Finite-Time Consensus for Switching Network Topologies with Disturbances

Mauro Franceschelli[a] Alessandro Giua[a,b] Alessandro Pisano[a]
Elio Usai[a]

[a]*Dept. of Electrical and Electronic Engineering, Univ. of Cagliari, Italy*
E-mail: {mauro.franceschelli,pisano,giua,eusai}@diee.unica.it
[b]*LSIS, Aix-Marseille University, France*

Abstract

In this paper we investigate the properties of a decentralized consensus algorithm for a network of continuous-time integrators subject to unknown-but-bounded time-varying disturbances. The proposed consensus algorithm is based on a discontinuous local interaction rule. Under certain restrictions on the switching topology, it is proven that after a finite transient time the agents achieve an approximated consensus condition by attenuating the destabilizing effect of the disturbances. This main result is complemented by an additional result establishing the achievement of consensus under different requirements on the switching communication topology. In particular, we provide a convergence result that encompasses situations in which the time varying graph is always disconnected. Lyapunov analysis are carried out to support the suggested algorithms and results. Simulative tests considering, as case study, the synchronization problem for a network of clocks are illustrated and commented to validate the developed analysis.

Key words: Finite time consensus, Multi-agent systems, Discontinuous control, Disturbance rejection, Clock synchronization

1 Introduction

The problem of reaching consensus, i.e., driving the state of a set of interconnected dynamical systems towards the same value, has received much attention due to its many applications in, both, the modeling of natural phenomena such as flocking (see e.g. [1–4]) and in the solution of several control problems involving synchronization or agreement between dynamical systems (see [5–7]).

In this paper we discuss a few approaches to reach consensus in a network of interacting agents whose dynamics are modeled by first order continuous time integrators subject to unknown perturbations. The approaches are based on a discontinuous local interaction rule using sliding mode control concepts and techniques (see [8,9]). Discontinuous local interactions rules have been already exploited in the framework of consensus algorithms to exploit the underlying finite-time convergence and robustness against disturbances and unmodeled dynamics. Several examples of applications to flocking or synchronization problems can be found in the literature (see [10]). Discontinuous local interactions were studied in [11], within a general framework of gradient flows, and several examples of discontinuous consensus protocols were analyzed there.

In [12], a finite-time consensus algorithm is proposed to address the leader-follower tracking problem in multi-robot systems with static topology but varying leader. In [13], [14] and [15], finite-time consensus algorithms are provided for networks of unperturbed integrators by exploiting discontinuous local interaction rules under time varying (both undirected and directed) network topologies.

The consensus problem in presence of measurement errors is studied in [16], in a discrete-time setting, with reference to linear consensus protocols with constant or vanishing weights. The authors derive explicit upper bounds to the maximum ultimate disagreement error in dependence of the bounds on the noise magnitude and of the smallest non-zero singular value of the network's state update matrix.

In [17] the authors suggest a class of non-linear continuous protocols that achieve the so-called "ϵ-consensus", namely an approximate agreement condition where all agents converge towards a set, in spite of the presence of additive disturbances. Our work differs from [17] in that we consider a discontinuous protocol, as opposed to continuous, that is able to achieve almost complete disturbance rejection up to an arbitrarily small error if the time-varying network is always connected.

A problem that shares some technical issues with the protocol proposed in this paper is the continuous-time consensus problem in presence of quantization errors. In [18] the continuous-time consensus problem is studied in the case of

quantized information exchange between agents, and this leads to an instance of discontinuous protocol where the effect of quantization can be regarded as a disturbance.

The approaches illustrated in this paper further differ from the above mentioned literature works in that we address the analysis of the practical stability and disturbance attenuation properties of finite-time consensus under the effect of unknown perturbations and, additionally, with a switching communication topology. In the present work the finite time transient to reach consensus can be made arbitrarily small by properly selecting the algorithm parameters. The disturbance rejection performance will primarily depend on the time-varying network connectivity properties. To the best of our knowledge, the above aspects were never simultaneously addressed and characterized in the existing literature.

The main result of the present work, outlined in Theorem 3.2, consists in proposing a feasible local interaction rule which provides finite time convergence of the state of the network to a condition of approximate agreement, by attenuating the effect of the disturbances. This result is subject to the requirement that the time varying graph defining the network switching interaction topology stays connected during, at least, a certain "minimal percentage" of time.

An additional result, outlined in Theorem 3.5, demonstrates the finite time attainment of the approximate consensus condition while allowing the graph to be always disconnected and introducing a different requirement on the switching topology. The conditions involved in this additional result are, however, not easily implementable and mostly of theoretical interest. Nevertheless, it is worth to stress that it appears to be the first result stating the finite time attainment of consensus by allowing the graph to be disconnected at all times.

This paper generalizes the preliminary results presented in [19] by extending the analysis to cover switching topologies in which the communication graph can be always disconnected. Furthermore, we consider a different Lyapunov function by means of which a less conservative tuning inequality for the algorithm parameter is derived.

The proposed framework is well suited to model a network of perturbed clocks in which the proposed local interaction rule improves the synchronization accuracy attenuating the disturbance effects. In the simulation section we shall provide a thoroughly discussed case study of "robust" clock synchronization.

The structure of the paper is as follows. In Section 2 we recall some basic definitions and formulate the problem under investigation. In Section 3 we describe the proposed local interaction rule and we investigate the associated convergence properties by stating the main result of this paper. Subsection 3.1

addresses the previously mentioned additional result which broadens the admitted switching topology, as compared to the main result, at the price of needing certain global state information to implement it. In Section 4 some simulation results are presented by considering the case study of the synchronization problem in a network of clocks. Finally, in Section 5, conclusions are drawn and possible future research directions are discussed.

2 Preliminaries and Problem statement

Let us consider n agents interacting through a communication network whose topology is described by a connected undirected graph $\mathcal{G} = \{V, E\}$, where $V = \{1, \ldots, n\}$ denotes the set of agents and $E \subseteq \{V \times V\}$ the set of edges representing communication channels between them. Let the (i, j) elements of E be ordered such that $i < j$. Furthermore, assume that graph $\mathcal{G}$ does not contain self loops. The i-th agent is modeled by the first-order dynamics

$$\dot{x}_i = \alpha + \nu_i(t) + u_i(t), \quad x_i(0) = x_{i0}, \quad i \in V, \tag{1}$$

where α is a constant reference slope, $\nu_i(t)$ is a bounded disturbance, $u_i(t)$ is the adjustable control input and x_{i0} is the initial condition.

Model (1) is well suited to represent the dynamics of a network of clocks where α is a desired clock skew, the same for each clock, $\nu_i(t)$ is a bounded disturbance corrupting the clock dynamics, $u_i(t)$ is the modifiable input, allowing to adjust the speed of the i-th clock, and x_{i0} is the initial clock off-set. Signal $\nu_i(t)$ is a general representation of all the possible modeled and unmodeled uncertainty sources such as time off-set, noise, etc. The next assumption is made on the disturbance signals $\nu_i(t)$

$$\exists \Pi \in \mathbb{R}^+ : \forall i \in V, \quad |\nu_i(t)| \leq \Pi. \tag{2}$$

It is worth to stress that the presented model also covers the case in which the clocks of the network have different skews, says α_i for $i \in V$. To this end one can just assume that a constant bias $\alpha_i - \alpha$ is part of the disturbance term ν_i, which would be compatible with the above boundedness constraint (2). The present formulation of the model is preferred as it brings some simplification in the underlying convergence proof of the consensus protocol.

At each time instant, only a subset of the available communication channels in $\mathcal{G}$ is active for information exchange between agents. Let $\hat{\mathcal{G}}(t) = \{V, E(t)\}$ be a time varying graph representing the instantaneous topology of active links, where $E(t) \subseteq E$ is the subset of active edges at time t.

4

Let $\mathcal{N}_i(t) = \{j \in V : (j,i) \in E(t)\}$ denote the set of neighbors of node i at time t. Denote

$$r_{ik}(t) = \begin{cases} 1 & \text{if} \quad k \in \mathcal{N}_i(t) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Since graph $\hat{\mathcal{G}}(t)$ is undirected, it follows that

$$r_{ik}(t) = r_{ki}(t), \quad \forall i, k, \in V, \ i \neq k. \tag{4}$$

Our objective is to design a local interaction rule $u_i(t)$, compatible with the time-varying topology of graph $\hat{\mathcal{G}}(t)$, which can guarantee, under suitable assumptions on $\hat{\mathcal{G}}(t)$, the achievement of the next **practical finite-time consensus** condition

$$\exists M, t_r \in \mathbb{R}^+ : \quad \forall t > t_r, \ \forall i, j \in V, \quad |x_i(t) - x_j(t)| \leq M, \tag{5}$$

where M and t_r are positive constants.

We consider a practical, rather than exact, consensus condition because we are interested in the case in which the network topology may be disconnected for several intervals of time. For this reason, since disturbances are unknown, an exact consensus condition can not be guaranteed by any consensus protocol.

3 Main result

The proposed discontinuous local interaction rule takes the form

$$u_i(t) = -\lambda \sum_{k \in \mathcal{N}_i(t)} sign(x_i(t) - x_k(t)), \quad \lambda > 0, \tag{6}$$

where λ is a positive tuning constant of the algorithm and function $sign(z)$, $z \in R$, is defined as follows

$$sign(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \\ -1 & z < 0 \end{cases} \tag{7}$$

We adopt the following equivalent notation for the local interaction rule (6)-(7)

$$u_i(t) = -\lambda \sum_{k \in V, k \neq i} r_{ik}(t) sign(x_i(t) - x_k(t)), \quad \lambda > 0. \tag{8}$$

Remark 3.1 *Due to the concurrent effect of the suggested discontinous local interaction rule, the switching network topology, and the possibly discontinuous nature of the external disturbances (which are only supposed to be uniformly bounded), the closed loop network dynamics will be discontinuous and the resulting solution notion needs to be discussed and clarified. For a differential equation with discontinuous right-hand side, following [20], we understand the resulting solution in the so-called "Filippov sense" as the solution of an appropriate differential inclusion, the existence of which is guaranteed (owing on certain properties of the associated set-valued map) and for which noticeable properties, such as absolute continuity, are in force. The reader is referred to [21] for a comprehensive account of the notions of solution for discontinuous dynamical systems.* ∎

From now on we investigate the conditions under which the local interaction protocol (8) can achieve the approximate consensus conditions (5).

Let us define an **error variable** for each edge in the network

$$\delta_{ij}(t) = x_i(t) - x_j(t), \quad (i,j) \in E. \tag{9}$$

The dynamics of $\delta_{ij}(t)$ are easily obtained by differentiating (9) and considering the closed loop dynamics of each agents

$$\dot{x}_i(t) = \alpha + \nu_i - \lambda \sum_{k \in V, k \neq i} r_{ik}(t) sign(x_i(t) - x_k(t)), \quad i \in V. \tag{10}$$

Simple manipulations yield

$$\dot{\delta}_{ij}(t) = \nu_i - \nu_j - \lambda \sum_{k \in V, k \neq i} r_{ik}(t) sign(\delta_{ik}(t))$$
$$+ \lambda \sum_{k \in V, k \neq j} r_{jk}(t) sign(\delta_{jk}(t)). \tag{11}$$

The requirement concerning the switching communication topology is that the time varying graph $\hat{\mathcal{G}}(t)$ stays connected during, at least, a certain "minimal percentage" of time. This is formalized by the next Assumption.

Assumption 1 *There are positive constants ϵ and T, with $\epsilon \leq T$, such that during the receding horizon time interval $\mathcal{I}(t) = (t, t+T)$, $\hat{\mathcal{G}}(t)$ is connected along a subinterval $\mathcal{S}(t) \subseteq \mathcal{I}(t)$, possibly formed by the union of disjoint subintervals, whose overall length is at least equal to ϵ.*
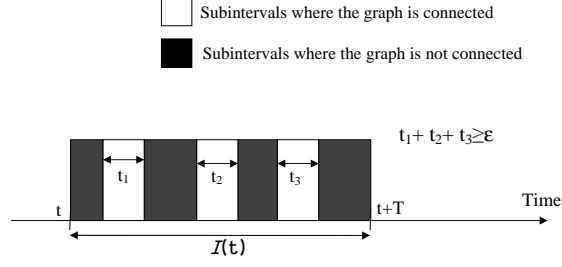
Fig. 1. Changes in network topology and communication constraints.

The meaning of Assumption 1 is clarified by the Figure 1, namely the overall duration of the disjoint white subintervals during which the instantaneous graph $\hat{\mathcal{G}}(t)$ is connected should be not less than the constant ϵ. We are now in a position to state the main result of the paper.

Theorem 3.2 *Consider the agents' dynamics (1), which satisfies (2), and let Assumption 1 be in force. Then, the discontinuous local interaction rule (8) with tuning parameter λ selected according to*

$$\lambda \geq \; \Pi\frac{T}{\epsilon} + \mu^2, \quad \mu \neq 0, \tag{12}$$

provides for the approximate consensus condition (5) with

$$M = [2(T - \varepsilon) + \xi]\Pi, \tag{13}$$

where ξ is an arbitrarily small positive real number and the finite transient time t_r is such that

$$t_r \leq \frac{T/\varepsilon}{2\mu^2} max_{i,j \in V \times V} |x_i(0) - x_j(0)|, \tag{14}$$

where μ is an arbitrary nonzero constant allowing to tune the transient time as specified by eq. (14).

    Proof:

Consider

$$V(t) = |\delta_{ij}(t)| \tag{15}$$

as a candidate Lyapunov function, where

$$(i, j) = argmax_{(i,j) \in \mathcal{V} \times \mathcal{V}} |\delta_{ij}(t)| \tag{16}$$

in such a way that, without loss of generality, index "i" will correspond to an agent carrying the maximal value at time t among all the agents in the network,

and, dually, index "j" will correspond to an agent carrying the minimal value, i.e.

$$x_{\mathrm{i}}(t) = \sup_{h \in \mathcal{V}} x_h(t), \quad x_{\mathrm{j}}(t) = \inf_{h \in \mathcal{V}} x_h(t) \qquad (17)$$

Let us preliminarily address the case in which $\epsilon < T$. It is worth to emphasize that the chosen Lyapunov function (15) is continuous at those time instants at which either i or j will change its value. Clearly, the vanishing of $V(t)$ implies the exact consensus condition among the agents of the network, while small values for $V(t)$ correspond to a practical consensus condition as in (5). Note that the considered Lyapunov function is locally Lipschitz and it is not differentiable when $\delta_{\mathrm{ij}}(t) = 0$. Thus, we refer for stability analysis to the *Lyapunov Generalized Theorem* for non-smooth analysis reported in [22], which makes use of the *Clarke's Generalized Gradient* [23]. However, we can observe that $\delta_{\mathrm{ij}}(t) = 0$ holds only when the exact consensus condition is in force, which will bring some useful simplification in the stability analysis.

In the remainder, we refer to the computation method illustrated in [22], where a Lyapunov analysis based on a similar Lyapunov function containing absolute value functions was dealt with. All the necessary technicalities justifying the correctness of adopting the chain rule to compute the time derivative of $V(t)$, which exists almost everywhere in the form of a suitable set-valued map, are not reported here, and the reader is referred to [11,22,24] where discontinuous systems and non-smooth Lyapunov tools analogous to those involved in the present analysis were discussed in more detail.

The time-derivative of $V(t)$ along the solutions of the deviation error dynamics (11) takes the following set-valued form

$$\begin{aligned} \dot{V}(t) &= \mathrm{SGN}(\delta_{\mathrm{ij}}(t)) \cdot \dot{\delta}_{\mathrm{ij}}(t) \\ &= \mathrm{SGN}(\delta_{\mathrm{ij}}) \cdot (\nu_{\mathrm{i}} - \nu_{\mathrm{j}}) - \lambda \cdot \mathrm{SGN}(\delta_{\mathrm{ij}}) \sum_{k \in \mathcal{V}, k \neq \mathrm{i}} r_{ik} \cdot \mathrm{sign}(\delta_{ik}) \\ &\quad + \lambda \cdot \mathrm{SGN}(\delta_{\mathrm{ij}}) \sum_{k \in \mathcal{V}, k \neq \mathrm{j}} r_{jk} \cdot \mathrm{sign}(\delta_{jk}) \end{aligned} \qquad (18)$$

where $\mathrm{SGN}(\delta_{\mathrm{ij}}(t))$, the generalized gradient of $V(t)$ (see [22]), is the multi-valued function

$$\mathrm{SGN}\left(\delta_{\mathrm{ij}}(t)\right) = \begin{cases} 1 & \text{if} \quad \delta_{\mathrm{ij}}(t) > 0 \\ [-1, 1] & \text{if} \quad \delta_{\mathrm{ij}}(t) = 0 \\ -1 & \text{if} \quad \delta_{\mathrm{ij}}(t) < 0 \end{cases} \qquad (19)$$

It is apparent from (18)-(19) that the Lyapunov function $V(t)$ is non smooth only when $\delta_{\mathrm{ij}}(t) = 0$ for some $i \neq j$.

Note that by definition, and considering (17), as long as $V(t) \neq 0$ we have $\mathrm{SGN}(\delta_{\mathrm{ij}}(t)) = 1$. Furthermore due to the uniform boundedness of the disturbance (2), the next estimation is in force

$$|\nu_{\mathrm{i}} - \nu_{\mathrm{j}}| \leq 2\Pi \tag{20}$$

Thus, we can manipulate (18) so as to obtain

$$\dot{V}(t) \leq 2 \cdot \Pi - \lambda \sum_{k \in \mathcal{V}, k \neq \mathrm{i}} r_{\mathrm{i}k} \cdot \mathrm{sign}(\delta_{\mathrm{i}k}) + \lambda \sum_{k \in \mathcal{V}, k \neq \mathrm{j}} r_{\mathrm{j}k} \cdot \mathrm{sign}(\delta_{\mathrm{j}k}) \tag{21}$$

Note that, in light of (17), irrespectively of the instantaneous current graph topology, all the state-dependent feedback terms in the right hand side of (21) are nonnegative, i.e.

$$-\lambda \sum_{k \in \mathcal{V}, k \neq \mathrm{i}} r_{\mathrm{i}k} \cdot \mathrm{sign}(\delta_{\mathrm{i}k}) + \lambda \sum_{k \in \mathcal{V}, k \neq \mathrm{j}} r_{\mathrm{j}k} \cdot \mathrm{sign}(\delta_{\mathrm{j}k}) \leq 0 \tag{22}$$

The receding horizon time interval $\mathcal{I}(t) = (t, t + T)$ is divided into the union of subinterval $\mathcal{S}(t)$, along which the graph is guaranteed to be connected, and the complementary interval $\mathcal{I}(t) \setminus \mathcal{S}(t)$ during which nothing claimed about the connectivity properties of the switching graph. By virtue of (21) and (22) one can conclude that

$$\dot{V}(t) \leq 2 \cdot \Pi, \quad t \in \mathcal{I}(t) \setminus \mathcal{S}(t). \tag{23}$$

It shall be noted that the pair (i, j) is not uniquely defined and there can be multiple agents carrying the maximal or minimal values x_{i} and x_{j} at time t. At those time instants when $\hat{\mathcal{G}}(t)$ is connected, however, both the following conditions holds:

A - Among all agents carrying the maximal value, there is at least one of them which admits, among its neighbors, one agent with state value strictly less than x_{i}.

B - Among all agents carrying the minimal value, there is at least one of them which admits, among its neighbors, one agent with state value strictly greater than x_{j}.

Suppose "I_t" (resp., "J_t") is the agent for which the maximum (resp., minimum) is achieved at time t. If there are many such agents, we choose one, if

any, which share an active edge with a neighbor having state value strictly less (resp., greater) than x_i (resp., x_j). These agents always exist at those time instants at which $\hat{\mathcal{G}}(t)$ is connected according to previous statements A and B. If there are still many of such agents we choose any one of those, but commit to that until a new agent holds the maximum (resp., minimum) value.

As a consequence of the previous developments, at those time instants when $\hat{\mathcal{G}}(t)$ is connected there exists two agent indexes $\bar{k}_1, \bar{k}_2$, $\bar{k}_1 \neq i$, $\bar{k}_2 \neq j$, which satisfies both the following conditions:

$$r_{i\bar{k}_1}(t) = 1\,, \delta_{i\bar{k}_1} > 0, \tag{24}$$
$$r_{j\bar{k}_2}(t) = 1\,, \delta_{j\bar{k}_2} < 0. \tag{25}$$

When (24) and (25) are both in force, it follows that the right hand side of (21) can be upper-estimated as follows

$$\dot{V}(t) \leq 2 \cdot \Pi - 2\lambda, \quad t \in \mathcal{S}(t), \tag{26}$$

whenever $V(t) \neq 0$. By construction the next relation holds:

$$V(t+T) - V(t) = \int_{\mathcal{S}(t)} \dot{V}(\tau)\,d\tau \;+\; \int_{\mathcal{I}(t)\backslash\mathcal{S}(t)} \dot{V}(\tau)\,d\tau. \tag{27}$$

By noticing that the length of the subinterval $\mathcal{S}(t)$ is at least ε, according to the Assumption 1, it follows that the length of the interval $\mathcal{I}(t) \backslash \mathcal{S}(t)$ will not exceed the value of $T - \varepsilon$. Thus, in light of (23) and (26), one can manipulate (27) as

$$V(t+T) - V(t) \leq \epsilon\,(2\Pi - 2\lambda) + (T - \varepsilon)\,2\Pi =$$
$$= -2\varepsilon\lambda + 2T\Pi \tag{28}$$

By plugging (12) into (28) one obtains the next condition

$$V(t+T) - V(t) \leq -\mu^2 \varepsilon, \tag{29}$$

which will be satisfied as long as $V(\tau) \neq 0 \,\forall\, \tau \in (t, t+T)$, thereby guaranteeing the existence of a finite t_r such that $V(t_r) = 0$. In order to evaluate an upper bound to the transient time t_r, denote $V_\kappa = V(\kappa T)$, and express (29) in the form of the difference equation

$$V_{\kappa+1} = V_\kappa - \mu^2 \epsilon \tag{30}$$

which admits the solution

$$V_\kappa = V(0) - \kappa\mu^2 \epsilon \tag{31}$$

10

From (31) it can be readily concluded that

$$t_r \leq \left(\frac{T}{\epsilon \mu^2}\right) V(0) = \left(\frac{T}{\epsilon \mu^2}\right) \max_{i,j \in \mathcal{V} \times \mathcal{V}} |x_i(0) - x_j(0)| \tag{32}$$

which is according to (14). We now prove that, at all $t \geq t_r$, the Lyapunov function $V(t)$ undergoes bounded fluctuations preserving the consensus accuracy established by (5) and (13). Define

$$V_S = \sup_{t \geq t_r} V(t) \tag{33}$$

which sets the ultimate accuracy of the approximate consensus condition. If, at any time t' one has that $V(t') = 0$ then along the time interval $t \in (t', t'+T)$ the Lyapunov function $V(t)$ may deviate form zero, at most, by a quantity $2(T-\varepsilon)\Pi$, which is obtained by integrating (23) for a time $T-\varepsilon$ (the maximal consecutive time interval in which the graph is disconnected according to the Assumption 1) starting from the zero initial condition. Thereby, the domain

$$V(t) \leq 2(T - \varepsilon)\Pi. \tag{34}$$

is positively invariant at any $t \geq t_r$.

Now let us address the case in which $\varepsilon = T$, i.e. the time varying graph is connected at all times. The previous analysis has shown that there exists a finite time t_r, satisfying (14), at which exact consensus is achieved, i.e. $V(t_r) = 0$. Unfortunately, $V(t) = 0$ cannot be an equilibrium state at $t \geq t_r$ due to the fact that all the local control laws $u_i(t)$ are identically zero when $V(t) = 0$ (as a consequence of all δ_{ij}'s in (9) being zero and in view of the adopted definition (7) of the sign function) while the disturbances $\nu_i(t)$ are not. On the other hand, an infinitesimal deviation of $V(t)$ from zero will restore the convergence features of the algorithm, steering immediately $V(t)$ back to zero. This phenomenon, local instability of the ideal consensus condition $V(t) = 0$ when the disturbances are acting, can be characterized by an infinitesimal increase of Γ as follows:

$$\Gamma \leq [2(T - \varepsilon) + \xi]\Pi \tag{35}$$

where ξ is an arbitrarily small positive real number. Theorem 3.2 is proven.

$$\square$$

It is worth to remark that the tuning of the gain λ does not require the perfect knowledge of the time varying network topology, and it is carried out on the basis of an upper bound to the noise magnitude and an upper bound to the ratio T/ε that sets the relative amount of time during which the network is connected.
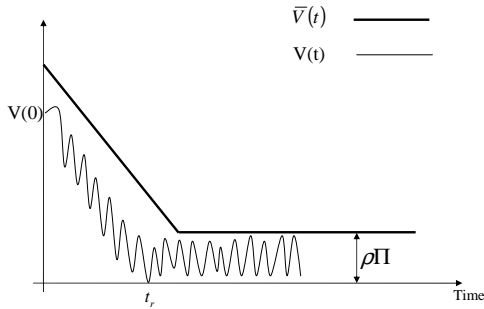
Fig. 2. Actual and majorant curves of V(t)

The relative amount of time in which the network is connected ε/T has to be estimated experimentally in practical applications. Then a "safety margin" could be taken into account for the control gain so that small deviations of the estimated parameter do not affect network performance. Since we do not model the network switching process we can not estimate this parameter before hand. If the network topology switches according to a stochastic or deterministic model then one can estimate an upperbound to this parameter and consider this value to set the control gain of the proposed consensus protocol.

Remark 3.3 *Note that the transient time, which satisfies (32), can be made arbitrarily small by taking the free design parameter μ large enough. It can be defined a μ-dependent majorant curve, illustrated in Figure 2, such that*

$$V(t) \le \bar{V}(t) = \max\left\{V(0) - \mu^2 \varepsilon \frac{t}{T} + \Gamma, \Gamma\right\}, \qquad (36)$$

Remark 3.4 *It is worth to stress that possible trajectories of the corresponding feedback system may not be unique, as it commonly happens with discontinuous dynamical systems. However, as our conclusions rely upon (non-smooth) Lyapunov-based arguments, all possible solutions of the corresponding Filippov differential inclusion will satisfy the demonstrated property of finite time convergence to the approximate-consensus condition.*

3.1 A further result

Within the present section, the achievement of the approximate consensus condition (5) is guaranteed under restrictions on the time varying connectivity graph that differ from those given in Theorem 3.2. Rather than assuming that the graph is connected for a guaranteed amount of time, we instead assume, qualitatively speaking, that arcs connecting the agent having the maximal (or minimal) value with a non-synchronized agent are active "sufficiently often". We are not aware of other results ensuring the finite time attainment of consensus by allowing the graph to be disconnected at all times and in the presence of disturbances as well.

12

This result is formalized by means of the next Theorem:

Theorem 3.5 *Consider the network of agents (1), which satisfies (2), along with the discontinuous local interaction rule (8).*

Let $T > 0$ be an arbitrary constant. Consider an infinite number of contiguous disjoint time intervals $\mathcal{I}(t_i) = [t_i, t_i + T)$ $(i = 1, 2, \ldots)$, and let $\varepsilon_i \in [0, T]$ be the union of subsets of $\mathcal{I}(t_i)$ during which at least one edge joining two non-synchronized agents is active where one of the agents holds a value either equal to $\sup_{h \in \mathcal{V}} x_h(t)$, or $\inf_{h \in \mathcal{V}} x_h(t)$.

If, for all $\mathcal{I}(t_i)$, there exist $k^* < \infty$ and $\beta > 0$ such that the next relation holds

$$\sum_{j=1}^{k^*} \varepsilon_j \geq \beta. \tag{37}$$

then, provided the algorithm parameter λ satisfies the inequality

$$\lambda \geq \frac{2k^* \, \Pi \, T}{\beta} + \mu^2, \quad \mu \neq 0, \tag{38}$$

the collective dynamics (10) reaches the approximate consensus condition (5) in finite time.

Proof: See A. □

In Theorem 3.5 we have proven finite-time convergence despite the possibility of an always-disconnected time varying graph. This result is, however, not easy to implement in practice as it might require possibly infinite-frequency topology switchings, and, for this reason, Theorem 3.3 qualifies as mainly of theoretical interest. Nevertheless it appears to relax the theoretical requirements for finite-time convergence in consensus algorithms with respect to the current state of the art.

4   Numerical simulations

A network of 20 clocks is considered, governed by model (1), with $\alpha = 1$ and with the chosen disturbance model of the form

$$\nu_i(t) = n_{ir}(t) + \beta_i + k_i sin(20t + \phi_i) \quad i \in \{1, 2, \ldots, 20\} \tag{39}$$

where $n_{ir}(t)$ is a uniformly distributed bounded random noise signal, and parameters β_i, ϕ_i and k_i were randomly chosen in such a way to enforce the relation $|\nu_i(t)| \leq \Pi = 3 \; \forall i \in V$. Initial conditions were also chosen at random

in the range $[0, 10]$. The instantaneous communication topology is set by a randomly chosen time-varying graph $\mathcal{G}(t)$. We assume that the switching policy is such that the graph stays connected for at least 30% of time, i.e. $\varepsilon \geq 0.3T$, which means that $T/\varepsilon \leq 10/3$. Furthermore, the free parameter μ^2 in (12) is set to the unit value. Under these assumptions, the gain parameter λ of the discontinuous local interaction rule (4)-(8) can be set in accordance with

$$\lambda \geq 3\frac{10}{3} + 1 = 11 \tag{40}$$

The value $\lambda = 11$ will be used in all tests presented here. Parameter T is set as $T = 2 \cdot 10^{-2}s$, and the sampling time of the fixed-step Euler solver used in the simulations is set as $T_s = 2 \cdot 10^{-4}s$. The time varying graph is updated every $10 \cdot T_s$ seconds. After each updating of the graph, during the subintervals where it must be connected (according to the T/ε requirement) the graph is checked for connectivity and if connectivity is not in force a subset of edges forming a spanning tree is added to ensure it.

The outline of the performed simulations is as follows. In the first simulation (Test 1) the graph $\hat{\mathcal{G}}(t)$ is randomly chosen so that at each time instant it is connected, i.e., $\varepsilon = T$. In the second simulation (Test 2), the switching policy is modified in such a way that $\varepsilon = 0.3T$. In Test 3, the same switching policy of Test 1 is used, and the disturbance signals are removed to show the effects of different sampling times on the steady state. In Test 4, a simulation with the same parameters as Test 1 is performed but with a different edge selection process to corroborate the analysis presented in Section 3.1. In particular in Test 4 only one edge of graph $\mathcal{G}(t)$ is active at any time but it is ensured that such edge is always incident on non-synchronized clocks one of which is, either, the agent having the maximal or minimal value between all agents. Finally, in Test 5 the performance of the algorithm is case of a time varying directed graph in which there is always a directed spanning tree. It is worth to stress that the case of directed graphs is not covered by the present analysis, and Test 5 is carried out just to inspect the performance of the algorithm in such a condition.

In Figure 3-left the time evolution of the clock variables relative to Test 1 are shown. It clearly emerges that after a finite time transient the clocks will be almost exactly synchronized, in accordance with (5) and (13). Figure 3-right shows the corresponding time evolution of the Lyapunov function $V(t)$.

In Figure 4-left, the time evolution of the clock variables relative to Test 2 is shown, with the same network parameters as those used in Test 1 except the value $\epsilon = 0.3T$. Figure 4-right depicts the time evolution of the Lyapunov function $V(t)$.

In Figure 5-left, the clock variables time evolutions relative to Test 3, having
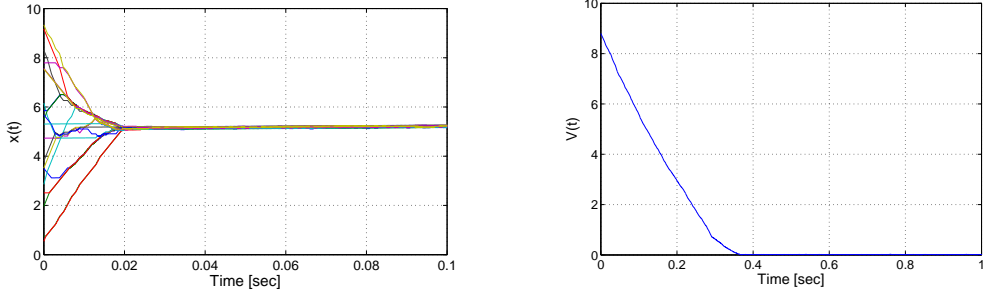
14

Fig. 3. Test 1. Time evolution of the clock variables (left) and of the Lyapunov function $V(t)$ (right).
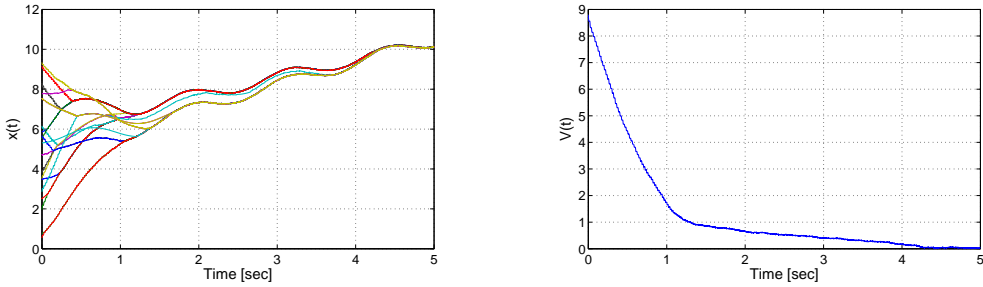

Fig. 4. Test 2. Time evolution of the clock variables (left) and steady state accuracy of the Lyapunov function $V(t)$ (right).

removed any additive disturbance, are displayed. Due to the discretized approximation of the continuous-time evolution, made by the Euler method, we may still observe some small residual errors in the steady state even in presence of theoretically exact synchronization. These errors, however, tend to vanish when the sampling time is progressively reduced as shown in Figure 5-right which displays the steady state evolutions of the Lyapunov function relative to Test 3 using two different sampling times T_s.

Inspection of the results of Test 2 (cfr. Figure 4) shows that the clock variables feature oscillations. This is entirely due to the chosen sinusoidal form for the disturbances, which affect the overall synchronized motion by enforcing oscillations on it. As a matter of fact, Figure 5, which refers to Test 3 in which the disturbances were removed, shows no oscillations and a strictly increasing behavior of the clock variables.

The results of Test 4, displayed in the Figures 6 demonstrate robust convergence also in the case in which only one edge at a time is active and thus the graph $\hat{\mathcal{G}}(t)$ is always disconnected.

Finally, we performed Test 5 with a time-varying directed graph in which there always exist a directed spanning tree. The simulation results (see Figure 7-left) show that consensus is achieved also in this condition even if the presented theoretical developments do not ensure it.
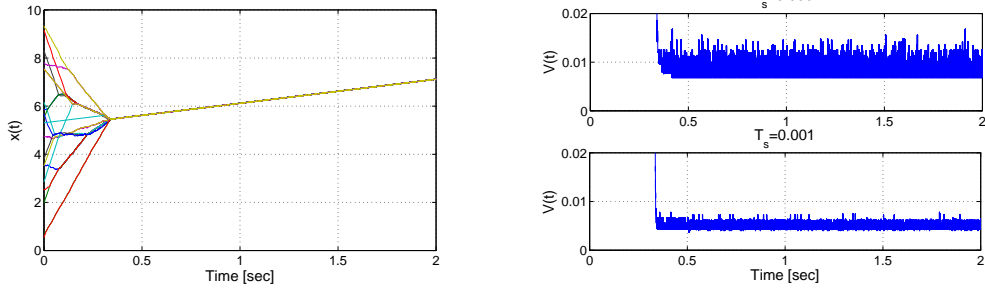
15

Fig. 5. Test 3. Time evolution of the clock variables (left) and steady state accuracy of the Lyapunov function $V(t)$ with different sampling times (right).
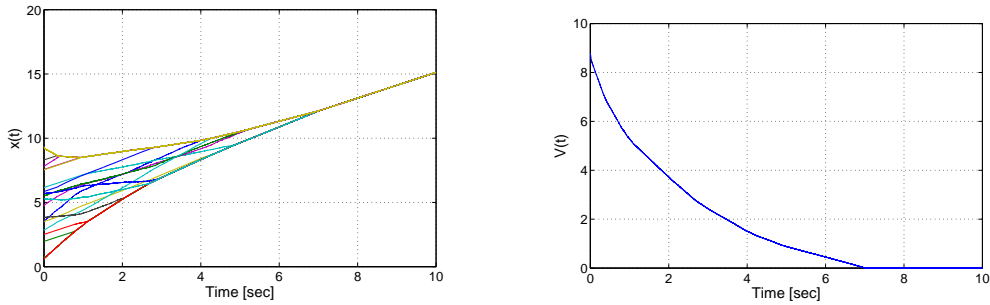

Fig. 6. Test 4. Time evolution of the clock variables (left) and of the Lyapunov function $V(t)$ (right).
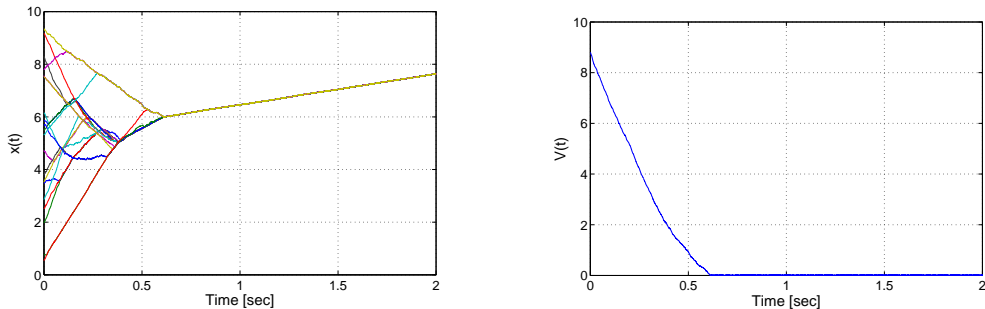

Fig. 7. Test 5. Time evolution of the clock variables (left) and of the Lyapunov function $V(t)$ (right).

5 Conclusions and future work

In this paper we have studied a distributed algorithm, based on a discontinuous local interaction rule, solving the finite-time consensus problem in a network of continuous time perturbed integrators with additive disturbance signals. It has been proven that the proposed local interaction rule is robust against bounded disturbance signals and the system converges in finite time to an approximate consensus state in which each system evolves at the same speed with a bounded error that can be made arbitrarily small. Additionally, we have proven finite-time convergence under different restrictions on the net-

work topology encompassing the possibility of an always-disconnected time varying graph. Numerical simulations have been provided to corroborate the analytical results by considering the case study of a synchronization problem in a network of clocks. Among the most interesting directions for next research, more complex agent's dynamics are under investigation as well as the extension to directed graphs, which could be possibly treated by considering a switching directed spanning tree and elaborating a different proof. Convergence under discrete time implementation of the proposed interaction rule, whose proof demands different and more involved Lyapunov analysis, is under study as well.

References

[1] C. Reynolds, Flocks, herds and schools: A distributed behavioral model, in: ACM SIGGRAPH Computer Graphics, Vol. 21, 1987, pp. 25–34.

[2] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, Physical Review Letters 75 (6) (1995) 1226–1229.

[3] A. Jadbabaie, J. Lin, A. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Transactions on Automatic Control 48 (6) (2003) 988–1001.

[4] J. Toner, Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking, Physical Review E 58 (4) (1998) 4828.

[5] R. Olfati-Saber, J. Fax, R. Murray, Consensus and cooperation in networked multi-agent systems, Proceedings of the IEEE 95 (1) (2007) 215–233.

[6] W. Ren, R. Beard, Consensus seeking in multiagent systems under dynamically changing interaction topologies, IEEE Transactions on Automatic Control 50 (5) (2005) 655–661.

[7] F. Dorfler, F. Bullo, Synchronization and transient stability in power networks and non-uniform kuramoto oscillators, in: IEEE American Control Conference, 2010, pp. 930–937.

[8] V. Utkin, Sliding modes in control and optimization, Springer-Verlag Berlin, 1992.

[9] A. Pisano, E. Usai, Sliding mode control: a survey with applications in math, Mathematics and Computers in Simulation, 81 (2011) 954979.

[10] V. Gazi, B. Fidan, Y. Hanay, M. Köksal, Aggregation, foraging, and formation control of swarms with non-holonomic agents using potential functions and sliding mode techniques, Turk J Elec Engin 15 (2) (2007) 149–168.

[11] J. Cortés, Finite-time convergent gradient flows with applications to network consensus, Automatica 42 (11) (2006) 1993–2000.

[12] S. Khoo, L. Xie, Z. Man, Robust finite-time consensus tracking algorithm for multirobot systems, IEEE/ASME Transactions on Mechatronics, 14 (2) (2009) 219–228.

[13] L. Wang, F. Xiao, Finite-time consensus problems for networks of dynamic agents, IEEE Transactions on Automatic Control 55 (4) (2010) 950–955.

[14] P. Menon, C. Edwards, A discontinuous protocol design for finite-time average consensus, in: IEEE Conference on Control Applications, 2010, pp. 2029–2034.

[15] S. Rao, D. Ghose, Sliding mode control-based algorithms for consensus in connected swarms, International Journal of Control 84 (9) (2011) 1477–1490.

[16] A. Garulli, A. Giannitrapani, Analysis of consensus protocols with bounded measurement errors, Systems & Control Letters 60 (1) (2011) 44–52.

[17] D. Bauso, L. Giarré, R. Pesenti, Consensus for networks with unknown but bounded disturbances, Siam Journal on Control and Optimization 48 (2009) 1756–1770.

[18] P. Frasca, Continuous-time quantized consensus: Convergence of krasovskii solutions, Systems & Control Letters 61 (2) (2012) 273 – 278.

[19] M. Franceschelli, A. Pisano, A. Giua, E. Usai, Finite-time consensus based clock synchronization by discontinuous control, in: 4th IFAC Conference on Analysis and Design of Hybrid Systems, June 6-8, Eindhoven, Netherlands, 2012.

[20] A. Filippov, Differential Equations with Discontinuous Righthand Sides,, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[21] J. Cortes, Discontinuous dynamical systems, IEEE Control Systems Magazine, 28 (3) (2008) 36–73.

[22] B. Paden, S. Sastry, A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators, IEEE Transactions on Circuits and Systems 34 (1) (1987) 73–82.

[23] F. Clarke, Optimization and Nonsmooth Analysis,, Wiley & Sons, New York, 1983.

[24] D. Shevitz, B. Paden, Lyapunov stability theory of nonsmooth systems, IEEE Transactions on Automatic Control 39 (9) (1994) 1910–1914.

A   Proof of Theorem 3.5

The proof of this theorem makes use of the same Lyapunov function (15) used in proof of Theorem 3.2 and shares the same treatment until eq. (26), which holds for $V(t) \neq 0$, and is recalled as follows

$$\dot{V}(t) \le 2 \cdot \Pi - \lambda \quad t \in \mathcal{S}(t). \tag{A.1}$$

The variation of $V(t)$ across the time interval $\mathcal{I}(t_i)$ can be split into two integrals. Denote by $\mathcal{S}(t_i)$ the subdomain of $\mathcal{I}(t_i)$ during which at least one edge joining non-synchronized agents, one of which holds the maximum or minimum value is active. Then, one has that

$$V(t_i + T) - V(t_i) = \int_{\mathcal{S}(t_i)} \dot{V}(\tau)d\tau + \int_{\mathcal{I}(t_i) \backslash \mathcal{S}(t_i)} \dot{V}(\tau)d\tau. \tag{A.2}$$

The two integral terms in (A.2) can be estimated as

$$\int_{\mathcal{I}(t_i) \backslash \mathcal{S}(t_i)} \dot{V}(\tau)d\tau \le 2(T - \varepsilon_i)\Pi, \tag{A.3}$$

and

$$\int_{\mathcal{S}(t_i)} \dot{V}(\tau)d\tau \le \varepsilon_i 2\Pi - \lambda \varepsilon_i. \tag{A.4}$$

Combining (A.3) and (A.4) one obtains

$$V(t_i + T) - V(t_i) \le - (\lambda \varepsilon_i - 2T\Pi). \tag{A.5}$$

It follows from (A.5) that for any $k > 0$ the next inequality holds true

$$V(t_{k+j}) \le V(t_j) - \sum_{i=j}^{j+k} \lambda \varepsilon_i + 2kT\Pi, \tag{A.6}$$

By virtue of eq. (37), once evaluated for $k = k^*$ relation (A.6) yields

$$V(t_{k^*+j}) - V(t_j) \le - (\lambda \beta - k^* 2T\Pi). \tag{A.7}$$

Thus, if λ satisfies (38) the next conditions holds

$$V(t_{k+j}) - V(t_j) \le -\mu^2 \beta, \tag{A.8}$$

which is analogous to eq. (29) thereby proving the achievement of the approximate consensus condition (5) by following similar steps as those made in the proof of Theorem 3.2.