (21).

Case 2. This case may only happen in the (unlikely) event that more than one agent hold the same value of their corresponding initial state, $x_i(0)=z_i$, and additionally this value corresponds to the median value. We now prove that if condition (69) holds then $c(t)=m(z)$, i.e, the network has already achieved consensus on the median value.

Define

$$k_{up} = \min_{k\in I_{up}} k, \qquad k_{down} = \max_{k\in I_{down}} k. \tag{88}$$

Due to (8), and owing on the definitions (58), one has that

$$k_{up} > k_{down}, \tag{89}$$

$$|I_{up}| = n - k_{up} + 1, \tag{90}$$

$$|I_{down}| = k_{down}. \tag{91}$$

By substituting (90) and (91) into the next relation

$$|I_{up}| + |I_{down}| + |I_{equal}| = n, \tag{92}$$

which is verified by construction, one obtains

$$|I_{equal}| = (k_{up} - k_{down}) - 1. \tag{93}$$

Relation (69) yields the two inequalities

$$|I_{up}| - |I_{down}| \leq |I_{equal}| \quad if\ |I_{up}| > |I_{down}| \tag{94}$$

$$|I_{down}| - |I_{up}| \leq |I_{equal}| \quad if\ |I_{up}| < |I_{down}| \tag{95}$$

By substituting (90), (91) and (93) into (94) it yields

$$k_{up} \geq \frac{n}{2} + 1. \tag{96}$$

By substituting (96) into (90) we get

$$|I_{up}| \leq \frac{n}{2} \tag{97}$$

Since we are investigating the case $|I_{up}| > |I_{down}|$, it derives from (97) and (91) that

$$k_{down} < \frac{n}{2}. \tag{98}$$

Thus, if $|I_{up}| > |I_{down}|$ then the set I_{equal} satisfies

$$\max_{k \in I_{equal}} k = k_{up} - 1 \geq \frac{n}{2}, \tag{99}$$

and

$$\min_{k \in I_{equal}} k = k_{down} + 1 < \frac{n}{2} + 1. \tag{100}$$

By applying similar manipulations to (95), one derives the same conditions (99)-(100), which, considered together, imply that $c(t) = m(z)$. Theorem 4.3 is proven. □

D. *Proof of Theorem 5.2*

Let

$$\begin{array}{l} I^c_{max}(t) = \{k \in \mathcal{V}_c : x_k = \max_{i \in \mathcal{V}_c} x_i(t)\}, \\ I^c_{min}(t) = \{k \in \mathcal{V}_c : x_k = \min_{i \in \mathcal{V}_c} x_i(t)\}. \end{array} \tag{101}$$

The proof develops along the same lines followed in the proof of Theorem 4.1 by considering the set of agents $\mathcal{V}_c \subseteq \mathcal{V}$ as opposed to set $\mathcal{V}$.

Consider the next non-smooth Lyapunov candidate function

$$V^c(x(t)) = \frac{\sum_{i \in I^c_{max}} x_i(t)}{|I^c_{max}|} - \frac{\sum_{i \in I^c_{min}} x_i(t)}{|I^c_{min}|}. \tag{102}$$

Similarly to the proof of Theorem 4.1, sets I^c_{max} and I^c_{min} may change cardinality only at isolated instants of time. Furthermore, we can use the same reasoning to get an expression of the generalized time derivative of $V^c(x(t))$ as follows

$$\frac{d}{dt}\left(V^c(x(t))\right) = \frac{\sum_{i \in I^c_{max}} \dot{x}_i(t)}{|I^c_{max}|} - \frac{\sum_{i \in I^c_{min}} \dot{x}_i(t)}{|I^c_{min}|}. \tag{103}$$

Due to the symmetry of local interactions between the cooperative agents it holds

$$\sum_{i \in I^c_{max}} \left(\sum_{j \in N_i \bigcap I^c_{max}} \lambda_{ij} \text{sign}\left(x_i(t) - x_j(t)\right) \right) = 0. \tag{104}$$

Thus, considering the corresponding set-valued map, it holds

$$\begin{array}{rl} \frac{d}{dt}\left(V^c(x(t))\right) \in & \frac{1}{|I^c_{max}|}\left(- \sum_{i \in I^c_{max}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right) \right. \\ & - \sum_{i \in I^c_{max}} \sum_{j \in \mathcal{V}_c \bigcap N_i \setminus I^c_{max}} \lambda_{ij} \text{SIGN}\left(x_i(t) - x_j(t)\right) \\ & \left. - \sum_{i \in I^c_{max}} \sum_{j \in \mathcal{V}_u \bigcap N_i \setminus I^c_{max}} \lambda_{ij} \text{SIGN}\left(x_i(t) - x_j(t)\right) \right) \\ & - \frac{1}{|I^c_{min}|}\left(- \sum_{i \in I^c_{min}} \alpha_i \text{SIGN}\left(x_i(t) - z_i\right) \right. \\ & - \sum_{i \in I^c_{min}} \sum_{j \in \mathcal{V}_c \bigcap N_i \setminus I^c_{min}} \lambda_{ij} \text{SIGN}\left(x_i(t) - x_j(t)\right) \\ & \left. - \sum_{i \in I^c_{min}} \sum_{j \in \mathcal{V}_u \bigcap N_i \setminus I^c_{min}} \lambda_{ij} \text{SIGN}\left(x_i(t) - x_j(t)\right) \right). \end{array} \tag{105}$$

By definition of set I^c_{max}, any agent j which belongs also to set $\mathcal{V}_c \bigcap N_i \setminus I^c_{max}$ is such that if the network is not at consensus then $x_i(t) > x_j(t)\ \forall i \in I^c_{max}$, which implies that $\text{SIGN}\left(x_i(t) - x_j(t)\right) = 1\ \forall i \in I^c_{max}$. On the contrary, nothing can be said about the sign of $x_i(t) - x_j(t)$ when $j \in \mathcal{V}_u \bigcap N_i \setminus I^c_{max}$. The number of uncooperative agents is δ (a value unknown a-priori) whereas Δ_{max} is the maximum degree of agents in the set V_{unsafe}. Since the network is k-safe each uncooperative agent is connected only to cooperative agents which belong to set $\mathcal{V}_{safe}$.

Furthermore, if the network is not in the consensus state, then either I^c_{max} contains only cooperative agents belonging to set set V_{unsafe} and thus only connected to other cooperative agents in the set $\mathcal{V}_{safe}$ or I^c_{max} contains at least one node in the set $\mathcal{V}_{safe}$. In the first case, nodes in I^c_{max} are not connected among themselves therefore, it can be shown that they converge toward nodes in thr set V_{safe} with speed at worst equal to $\alpha_{max} - \lambda^{unsafe}_{min}$.

In the second case, since there exist at least k edges connecting nodes in I^c_{max} with nodes having a different state value, the following estimate can be derived by (103) and (105)

$$\begin{aligned}\frac{d}{dt}\left(V^c(x(t))\right) \leq \quad & \alpha_{max} - \frac{k\lambda^{safe}_{min} - \Delta_{max}\delta\lambda^{unsafe}_{max}}{|I^c_{max}|} \\ & + \alpha_{max} - \frac{k\lambda^{safe}_{min} - \Delta_{max}\delta\lambda^{unsafe}_{max}}{|I^c_{min}|}.\end{aligned} \tag{106}$$

Additionally, if the network is not in the consensus state then $|I^c_{max}| = p < n$ and $|I^c_{min}| \leq n - \delta - p$. Thus, (106) is further elaborated as

$$\frac{d}{dt}\left(V^c(x(t))\right) \leq 2\alpha_{max} - \frac{(n-\delta)(k\lambda^{safe}_{min} - \delta\lambda^{unsafe}_{max})}{p(n-\delta-p)}. \tag{107}$$

The value of p which maximizes the right-hand side of (107) is $p = \frac{n-\delta}{2}$, therefore by exploiting relation $\delta \leq \bar{\delta}$ we further manipulate (107) as follows

$$\frac{d}{dt}\left(V^c(x(t))\right) \leq 2\alpha_{max} - \frac{4(k\lambda^{safe}_{min} - \Delta_{max}\bar{\delta}\lambda^{unsafe}_{max})}{n-\bar{\delta}}. \tag{108}$$

If the tuning conditions (25) and (26), then inequality (108) is rewritten as follows

$$\frac{d}{dt}\left(V^c(x(t))\right) \leq -\beta^2, \tag{109}$$

where $\beta^2 = 2\left(\frac{2(k\lambda^{safe}_{min} - \bar{\delta}\lambda^{unsafe}_{max})}{n-\bar{\delta}} - \alpha_{max}\right) > 0$, which proves the finite-time convergence of $V^c(x(t))$ to zero according to Theorem 2.6. To evaluate the convergence time, we follow analogous developments as those made in the end of the proof of Theorem 4.1, yielding that the finite-time consensus condition (11) is achieved within a maximal transient time T_1 satisfying (28). Theorem 5.2 is proved. □

E. Proof of Theorem 5.3

Consider the following Lyapunov function

$$V_3(x(t)) = |c(t) - \bar{c}|, \quad \bar{c} = \frac{z^c_{max} + z^c_{min}}{2}, \quad t \geq T. \tag{110}$$

By (31), and taking into account that $|\mathcal{V}_c| = n - \delta$ (where δ is the actual number of uncooperative agents) one has that

$$c(t) = \frac{\sum_{i\in\mathcal{V}_c} x_i(t)}{n-\delta}. \tag{111}$$

The generalized gradient of $V_3(x(t))$ takes the form $\partial V_3(x) = \text{SIGN}(c(t) - \bar{c})$. Function $c(t)$, the time varying consensus value, obeys the discontinuous differential equation

$$\begin{aligned}\dot{c}(t) = \quad & \frac{\sum\limits_{i\in\mathcal{V}_c} \dot{x}_i(t)}{n-\delta} = -\frac{\sum\limits_{i\in\mathcal{V}_c} \alpha_i \text{sign}\left(x_i(t) - z_i\right)}{n-\delta} \\ & -\frac{\sum\limits_{i\in\mathcal{V}_c}\sum\limits_{j\in\mathcal{V}_c\bigcap N_i} \lambda_{ij}\text{sign}\left(x_i(t) - x_j(t)\right)}{n-\delta} \\ & -\frac{\sum\limits_{i\in\mathcal{V}_c}\sum\limits_{j\in\mathcal{V}_u\bigcap N_i} \lambda_{ij}\text{sign}\left(x_i(t) - x_j(t)\right)}{n-\delta}.\end{aligned} \tag{112}$$

Due to the symmetry of local interactions between the cooperative agents, it holds

$$\sum_{i\in\mathcal{V}_c}\left(\sum_{j\in\mathcal{V}_c\bigcap N_i} \lambda_{ij}\text{sign}\left(x_i(t) - x_j(t)\right)\right) = 0,$$

therefore the second term in the right hand side of (112) is identically zero, and (112) simplifies to

$$\dot{c}(t) = -\frac{\sum_{i\in\mathcal{V}_c}\alpha_i \text{sign}\left(x_i(t)-z_i\right)}{n-\delta} - \frac{\sum_{i\in\mathcal{V}_c}\sum_{j\in\mathcal{V}_u\bigcap N_i}\lambda_{ij}\text{sign}\left(x_i(t)-x_j(t)\right)}{n-\delta}. \tag{113}$$

The Filippov solutions of (113) are governed by the differential inclusion $\dot{c}(t)\in K(x)$, with the set-valued map

$$K(x) = -\frac{1}{n-\delta}\sum_{i\in\mathcal{V}_c}\alpha_i \text{SIGN}\left(x_i(t)-z_i\right) - \frac{\sum_{i\in\mathcal{V}_c}\sum_{j\in\mathcal{V}_u\bigcap N_i}\lambda_{ij}\text{SIGN}\left(x_i(t)-x_j(t)\right)}{n-\delta}. \tag{114}$$

Let $I^c_{up}=\{k\in\mathcal{V}_c : x_k<z_k\}$, $I^c_{down}=\{k\in\mathcal{V}_c : x_k>z_k\}$, $I^c_{equal}=\{k\in\mathcal{V}_c : x_k=z_k\}$. We point out that sets $I^c_{up}, I^c_{down}, I^c_{equal}$ are here defined over the subset $\mathcal{V}_c$ of collaborative agents instead of over the full set $\mathcal{V}$ of agents as it as was done in the proof of Theorem 4.3.

We can thus decompose the Filippov map $K(x)$ in (114) as follows

$$\begin{aligned} K(x) = & -\frac{1}{n-\delta}\Bigg(\sum_{i\in I^c_{up}}\alpha_i\text{SIGN}\left(x_i(t)-z_i\right) \\ & +\sum_{i\in I^c_{down}}\alpha_i\text{SIGN}\left(x_i(t)-z_i\right) \\ & +\sum_{i\in I^c_{equal}}\alpha_i\text{SIGN}\left(x_i(t)-z_i\right) \\ & -\sum_{i\in\mathcal{V}_c}\sum_{j\in\mathcal{V}_u\bigcap N_i}\lambda_{ij}\text{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg), \end{aligned} \tag{115}$$

where, by construction

$$\text{SIGN}\left(x_i(t)-z_i\right) = -1 \qquad \forall i\in I^c_{up}, \tag{116}$$

$$\text{SIGN}\left(x_i(t)-z_i\right) = 1 \qquad \forall i\in I^c_{down}. \tag{117}$$

Substituting (115)-(117) into (114) we obtain

$$\begin{aligned} K(x) = & -\frac{1}{n-\delta}\Bigg(\sum_{i\in I^c_{down}}\alpha_i-\sum_{i\in I^c_{up}}\alpha_i \\ & +\sum_{i\in I^c_{equal}}\alpha_i\text{SIGN}(x_i(t)-z_i(t)) \\ & -\sum_{i\in\mathcal{V}_c}\sum_{j\in\mathcal{V}_u\bigcap N_i}\lambda_{ij}\text{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg). \end{aligned} \tag{118}$$

and the set valued Lie derivative of $V_3(x(t))$ correspondingly takes the form

$$\begin{aligned} \widetilde{\mathcal{L}}V_3(x(t)) = & -\frac{1}{n-\delta}\text{SIGN}(c(t)-\bar{c})\Bigg(\sum_{i\in I^c_{down}}\alpha_i-\sum_{i\in I^c_{up}}\alpha_i \\ & +\sum_{i\in I^c_{equal}}\alpha_i\text{SIGN}(x_i(t)-z_i(t)) \\ & +\sum_{i\in\mathcal{V}_c}\sum_{j\in\mathcal{V}_u\bigcap N_i}\lambda_{ij}\text{SIGN}\left(x_i(t)-x_j(t)\right)\Bigg). \end{aligned} \tag{119}$$

We now assume that $V_3(x(t))>\frac{z^c_{max}-z^c_{min}}{2}$, which happens when $c(t)>z^c_{max}$ or when $c(t)<z^c_{min}$. If $c(t)>z^c_{max}$ it holds $|I^c_{down}|=n-\delta$, whereas if $c(t)<z^c_{min}$ then $|I^c_{up}|=n-\delta$. In both cases, it holds $|I^c_{down}|-|I^c_{up}|=n-\delta$ and $|I^c_{equal}|=0$. Since, by assumption, each uncooperative agent is connected only to cooperative agents and the trajectories of the uncooperative agents cross the trajectories of the cooperative agents only at isolated instants of time, these instants of time can

thus be disregarded in the analysis. Thus, if $\frac{\alpha_{max}-\alpha_{min}}{\alpha_{min}} < \frac{1}{n}$, by manipulating eq. (119) as done in the proof of Theorem 4.3 to derive inequality (81), and considering the relation $\delta \leq \bar{\delta}$ it derives that

$$\max_{\xi \in \widetilde{\mathcal{L}} V_3(x(t))} \xi \leq -\frac{\alpha_{max}}{2} + \lambda_{max}^{unsafe} \frac{\Delta_{max}\bar{\delta}}{n-\bar{\delta}}. \tag{120}$$

Thus, taking into account (30), it follows from (120) that

$$\frac{d}{dt}\left(V_3(x(t))\right) \in \widetilde{\mathcal{L}} V_3(x) \leq -\frac{\alpha_{max}}{2} + \lambda_{max}^{unsafe} \frac{\Delta_{max}\bar{\delta}}{n-\bar{\delta}}. \tag{121}$$

which implies that the level set $V_3(x) \leq \frac{z_{max}-z_{min}}{2}$ is invariant and finite-time attracting with a transient time $T + \bar{T}$ fulfilling the estimation (34). Theorem 5.3 is proven. □

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