Optimal model predictive control of Timed Continuous Petri nets

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Abstract

This paper addresses the optimal control problem of timed continuous Petri nets under infinite servers semantics. In particular, our goal is to find a control input optimizing a certain cost function that permits the evolution from an initial marking (state) to a desired steady-state. The solution we propose is based on a particular discrete-time representation of the controlled continuous Petri net system, as a certain linear constrained system. An upper bound on the sample period is given in order to preserve important information of the timed continuous net, in particular the positiveness of the markings. The reachability space of the sampled system in relation to autonomous continuous Petri nets is also studied. Based on the resulting linear constrained model, the optimal control problem is studied through Model Predictive Control (MPC). Implicit and explicit procedures are presented together with a comparison between the two schemes. Stability of the closed-loop system is also studied.

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I. INTRODUCTION

Petri Nets (PN) are discrete event models in which the distributed state is a vector of non-negative integers. As any other model of concurrent systems, discrete PN may suffer from the so-called *state explosion* problem. One way to tackle this difficulty consists in the relaxation of the original integrality constraints, giving a *fluid* (i.e., continuous) approximation of the discrete event dynamics [1], [2]. The *continuous Petri net* (contPN) model we consider in this paper, has mainly been used in the manufacturing domain, see e.g. [3], even if some other interesting applications have been presented dealing with biological systems, transportation systems [4] or supply chains [5].

In this paper we consider *timed* contPN systems under *infinite server semantics* and subject to *external control actions*: we assume that the only admissible control law consists in slowing down the firing speed of transitions [2]. We first observe that such a system can be represented by a particular *hybrid* model: a *piecewise linear* model with autonomous switches and with constraints on the state and control input space [6]. Then, we prove that by a suitable change of variables, it is also possible to further simplify the model into a *linear one* with inequality constraints on the state and input space.

A steady state for such a system represents a stable operation point where the system can work indefinitely: the existence and choice of an optimal steady state has been addressed in [6]. Here we assume such a steady state is given and our goal is to reach it from a given initial marking, while optimizing a quadratic performance index.

The solution we propose is based on a *discrete-time* version of the above constrained linear model, thus we need to be sure that the discretization does not produce *spurious* markings, in particular negative markings. To this aim an upper bound on the sampling period is given. Moreover, for the sampled timed contPN, some "equivalence results" regarding the reachability space of sampled timed contPN and (autonomous) contPN are also presented. The results obtained here, together with the ones in [6], ensure the equivalence conditions for the reachability spaces of sampled and continuous time system.

Starting from the discrete-time linear model of the contPN we propose an optimal control strategy based on *Model Predictive Control* (MPC) [7]. In particular, we investigate the possibility of using both an *implicit* and an *explicit* [8] MPC control strategy.

We also discuss some properties of the system controlled via MPC, such as feasibility and asymptotic stability. We prove that for contPN systems feasibility is always guaranteed, while asymptotic stability is not ensured. Different approaches are investigated in order to guarantee this property. One of them consists in the introduction of an appropriate terminal constraint, and in such a case asymptotic stability can be guaranteed under appropriate assumptions on the initial state and on the moving horizon.

II. CONTINUOUS PETRI NETS

A. Untimed Continuous Petri nets

***Definition** 2.1:* A contPN system is a pair $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$, where $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is the net structure (with set of places P, set of transitions T, pre and post incidence matrices $\boldsymbol{Pre}, \boldsymbol{Post}$: $P \times T \to \mathbb{N}$), and $\boldsymbol{m_0} : P \to \mathbb{R}_{\geq 0}$ is the initial marking (or distributed state).

The token load contained in place p_i at marking $\boldsymbol{m}$ is denoted m_i, and *preset* and *postset* of a node $X \in P \cup T$ are denoted ${}^\bullet X$ and $X^\bullet$, respectively.

A transition $t_j \in T$ is *enabled* at $\boldsymbol{m}$ iff $\forall p_i \in {}^\bullet t_j, m_i > 0$, and its *enabling degree* is $enab(t_j, \boldsymbol{m}) = \min\limits_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i}{Pre(p_i,t_j)} \right\}$. An enabled transition t can fire in any real amount $0 \leq \alpha \leq enab(t, \boldsymbol{m})$ leading to a new marking $\boldsymbol{m'} = \boldsymbol{m} + \alpha \boldsymbol{C}(\cdot, t)$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *incidence matrix* (or the *token flow matrix*); this firing is denoted as $\boldsymbol{m}[t(\alpha)\rangle \boldsymbol{m'}$ or $\boldsymbol{m} \xrightarrow{t(\alpha)} \boldsymbol{m'}$.

If $\boldsymbol{m}$ is reachable from $\boldsymbol{m_0}$ through a sequence $\sigma = t_{r_1}(\alpha_1) t_{r_2}(\alpha_2) \ldots t_{r_k}(\alpha_k)$, then we can write: $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} : T \to \mathbb{R}_{\geq 0}$ is the *firing count vector* and expresses the cumulative amount of firing per transition. This is called the *fundamental equation*.

The basic difference between classical discrete and continuous PN is that now the components of the markings and firing count vectors are not restricted to take value in the set of natural numbers but may take any non-negative real value.

***Definition** 2.2:* Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a contPN system. A marking $\boldsymbol{m} \in \mathbb{R}^{|P|}_{\geq 0}$ is reachable if a finite sequence $\sigma = t_{a_1}(\alpha_1) \cdots t_{a_k}(\alpha_k)$ exists, and $\boldsymbol{m_0} \overset{t_{a_1}(\alpha_1)}{\longrightarrow} \boldsymbol{m_1} \overset{t_{a_2}(\alpha_2)}{\longrightarrow} \boldsymbol{m_2} \cdots \overset{t_{a_k}(\alpha_k)}{\longrightarrow} \boldsymbol{m_k} = \boldsymbol{m}$, where $t_{a_i} \in T$ and $\alpha_i \in \mathbb{R}^+$. $RS^{ut}(\mathcal{N}, \boldsymbol{m_0})$ is the *set of reachable markings*.

A relaxation of this space can be considered allowing an infinite firing sequence [9].

***Definition** 2.3:* Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a contPN system. Then $\boldsymbol{m}$ is *lim-reachable* if a sequence of reachable markings $\{\boldsymbol{m_i}\}_{i \geq 1}$ exists such that $\boldsymbol{m_0} \xrightarrow{\sigma_1} \boldsymbol{m_1} \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_i} \boldsymbol{m_i} \cdots$ and $\lim\limits_{i \to \infty} \boldsymbol{m_i} = \boldsymbol{m}$.

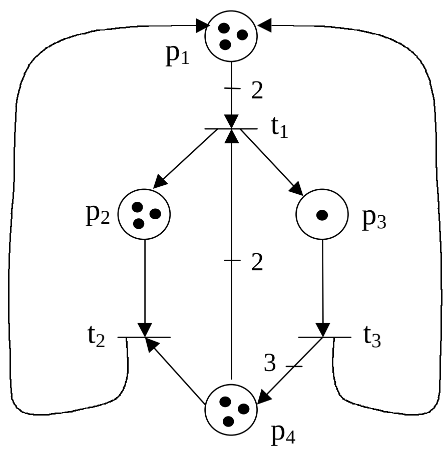

Fig. 1. ContPN system.

B. (Unforced) Timed Continuous Petri nets

***Definition** 2.4:* A (deterministically) timed contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ is a contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ together with a vector $\boldsymbol{\lambda} : T \to \mathbb{R}_{>0}$, where λ_j is the firing rate of transition t_j.

Now, the fundamental equation depends on time: $\boldsymbol{m}(\tau) = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$, where $\boldsymbol{\sigma}(\tau)$ denotes the firing count vector in the interval $[0, \tau]$. Differentiating it with respect to time we obtain: $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. The derivative of firing vector represents the *flow* of the timed model $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$. Depending on how this flow is defined many firing semantics are possible. This paper deals with *infinite server semantics* in which the flow of transition t_j is given by:

$$f_j = \lambda_j \min_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i}{Pre(p_i, t_j)} \right\} \tag{1}$$

Because the flow of a transition depends on its enabling degree, which is based on the minimum function, a timed contPN under infinite servers semantics is a piecewise linear system.

For example, in the system sketched in Fig. 1 the flow of t_1 can be restricted by the marking of p_1 or p_4 and the flow of t_2 can be restricted by the marking of p_2 or p_4. Thus, the number of embedded linear systems in this case is 4.

C. Controlled Timed Continuous Petri nets

We now consider net systems subject to external control actions, and assume that the only admissible control law consists in *slowing down* the firing speed of transitions [2].

***Definition** 2.5:* The flow of the *controlled timed contPN* (ct-contPN) is denoted as $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$, with $0 \leq \boldsymbol{u}(\tau) \leq \boldsymbol{f}(\tau)$, where $\boldsymbol{u}(\tau)$ represents the control input.

Therefore, the control input will be dynamically upper bounded by the flow of the corresponding unforced system. Under these conditions, the overall behavior of the system is ruled by the following system [6]:

$$\begin{cases} \dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot (\boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)) \\ 0 \leq \boldsymbol{u}(\tau) \leq \boldsymbol{f}(\tau) \end{cases} \tag{2}$$

This is a particular hybrid system: piecewise linear with autonomous switches and dynamic (or state-based) constraints in the input.

In this paper we assume that all transitions are *controllable*, i.e., can be slowed down by an external controlling agent. It may also be possible to extend the approach to deal with uncontrollability of certain transitions. If transition t_j cannot be controlled, then it is obvious that the control input must be $u_j = 0$ at every time instant.

III. Constrained linear representations of controlled systems

A. *A constrained linear representation of continuous Petri nets*

The system in (2) is a *piecewise linear* system with a dynamical constraint on the control input $\boldsymbol{u}$ that depends on the current value of the system state $\boldsymbol{m}$ [6]. For our control purposes, in this section we provide an alternative expression that takes the form of a *linear* system with dynamical inequalities constraints on the control input.

***Proposition** 3.1:* [10] Any piecewise linear constrained model of the form (1)–(2) can be rewritten, as a linear constrained model of the form:

$$\begin{cases} \dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \boldsymbol{w}(\tau) \\ \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}(\tau) \\ \boldsymbol{m}(\tau) \end{bmatrix} \leq \boldsymbol{0} \\ \boldsymbol{w}(\tau) \geq \boldsymbol{0} \end{cases} \tag{3}$$

with $\boldsymbol{G} = [\boldsymbol{\Delta} \ \ -\boldsymbol{\Gamma}]$, $\boldsymbol{\Delta}$ $(q \times |T|)$ and $\boldsymbol{\Gamma}$ $(q \times |P|)$, $q = \sum_{t \in T} |^{\bullet}t|$, (that is, they have as many rows as there are "pre" arcs in the net), and for any pre arc (p_i, t_j) the corresponding rows of $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ are respectively the vectors

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0 \ 1}_{j} & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\left[\underbrace{0 \quad \cdots \quad 0 \quad \frac{\lambda_j}{Pre(p_i, t_j)}}_{i} \quad 0 \quad \cdots \quad 0\right].$$

The initial value of the state system is $\boldsymbol{m}(0) = \boldsymbol{m_0} \geq \boldsymbol{0}$.

Proof: The equivalence of the dynamic equations is immediate if $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$. Concerning the constraints on the input, $0 \leq \boldsymbol{u} \leq \boldsymbol{f}$ can be rewritten as $0 \leq \boldsymbol{w} \leq \boldsymbol{f}$. Replacing $\boldsymbol{f}$ according to (1), $\forall j = 1, \cdots, |T|$ $\boldsymbol{0} \leq w_j \leq \lambda_j \min_{p_i \in {}^\bullet t_j} \left(\frac{m_i}{Pre(p_i, t_j)}\right)$; but that is equivalent to $\boldsymbol{0} \leq w_j \leq \lambda_j \frac{m_i}{Pre(p_i, t_j)} (\forall p_i \in {}^\bullet t_j)$. All these equations can be combined as $\boldsymbol{0} \leq \boldsymbol{\Delta} \cdot \boldsymbol{w} \leq \boldsymbol{\Gamma} \cdot \boldsymbol{m}$. ■

B. On sampled (or discrete-time) continuous Petri nets models

Let us obtain a discrete-time representation of continuous-time contPN under infinite servers semantics. Sampling should preserve the important information of the original model (for example the positiveness of the markings). This is directly studied in the next section through the equivalence of the reachability graph of the discrete-time model and the untimed model. In [6] the reachability space equivalence between continuous-time model and untimed model was studied and the equivalence was proved under the same conditions as in this case. Hence, the results in [6], together with those presented in this paper, provide as immediate conclusion that the reachability space of continuous-time and discrete-time are the same. In this section the discretization is defined together with a bound for the sampling period.

***Definition** 3.2:* Consider a ct-contPN as in eq. (3) and let Θ be a sampling period ($\tau = k \cdot \Theta$). The *discrete-time controlled contPN* or *dt-contPN* $\langle \mathcal{N}, \lambda, \boldsymbol{m_0}, \theta \rangle$ can be written as follows:

$$\begin{cases} \boldsymbol{m}(k+1) = \boldsymbol{m}(k) + \Theta \cdot \boldsymbol{C} \cdot \boldsymbol{w}(k) \\ \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}(k) \\ \boldsymbol{m}(k) \end{bmatrix} \leq \boldsymbol{0} \\ \boldsymbol{w}(k) \geq \boldsymbol{0} \end{cases} \tag{4}$$

The initial value of the state of this system is $\boldsymbol{m}(0) = \boldsymbol{m_0} \geq \boldsymbol{0}$.

The reachability space of dt-contPN can be defined as follows.

***Definition** 3.3:* We denote $RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta)$ the set of markings $\boldsymbol{m} \in \mathbb{R}_{\geq 0}$ such that there exists a finite input sequence $\boldsymbol{w} = \boldsymbol{w}(0) \cdots \boldsymbol{w}(k)$ and $\boldsymbol{m}(0) \xrightarrow{\boldsymbol{w}(0)} \boldsymbol{m}(1) \xrightarrow{\boldsymbol{w}(1)} \boldsymbol{m}(2) \cdots \xrightarrow{\boldsymbol{w}(k-1)}$

$\boldsymbol{m}(k) = \boldsymbol{m}$, where $\mathbf{0} \leq \boldsymbol{w}(j) \leq \boldsymbol{f}(j)\ \forall j$, and $\boldsymbol{f}(j)$ is the flow of the unforced system at time $j \cdot \Theta$.

It is important to stress that, although the evolution of a sampled contPN occurs in discrete steps, *discrete time evolutions* and *untimed evolutions* are not necessarily the same. As an example, while an untimed net can be seen evolving sequentially, executing a single transition firing at each step, a dt-contPN may evolve in concurrent steps where more than one transition fires. We denote such a concurrent step as $\boldsymbol{m}[\{t_{i_1}(\alpha_1), t_{i_2}(\alpha_2), \ldots, t_{i_k}(\alpha_k)\}\rangle \boldsymbol{m'}$.

In unforced ct-contPN under infinite servers semantics, the positiveness of the marking is ensured if the initial marking $\boldsymbol{m_0}$ is positive, because the flow of a transition goes to zero whenever one of the input places is empty [2]. In a dt-contPN, this is not always true. For example, let us consider the net in Fig. 1, with $\boldsymbol{m_0} = [0.1, 5.9, 1, 5.9]^T$, $\boldsymbol{\lambda} = \mathbf{1}^T$, $\Theta = 2$. Assume transitions t_2 and t_3 are stopped ($w_2(0) = w_3(0) = 0$), then $m_1(1) = m_1(0) - 2 \cdot \Theta \cdot w_1(0) = 0.1 - 4 \cdot w_1(0)$. But $w_1(0)$ is upper bounded by $\frac{\lambda_1}{2} \cdot m_1(0) = 0.5 \cdot 0.1 = 0.05$. If the maximum value is chosen, then $m_1(1)$ will be negative!!!

However, this can be avoided if the sampling period is small enough.

***Proposition** 3.4:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ be a dt-contPN system with $\boldsymbol{m_0} \geq \mathbf{0}$ where the sampling period Θ is such that:

$$\forall\, p \in P\ :\ \sum_{t_j \in p^\bullet} \lambda_j \Theta < 1. \tag{5}$$

Then the following statements hold.

1) Any marking reachable from $\boldsymbol{m_0}$ is non negative, i.e., $RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta) \subseteq \mathbb{R}^m_{\geq 0}$.
2) A place cannot be emptied with a finite sequence of firings, i.e., if $m(p) > 0$, then $\forall\, \boldsymbol{m'} \in RS^{dt}(\mathcal{N}, \boldsymbol{m}, \Theta)$ it also holds $m'(p) > 0$.

Proof: Let us consider a place p_i with ${p_i}^\bullet = \{t_1, t_2, \cdots, t_j\}$ and $m_k(p) > 0$.

Then $m_{k+1}(p_i) = m_k(p_i) + \Theta \boldsymbol{C}(i,:)\boldsymbol{w}(k) \geq m_k(p_i) - \Theta(\lambda_1 + \lambda_2 + \cdots + \lambda_j)m_k(p_i) = m_k(p_i)\Big(1 - \sum_{t_j \in {p_i}^\bullet} \lambda_j \Theta\Big) \geq 0$. Moreover, $m_{k+1}(p_i)$ is positive if $m_k(p_i)$ is positive. ■

In the rest of the paper we will assume that all nets are sampled with a sampling period Θ that satisfies (5).

***Proposition** 3.5:* If $\boldsymbol{m}$ is reachable in a dt-contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ with Θ verifying (5), then $\boldsymbol{m}$ is reachable in the underlying untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$, i.e.

$$RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta) \subseteq RS^{ut}(\mathcal{N}, \boldsymbol{m_0}).$$

Proof: In dt-contPN, transitions can fire concurrently and in order to prove that a marking is reached in the untimed contPN it is necessary to prove the existence of a sequence of transition firings leading to the same marking. This sequence exists due the fact that (5) implies $\boldsymbol{m}(k) - \boldsymbol{Pre} \cdot \Theta \cdot \boldsymbol{w}(k) \geq \boldsymbol{0}$ at any marking $\boldsymbol{m}(k)$. ■

In general the converse of Prop. 3.5 is not true: in fact, the second item of Prop. 3.4 shows that in a dt-contPN with Θ satisfying (5) it is never possible to empty a place (only at the limit, thus timed contPN can be deadlocked only at the limit), while this may be possible in an untimed net system. As an example, in the untimed net system in Fig. 1 from the marking shown it is possible to fire $t_1(2)t_1(0.5)$, thus emptying place p_1. This marking is clearly not reachable on the same net system if we associate to it a firing rate vector and choose Θ satisfying (5).

In the next section, two relaxations are studied: (1) considering in the untimed case only those sequences that never empty a marked place or (2) allowing the lim-reachable markings of the discrete-timed model. These relaxations are the same as in continuous-time case [6]. In fact we will prove that under any of them, and with the sampling period as in (5), the reachability space of the discrete-time and continuous-time models will be the same.

IV. Reachability "equivalence" between sampled and continuous models

Let us now characterize the reachability set of dt-contPN, first looking to a sequence with only one firing, then to more general sequences.

***Lemma** 4.1:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ be a dt-contPN system (with Θ satisfying (5)). Assume that in the underlying untimed net system it is possible from $\boldsymbol{m}$ to fire the sequence $\boldsymbol{m}[t_j(\alpha)\rangle\boldsymbol{m'}$ and that for a certain $\delta > 1$, for all $p \in {}^\bullet t_j$ it holds $m'(p) \geq m(p)/\delta$. Then marking $\boldsymbol{m'}$ is reachable from $\boldsymbol{m}$ with a finite sequence of length $r = \left\lceil \dfrac{\delta}{\Theta \lambda_j} \right\rceil$.

Proof: Let us first prove by induction that the firing of t_j with $w_j = \frac{\alpha\lambda_j}{\delta}$ can at least be repeated $r-1$ times in the discrete-time net, and that at any intermediate step, $\boldsymbol{m}(k) = \left(k \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m'} + \left(1 - k \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m}$. Observe first that $\boldsymbol{m'} = \boldsymbol{m} + \alpha \boldsymbol{C}(\cdot, j) \Longrightarrow \alpha \boldsymbol{C}(\cdot, j) = \boldsymbol{m'} - \boldsymbol{m}$.

- (Basic step) Since $t_j(\alpha)$ can be fired in the untimed net, and $\delta \geq 1$, for any $p_i \in {}^\bullet t_j$, $\lambda_j \frac{m_i}{\boldsymbol{Pre}(p,t_j)} \geq \lambda_j \alpha \geq \frac{\alpha\lambda_j}{\delta} = w_j(0)$. So, this is fireable in the discrete timed net. The new marking is $\boldsymbol{m}(1) = \boldsymbol{m} + \alpha \cdot \frac{\Theta\lambda_j}{\delta} \boldsymbol{C}(\cdot, j) = \boldsymbol{m} + \frac{\Theta\lambda_j}{\delta} \cdot (\boldsymbol{m'} - \boldsymbol{m}) = \left(\frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m'} + \left(1 - \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m}$.
- (Inductive step) Assume it holds for k. Observe that for all $p_i \in {}^\bullet t_j$, since $k \leq \frac{\delta}{\Theta\lambda_j}$, $\lambda_j \frac{m_i(k)}{\boldsymbol{Pre}(p,t_j)} \geq \lambda_j \frac{m_i}{\delta \boldsymbol{Pre}(p,t_j)} \geq \frac{\alpha\lambda_j}{\delta} = w_j(k)$. Moreover, $\boldsymbol{m}(k) + \alpha \cdot \frac{\Theta\lambda_j}{\delta} \boldsymbol{C}(\cdot, j) = \left(k \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot$

$$\boldsymbol{m}' + \left(1 - k \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m} + \frac{\Theta\lambda_j}{\delta} \cdot (\boldsymbol{m}' - \boldsymbol{m}) = \left((k+1) \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m}' + \left(1 - (k+1) \cdot \frac{\Theta\lambda_j}{\delta}\right) \cdot \boldsymbol{m}.$$

Therefore, $\boldsymbol{m}(r-1) = (r-1)\frac{\Theta\lambda_j}{\delta}\boldsymbol{m}' + (1-(r-1)\frac{\Theta\lambda_j}{\delta})\boldsymbol{m}$. To reach $\boldsymbol{m}'$ in one step we just have to prove that $w_j(r-1) = \left(1 - (r-1)\frac{\Theta\lambda_j}{\delta}\right)\frac{\alpha}{\Theta}$ can be fired. For all $p_i \in {}^\bullet t_j$, since $r - 1 < \frac{\delta}{\Theta\lambda_j} \leq r$, $\lambda_j \frac{m_i(r-1)}{\boldsymbol{Pre}(p,t_j)} \geq \lambda_j \frac{m_i}{\delta \boldsymbol{Pre}(p,t_j)} \geq \frac{\alpha\lambda_j}{\delta} \geq \frac{\alpha}{r\Theta} \geq \frac{\alpha}{\Theta}(1 - \frac{r-1}{r}) \geq \frac{\alpha}{\Theta}(1 - (r-1)\frac{\Theta\lambda_j}{\delta}) = w_j(r-1)$, and so this firing can be done. ■

Theorem *4.2:* A marking $\boldsymbol{m}$ is reachable in a dt-contPN $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ system (with Θ satisfying (5)) iff it is reachable in the underlying untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ with a sequence that never empties an already marked place.

Proof: A sequence $\boldsymbol{m}[t_{i_1}(\alpha_1)\rangle \boldsymbol{m_1}[t_{i_2}(\alpha_2)\rangle \boldsymbol{m_2} \cdots [t_{i_k}(\alpha_k)\}\rangle\ \boldsymbol{m_k} = \boldsymbol{m}'$ never empties a marked place if $(\forall j = 1, \ldots, k), (\forall p \in {}^\bullet t_{i_j})\ \boldsymbol{m_j}(p) > 0$.

(If) Applying Lemma 4.1 to $\boldsymbol{m_1}$, $\boldsymbol{m_2}$, $\cdots$, $\boldsymbol{m_k}$, $\boldsymbol{m}'$ is reachable with a finite sequence.

(Only if) Assume there is a finite sequence that reaches $\boldsymbol{m}$ in the dt-contPN, then there exists an equivalent firing sequence for the untimed net system, according to Prop. 3.5. It is also immediate to observe that the non emptying condition holds because in the dt-contPN a place cannot be emptied with a finite sequence, according to Prop. 3.4 part 2. ■

One may wonder what happens if a marking $\boldsymbol{m}$ is reachable in the untimed PN but there exists no sequence satisfying the non emptying condition. It can be proved that the marking is *lim*-reachable in the timed net, i.e., it is reachable with an infinite sequence of steps.

Theorem *4.3:* [10] If a marking $\boldsymbol{m}$ is reachable in the untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$, then it is lim-rechable in a dt-contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ with Θ satisfying (5).

Sketch of the proof: It is immediate if the sequence to reach the marking is such that Lemma 4.1 can be applied for each transition. Otherwise, the idea is to fire each transition in the sequence, but in an amount small enough so that the lemma can be used, and repeat the process. Moreover, the amount of firing of each transition can be defined in such a way that the sum converges to its firing in the sequence. See [11] for a complete proof. ■

Putting together Prop. 3.5 and Th. 4.2 with Prop.14 in [6], the equivalence between continuous and discrete time system is obtained.

Corollary *4.4:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a ct-contPN system and $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ with Θ satisfying (5) its discrete time approximation, then $RS^{ct}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}) = RS^{dt}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta)$, where $RS^{ct}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0})$ is the reachability space of the ct-contPN system.

Proof: According to Th. 4.2, $\boldsymbol{m} \in RS^{dt}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta)$ iff $\boldsymbol{m}$ is reachable in the untimed system with a sequence that never empties an already marked place. According to Lemma 13 in [6], if $\boldsymbol{m}$ is reachable in untimed system with a sequence that never empties any place, then this sequence can be fired in the ct-contPN system, so $\boldsymbol{m} \in RS^{ct}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0})$. On the other hand, if $\boldsymbol{m} \in RS^{ct}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0})$ then the empty places at $\boldsymbol{m}$ are also empty at $\boldsymbol{m_0}$ because a marked place cannot be emptied in the ct-contPN system. Taking the integral of the flow (see Prop. 14.1 in [6]), a firing sequence is obtained that does not empty an already marked place. Therefore, $\boldsymbol{m}$ is reachable in the dt-contPN system. ■

V. Optimal transient control via MPC

Steady state optimal control of contPN was studied in [6] and if all transitions can be controlled and the objective function is linear, the problem can be solved in *polynomial* time. The solution is an optimal marking and an optimal control input in steady state. In this paper we assume that the steady state condition ($\boldsymbol{m_f}$, $\boldsymbol{w_f}$) is known and our problem is how to reach it (from a given $\boldsymbol{m_0}$) in a finite time while optimizing a given performance index. The optimal control is studied using *Model Predictive Control* (MPC) [7]. MPC algorithms use different cost functions to obtain the control action. We consider the following standard quadratic form:

$$\begin{aligned} J(\boldsymbol{m}(k), N) \ &= (\boldsymbol{m}(k+N) - \boldsymbol{m_f})' \cdot \boldsymbol{Z} \cdot (\boldsymbol{m}(k+N) - \boldsymbol{m_f})) + \\ &\sum_{j=0}^{N-1} [(\boldsymbol{m}(k+j) - \boldsymbol{m_f})' \cdot \boldsymbol{Q} \cdot (\boldsymbol{m}(k+j) - \boldsymbol{m_f}) + \\ &+ (\boldsymbol{w}(k+j) - \boldsymbol{w}_f)' \cdot \boldsymbol{R} \cdot (\boldsymbol{w}(k+j) - \boldsymbol{w}_f)] \end{aligned} \tag{6}$$

where $\boldsymbol{Z}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive definite matrices.

The constraints are derived from the dt-contPN definition, and at every step the new marking should respect (4). Thus, at each step the following problem needs to be solved:

$$\begin{aligned} \min \quad & J(\boldsymbol{m}(k), N) \\ \text{s.t.}: \quad & \boldsymbol{m}(k+j+1) = \boldsymbol{m}(k+j) + \Theta \cdot \boldsymbol{C} \cdot \boldsymbol{w}(k+j), \\ & \qquad j = 0, \ldots, N-1, \\ & \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}(k+j) \\ \boldsymbol{m}(k+j) \end{bmatrix} \leq \boldsymbol{0}, \qquad j = 0, \ldots, N-1, \\ & \boldsymbol{w}(k+j) \geq \boldsymbol{0}, \qquad j = 0, \ldots, N-1. \end{aligned} \tag{7}$$

We denote as *implicit* MPC the control law computed solving on-line the optimization problem (7). An alternative to *implicit* MPC has been proposed in [8]. There the authors present a technique to compute *off-line* an *explicit* solution of the MPC control problem, based on multi-parametric linear programming (mp-LP) or quadratic programming (mp-QP). They split the maximum controllable set (i.e., all states that are controllable) into polytopes described by linear inequalities in which the control command is described as a piecewise affine function of the state. Thus, the control law results in a *state feedback* control law.

We have applied MPC with the above two approaches in the case of dt-contPN. Numerical examples are not reported here for the sake of brevity but can be found in [11]. In particular, in [11] we provided a detailed comparison among the results obtained using implicit and explicit MPC.

VI. Properties of the closed-loop system

A. Feasibility

In general, given an initial feasible state, there is no guarantee that the optimization problem we need to solve at each time step will remain feasible at all future time steps k, as the system might enter "blind alleys" where no solution to the optimization problem exists [8]. In terms of explicit MPC this translates into the fact that there is no guarantee that the resulting state space partition includes all reachable states. However, thanks to the particular structure of the constraints, in the case of contPN systems the following result can be proved.

***Proposition** 6.1:* The optimization problem (7) is feasible for any $\boldsymbol{m}(k) \geq \mathbf{0}$.

Proof: The solution $\boldsymbol{w}(k+j) = \mathbf{0}$ for $j = 0, 1, \ldots, N-1$ is feasible. In fact, $\boldsymbol{G} \cdot [\boldsymbol{w}'(k+j) \quad \boldsymbol{m}'(k+j)]' = [\boldsymbol{\Delta} - \boldsymbol{\Gamma}] \cdot [\boldsymbol{w}'(k+j) \quad \boldsymbol{m}'(k+j)]' = -\boldsymbol{\Gamma} \cdot \boldsymbol{m}(k+j) \leq \mathbf{0}$ since $\boldsymbol{\Gamma}$ is a matrix of non-negative numbers and $\boldsymbol{m}(k+j) = \boldsymbol{m}(k) \geq \mathbf{0}$ for any $j = 1, \ldots, N-1$. ■

B. Asymptotic stability

The feasibility of (7) is obviously a desirable property but it does not ensure the convergence of the optimal solution to the desired state, that is our main requirement. We investigate three different approaches to improve convergence, that are well known in the literature [12], [13].

The first approach consists in assuming that $\boldsymbol{w}(k+j) = \mathbf{0}$, $\forall\ j = N, \cdots, \infty$, and weighting the distance from the final marking not only for $j = 0, 1, \cdots, N-1$ but for any $j = 0, 1, \cdots, \infty$.

Obviously, such an approach can only be applied to asymptotically stable systems, that is not the case here, since the dynamical matrix is the identity matrix.

The second approach consists in assuming that $\boldsymbol{w}(k+j) = \boldsymbol{K}\boldsymbol{m}(k+j)$, $\forall\ j = N, \cdots, \infty$, and weighting the distance from the final marking not only for $j = 0, 1, \cdots, N-1$ but for any $j = 0, 1, \cdots, \infty$. In particular, matrix $\boldsymbol{K}$ is defined as in the unconstrained LQR problem with weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$. This is equivalent to an optimization problem of the form (7), where $\boldsymbol{Z}$ in the performance index is $\boldsymbol{Z} = \boldsymbol{P}$, and $\boldsymbol{P}$ is the solution of the unconstrained LQR. Using results from classical optimal control theory [14], we can guarantee convergence only if the region defined by the set of feasible state + input vectors is bounded and contains the final state + input in its interior. Therefore, such an approach does not apply to most control problems within the framework of contPN, because the desired marking is often not positive and/or the desired flow is set to its maximum allowable value. Note however that, if the final state + input is an interior point, and the moving horizon N is sufficiently large, this approach is surely the most convenient. In fact, it has the major advantage that the resulting strategy is indeed the optimal infinite horizon constrained LQR policy [8].

The third approach we consider consists in forcing the marking at time $k+N$ to belong to the straight path $\boldsymbol{m}(k)$ — $\boldsymbol{m}_f$. This is equivalent to adding a terminal constraint of the form

$$\begin{cases} \boldsymbol{m}(k+N) = \alpha \cdot \boldsymbol{m_f} + (1-\alpha) \cdot \boldsymbol{m}(k) \\ 0 \leq \alpha \leq 1 \end{cases} \tag{8}$$

to the optimization problem (7), where α is a free variable. This is always admissible when dealing with continuous Petri nets because the set of reachable markings is convex, thus if $m(k)$ and m_f are reachable, then all markings in the straight path $m(k) - -m_f$ are reachable as well.

Note that this constraint makes necessary to solve a certain number of bilinear (rather than linear) programming problems when using explicit MPC [8]. In particular, bilinear problems have to be solved when computing the Chebychev centers of the polytopic regions, where both the initial state and α are unknown. This approach was found to be satisfactory in several numerical examples, even if we have been able to prove asymptotic stability only under certain assumptions.

Proposition *6.2:* Let us consider a contPN system. Let $\boldsymbol{m_0}$ and $\boldsymbol{m_f}$ be the initial and final markings, respectively, with $\boldsymbol{m_0} > \boldsymbol{0}$ and $\boldsymbol{m_f}$ reachable from $\boldsymbol{m_0}$. Assume that the system is controlled using MPC with a terminal constraint of the form (8) and prediction horizon $N = 1$. Then the closed-loop system is asymptotically stable.

Proof: We prove the statement in three steps. We first prove that if $\boldsymbol{m_0} > \boldsymbol{0}$ then $\alpha > 0$ is feasible at any $k \geq 0$. Then, we define a quadratic function that we prove to be a Lyapunov function. Finally, we demonstrate that it is strictly decreasing.

— Observe that by item (2) of Prop. 3.4, if $\boldsymbol{m_0} > \boldsymbol{0}$ then $\boldsymbol{m}(k) > \boldsymbol{0}$ for any $k \geq 1$. Since $\boldsymbol{m_f}$ is reachable from $\boldsymbol{m_0}$, then it is also reachable from any marking in the straight path $\boldsymbol{m_f}$ — $\boldsymbol{m_0}$ being the reachability space convex. So, there exists $\boldsymbol{\sigma} \geq \boldsymbol{0}$ such that $\boldsymbol{m_f} = \boldsymbol{m}(k) + \boldsymbol{C} \cdot \boldsymbol{\sigma}$. Replacing in eq. (8), $\boldsymbol{m}(k+1) = \alpha \cdot \boldsymbol{m}(k) + \alpha \cdot \boldsymbol{C} \cdot \boldsymbol{\sigma} + (1-\alpha) \cdot \boldsymbol{m}(k) = \boldsymbol{m}(k) + \boldsymbol{C} \cdot (\alpha\ \boldsymbol{\sigma})$. But being $\boldsymbol{m}(k) > \boldsymbol{0}$, there always exists $\alpha > 0$ such that $\alpha\ \boldsymbol{\sigma}$ can be fired at $\boldsymbol{m}(k)$.

— Without loss of generality we assume that in eq. (7) it holds: (a) $\boldsymbol{m_f} = \boldsymbol{0}$; (b) $\boldsymbol{C}$ is full rank. If $\boldsymbol{m_f} \neq \boldsymbol{0}$ we can always redefine the state in eq. (7) by translation; if $\boldsymbol{C}$ is not full rank then the dimension of the state vector can be reduced until a full rank matrix is obtained.

Let $V(\boldsymbol{m}(k)) = \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k)$ where $\boldsymbol{Z}$ is the weighting matrix in the performance index (6). Obviously, $V(\boldsymbol{m}(k)) \geq 0$ for any $\boldsymbol{m}(k) \neq \boldsymbol{0}$, since $\boldsymbol{Z}$ is positive definite. Moreover, $V(\boldsymbol{m}(k+1)) \leq V(\boldsymbol{m}(k))$ for any $k \geq 0$. Indeed, under the assumption that $\boldsymbol{m_f} = \boldsymbol{0}$, by constraint (8) it holds $\boldsymbol{m}(k+1) = (1-\alpha) \cdot \boldsymbol{m}(k)$. Thus, $V(\boldsymbol{m}(k+1)) = \boldsymbol{m}(k+1)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k+1) = (1-\alpha)^2 \cdot \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k) = (1-\alpha)^2 \cdot V(\boldsymbol{m}(k)) \leq V(\boldsymbol{m}(k))$.

— We now prove that $\forall\ k \geq 0$ the optimal solution of problem (7) leads to $\alpha > 0$.

Let k be an arbitrary time instant. If $\alpha = 0$ the performance index (6) is $J' = \boldsymbol{m}(k)^T \cdot \boldsymbol{Q} \cdot \boldsymbol{m}(k) + \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k)$. If $\alpha > 0$, and taking into account that $\boldsymbol{m}(k+1) = (1-\alpha) \cdot \boldsymbol{m}(k)$, the performance index is $J'' = \boldsymbol{m}(k)^T \cdot \boldsymbol{Q} \cdot \boldsymbol{m}(k) + \boldsymbol{w}(k)^T \cdot \boldsymbol{R} \cdot \boldsymbol{w}(k) + \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k) - 2 \cdot \alpha \cdot \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k) + \alpha^2 \cdot \boldsymbol{m}(k)^T \cdot \boldsymbol{Z} \cdot \boldsymbol{m}(k)$. Moreover, since $\boldsymbol{m}(k+1) = (1-\alpha) \cdot \boldsymbol{m}(k) = \boldsymbol{m}(k) + \Theta \cdot \boldsymbol{C} \cdot \boldsymbol{w}(k)$, it follows that $\boldsymbol{m}(k) = -\dfrac{\Theta}{\alpha} \cdot \boldsymbol{C}\boldsymbol{w}(k)$.

Therefore

$$J'' - J' = \boldsymbol{w}(k)^T \cdot \boldsymbol{R} \cdot \boldsymbol{w}(k) - \boldsymbol{w}(k)^T \cdot \left[\left(\frac{2}{\alpha} - 1 \right) \cdot \Theta^2 \cdot \boldsymbol{C}^T \cdot \boldsymbol{Z} \cdot \boldsymbol{C} \right] \cdot \boldsymbol{w}(k).$$

But $\boldsymbol{C}^T \cdot \boldsymbol{Z} \cdot \boldsymbol{C}$ is always positive definite because $\boldsymbol{Z}$ is positive definite and $\boldsymbol{C}$ is a full rank matrix by assumption, so if α is small enough $J'' - J' < 0$. ■

***Remark** 6.3:* In general $\boldsymbol{m}(0) > \boldsymbol{0}$ is not a strict requirement in the above proposition. It is sufficient that for any $k \geq 0$ the optimization problem (plus terminal constraint) admits $\alpha > 0$

as a solution. Physically this means that we can move along the straight line $\boldsymbol{m}(0) - \boldsymbol{m_f}$. However, in general it is difficult to verify such a condition.

VII. Conclusions

Different ways of describing the behavior of controlled contPN with infinite server semantics are presented. The first one uses the "min" operator according to the definition of the semantics (Eq. (1) and (2)). Later, the "min" operator is substituted by linear inequalities obtaining a constrained linear form (Eq. (3)). Finally, in order to simplify the application of the MPC the system is discretized in time, leading to Eq. (4). After that, a *Sampling theorem* giving an upper bound on sampling period is provided. The purpose of this bound is to preserve reachability conditions (in particular non-negativity of markings), not to reconstruct the original signal from the sampled one. The reachability space of the sampled system is studied later and some relations between this space and the space of the underlying untimed contPN are provided. Then, optimal control laws based on both implicit and explicit MPC are investigated. Some aspects regarding the convergence of MPC are studied, and for a particular control law asymptotic stability is guaranteed. Our future efforts within this framework will be mainly devoted to the derivation of more general criteria that guarantee stability.

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