Decidability of Single–Rate Hybrid Petri Nets*

Carla Seatzu[†], Angela Di Febbraro[‡], Fabio Balduzzi[‡], Alessandro Giua[†]

[†]*Dip. di Ing. Elettrica ed Elettronica, Università di Cagliari, Italy*

email: {giua,seatzu}@diee.unica.it.

[‡] *Dip. di Automatica ed Informatica, Politecnico di Torino, Italy*

email: {balduzzi,difebbraro}@polito.it.

Abstract

In this paper we define the class of Single–Rate Hybrid Petri Nets. The continuous dynamics of these nets is such that the vector of the marking derivatives of the continuous places is constant but for a scalar factor. This class of nets can be seen as the counterpart of timed automata with skewed clocks. We prove that the reachability problem for this class can be reduced to the reachability problem of an equivalent discrete net and thus it is decidable.

1 Introduction

The purpose of a hybrid model is that of efficiently describing with a single formalism systems with both continuous–time and discrete–event dynamics.

Hybrid automata (HA) are one of the most well understood hybrid models. HA can be seen as a generalization of *timed automata* (TA) defined by Alur and Dill [1]. A hybrid automaton consists of a classic automaton extended with a continuous state that may continuously evolve in time with arbitrary dynamics or have discontinuous jumps at the occurrence of a discrete event. Several results concerning the decidability and the complexity of this model have been presented by different authors [6, 9].

In this paper we focus on a different hybrid model based on Petri nets (PNs) [7]. Petri nets are a family of models that have originally been introduced to describe and analyze discrete event systems. Recently, much effort has been devoted to apply these models to hybrid systems as well.

The Petri net formalism we use in this paper follows the basic model originally presented in [3] that was inspired from the approach of David and Alla [4]. This model, that will be called in the rest of this paper *Hybrid Petri Net* (HPN), consists of continuous places holding fluid, discrete places containing a non–negative integer number of tokens, and transitions, either discrete or continuous. Note that, unlike [3], we are assuming here that no timing structure is associated to the firing of discrete transitions. This is consistent with the definition of HA, where the variable "time" is only associated to the continuous evolution.

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The relationship between HPNs and HA warrants comments. One feature of HPNs is the fact that the discrete state space, i.e., the set of discrete markings of the net, may be infinite, while in HA the discrete state space, i.e., the set of locations, must be finite. In all other respects, however, a HPN is a special case of a HA, and any HPN with bounded discrete places may also be modeled by a HA.

The recent results on HA have shown that a trade-off between modeling power and analytical tractability is necessary. To this end, several special classes of HA have been studied: timed automata, timed automata with skewed clocks, multirate and rectangular automata (initialized or not) [9].

Since HPNs have only recently been introduced, very little is known about their decidability properties. We believe that even in this case it may be worth defining and exploring a hierarchy of models of increasing complexity.

In this paper we introduce a special class of HPNs, called *single–rate HPN* (SRHPN), that can be seen as the HPN counterpart of a timed automaton with skewed clocks. It consists of a HPN with a single continuous transition whose firing speed is constant. Thus the continuous dynamic is such that the marking of each continuous place increases with a single constant rate. Note however that all the results presented in this paper also hold if the firing speed of the continuous transition is not constant. In this case we still have a single–rate but for a scalar factor that may vary in time.

When comparing SRHPNs and TA with skewed clocks we observe that the two models are significantly different and neither one can be seen as a special case of the other one. TA can model "reset" of the continuous state, while SRHPNs can model "jumps of constant magnitude" of the continuous state (and, as in the general case, may also have an infinite discrete state space). We prove that the reachability problem is decidable for SRHPNs. This result is rather interesting, because the reachability problem for a TA with skewed clocks is known to be undecidable [5].

2 Hybrid Petri Nets

The Petri net formalism used in this paper can be seen as the "untimed" version of the model presented in [3]. For a more comprehensive introduction to place/transition Petri nets see [7].

A Hybrid Petri Net (HPN) is a structure $N = (P, T, Pre, Post, \mathcal{C})$.

The sets of *places* $P = P_d \cup P_c$ and transitions $T = T_d \cup T_c$ are partitioned into *discrete* places and transitions (represented as circles and boxes) and *continuous* places and transitions (represented as double circles and double boxes). The cardinality of P, P_d and P_c is denoted n, n_d and n_c. The cardinality of T, T_d and T_c is denoted q, q_d and q_c.

The pre- and post-incidence functions that specify the arcs are (here $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$) are $Pre, Post : \{P_d \times T \to \mathbb{N};\ P_c \times T \to \mathbb{R}_0^+$. We require (well-formed nets) that for all $t \in T_c$ and for all $p \in P_d$, $Pre(p,t) = Post(p,t)$.

The function $\mathcal{C} : T_c \to \mathbb{R}_0^+ \times \mathbb{R}_\infty^+$ specifies the firing speeds associated to continuous transitions (here $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$). For any continuous transition $t_j \in T_c$ we let $\mathcal{C}(t_j) = (V'_j, V_j)$, with $V'_j \leq V_j$. Here V'_j represents the minimum firing speed (mfs) and V_j represents the maximum firing speed (MFS).

We denote the preset (postset) of transition t as ${}^\bullet t$ ($t^\bullet$) and its restriction to continuous or discrete places as ${}^{(d)}t = {}^\bullet t \cap P_d$ or ${}^{(c)}t = {}^\bullet t \cap P_c$. Similar notation may be used for presets and postsets of places. The incidence matrix of the net is defined as $\boldsymbol{C}(p,t) = Post(p,t) - Pre(p,t)$. The restriction of $\boldsymbol{C}$ to P_X and T_Y $(X, Y \in \{c, d\})$ is denoted $\boldsymbol{C}_{XY}$. Note that by the well-formedness hypothesis $\boldsymbol{C}_{dc} = 0$.

A marking $\boldsymbol{m} : \{P_d \to \mathbb{N};\ P_c \to \mathbb{R}_0^+\}$ is a function that assigns to each discrete place a non-negative number of tokens, represented by black dots and assigns to each continuous place a fluid volume; m_p denotes the marking of place p. The value of a marking at time τ is denoted $\boldsymbol{m}(\tau)$. The restriction of $\boldsymbol{m}$ to P_d and P_c are denoted with $\boldsymbol{m}^d$ and $\boldsymbol{m}^c$, respectively. An HPN system $(N, \boldsymbol{m})$ is an HPN N with an initial marking $\boldsymbol{m}$.

The enabling of a discrete transition depends on the marking of all its input places, both discrete and continuous.

Definition 1. *Let $(N, \boldsymbol{m})$ be an HPN system. A discrete transition t is enabled at $\boldsymbol{m}$ if for all $p \in {}^\bullet t$, $m_p \geq Pre(p,t)$.* ■

A continuous transition is enabled only by the marking of its input discrete places. The marking of its input continuous places, however, is used to distinguish between strongly and weakly enabling.

Definition 2. *Let $(N, \boldsymbol{m})$ be an HPN system. A continuous transition t is enabled at $\boldsymbol{m}$ if for all $p \in {}^{(d)}t$, $m_p \geq Pre(p,t)$.*

We say that an enabled transition $t \in T_c$ is:
– strongly enabled at $\boldsymbol{m}$ if for all $p \in {}^{(c)}t$, $m_p > 0$;
– weakly enabled at $\boldsymbol{m}$ if for some $p \in {}^{(c)}t$, $m_p = 0$. ■

In the following we describe the hybrid dynamics of an HPN. We first consider the time–driven behavior associated to the firing of continuous transitions, and then the event–driven behavior associated to the firing of discrete transitions.

The instantaneous firing speed (IFS) at time τ of a transition $t_j \in T_c$ is denoted $v_j(\tau)$. We can write the equation which governs the evolution in time of the marking of a place $p \in P_c$ as

$$\dot{m}_p(\tau) = \sum_{t_j \in T_c} C(p, t_j) v_j(\tau). \tag{1}$$

Indeed Equation 1 holds assuming that at time τ no discrete transition is fired and that all speeds $v_j(\tau)$ are continuous in τ.

The enabling state of a continuous transition t_j defines its admissible IFS v_j.

Definition 3. (admissible IFS vectors)
Let $(N, \boldsymbol{m})$ be an HPN system. Let $T_{\mathcal{E}}(\boldsymbol{m}) \subset T_c$ ($T_{\mathcal{N}}(\boldsymbol{m}) \subset T_c$) be the subset of continuous transitions enabled (not enabled) at $\boldsymbol{m}$, and $P_{\mathcal{E}} = \{p \in P_c \mid m_p = 0\}$ be the subset of empty continuous places. Any admissible IFS vector $\boldsymbol{v}$ at $\boldsymbol{m}$ is a feasible solution of the following linear set:

$$\begin{cases} (a) & V_j - v_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\boldsymbol{m}) \\ (b) & v_j - V'_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\boldsymbol{m}) \\ (c) & v_j = 0 & \forall t_j \in T_{\mathcal{N}}(\boldsymbol{m}) \\ (d) & \sum_{t_j \in T_{\mathcal{E}}} C(p, t_j) v_j \geq 0 & \forall p \in P_{\mathcal{E}}(\boldsymbol{m}) \end{cases} \tag{2}$$

The set of all feasible solutions is denoted $\mathcal{S}(N, \boldsymbol{m})$. ■

	HPN	HA
state	$(\boldsymbol{m}^d, \boldsymbol{m}^c) \in \mathbb{N}^{n_d} \times (\mathbb{R}_0^+)^{n_c}$	$(l, \boldsymbol{x}) \in L \times \mathbb{R}^n$
activity	$\dot{\boldsymbol{m}}^c \in \{\boldsymbol{C}_{cc}\boldsymbol{v} \mid \boldsymbol{v} \in \mathcal{S}(N, \boldsymbol{m})\ \}$	$\dot{\boldsymbol{x}} \in act_l(\boldsymbol{x})$
invariant	$\boldsymbol{m}^c \in (\mathbb{R}_0^+)^{n_c}$	$\boldsymbol{x} \in inv_l$
guard	$\{\boldsymbol{m} \mid \boldsymbol{m} \geq Pre(\cdot, t)\}$	$g \subset \mathbb{R}^n$
jump	$\{(\boldsymbol{m}^c, \tilde{\boldsymbol{m}}^c) \mid \tilde{\boldsymbol{m}}^c = \boldsymbol{m}^c + \boldsymbol{C}_{cd}(\cdot, t)\}$	$j \subset \mathbb{R}^n \times \mathbb{R}^n$

Table 1: *Comparison between Hybrid Petri Nets and Hybrid Automata.*

Constraints of the form (2.a), (2.b), and (2.c) follow from the firing rules of continuous transitions. Constraints of the form (2.d) follow from (1), because if a continuous place is empty then its fluid content cannot decrease.

Note that the set $\mathcal{S}$ is a function of the marking of the net. Thus as $\boldsymbol{m}$ changes it may vary as well. In particular it changes at the occurrence of the following macro–events: (a) a discrete transition fires, thus changing the discrete marking and enabling/disabling a continuous transition; (b) a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak.

2.1 Firing sequence and reachability

Now, we provide some definitions that will be useful in the following sections.

Definition 4. *(Event Step) Let $(N, \boldsymbol{m})$ be a HPN system. If $t \in T_d$ is enabled at $\boldsymbol{m}$, t may fire yielding the marking $\tilde{\boldsymbol{m}} = \boldsymbol{m} + Post(\cdot, t) - Pre(\cdot, t)$ and we write $\boldsymbol{m}[t\rangle\tilde{\boldsymbol{m}}$.* ■

We can use a similar notation for the marking variation due to the firing of continuous transitions.

Definition 5. *(Time Step) Let $(N, \boldsymbol{m})$ be a HPN system. If $t \in T_c$ is enabled at $\boldsymbol{m}$ for a time interval of length $\overline{\tau} \in \mathbb{R}^+$, it may fire yielding the marking*

$$\begin{cases} \tilde{\boldsymbol{m}}^d = \boldsymbol{m}^d \\ \tilde{\boldsymbol{m}}^c = \int_0^{\overline{\tau}} \boldsymbol{C}_{cc}\boldsymbol{v}(\tau)d\tau + \boldsymbol{m}^c \geq \boldsymbol{0} \end{cases}$$

where $\boldsymbol{v} \in \mathcal{S}(N, \boldsymbol{m})$ and we write $\boldsymbol{m}[\overline{\tau}\rangle\tilde{\boldsymbol{m}}$. ■

Definition 6. *Let $(N, \boldsymbol{m})$ be a HPN system. A firing sequence $\sigma = \alpha_1, \cdots, \alpha_k \in (T_d \cup \mathbb{R}^+)^*$ is enabled at $\boldsymbol{m}$ if $\boldsymbol{m}[\alpha_1\rangle\boldsymbol{m}_1[\alpha_2\rangle\boldsymbol{m}_2 \cdots [\alpha_k\rangle\tilde{\boldsymbol{m}}$ holds and we write $\boldsymbol{m}[\sigma\rangle\tilde{\boldsymbol{m}}$.* ■

3 Hybrid Automata

A hybrid automaton [8, 9] is a structure $H = (L, act, inv, E)$ defined as follows.

- L is a finite set of locations.
- $act : L \to Inclusions$ is a function that associates to each location $l \in L$ a differential inclusion of the form $\dot{\boldsymbol{x}} \in act_l(\boldsymbol{x}) \subseteq \mathbb{R}^n$ where $act_l(\boldsymbol{x})$ is a set-valued map; if $act_l(\boldsymbol{x})$ is a singleton then it is a differential equation.

  A solution of a differential inclusion with initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ is any differentiable function $\boldsymbol{\phi}(\tau)$, where $\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^n$ such that $\boldsymbol{\phi}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{\phi}}(\tau) \in act_l(\boldsymbol{\phi}(\tau))$.

- $inv : L \to Invariants$ is a function that associates to each location $l \in L$ an invariant $inv_l \subset \mathbb{R}^n$.

  An invariant function is $\boldsymbol{x} \in inv_l$. The invariant function constrains the behaviour of the automaton state during time steps within a given subset of $\mathbb{R}^n$.

- $E \subset L \times Guards \times Jump \times L$ is the set of edges. An edge $e = (l, g, j, l') \subset E$ is an edge from location l to l' with guard g and jump relation j.

  A guard is $g \subset \mathbb{R}^n$. An edge is enabled when the state $\boldsymbol{x} \in g$.

  A jump relation is $j \subset \mathbb{R}^n \times \mathbb{R}^n$. During the jump, $\boldsymbol{x}$ is set to $\boldsymbol{x}'$ provided $(\boldsymbol{x}, \boldsymbol{x}') \in j$. When j is the identity relation, the continuous state does not change.

The state of the hybrid automaton is the pair $(l, \boldsymbol{x})$ where $l \in L$ is the discrete location, and $\boldsymbol{x} \in \mathbb{R}^n$ is the continuous state. The hybrid automaton starts from some initial state $(l_0, \boldsymbol{x}_0)$. The trajectory evolves with the location remaining constant and the continuous state $\boldsymbol{x}$ evolving within the invariant function at that location, and its first derivative remains within the differential inclusion at that location. When the continuous state satisfies the guard of an edge from location l to location l', a jump can be made to location l'. During the jump, the continuous state may get initialized to a new value $\boldsymbol{x}'$. The new state is the pair $(l', \boldsymbol{x}')$. The continuous state $\boldsymbol{x}'$ now moves within the invariant function with the new differential inclusion, followed some time later by another jump, and so on.

In this paper we are interested in a special class of hybrid automata called *Timed Automata with Skewed Clocks*. Let us first recall the definition of a *rectangle*.

Definition 7. *An n–dimensional* rectangle *is a set of the form $r = [l_1, u_1] \times \cdots \times [l_n, u_n] \subset \mathbb{R}^n$ with $l_i, u_i \in \mathbb{Z}_{\pm\infty}$. The i-th component of r is $r_i = [l_i, u_i]$. The set of all n-dimensional rectangles is $Rect_n$.* ■

Definition 8. *An n-dimensional* timed automaton with skewed clocks *$R = (L, act, E)$ is a hybrid automaton in which the set Inclusions contains the single element $\boldsymbol{v} \in (\mathbb{R}^+)^n$, i.e., $\dot{x}_i = v_i$ for each i at every location; $Guard = Rect_n$; $Jump = \{j \mid j = j_1 \times \cdots \times j_n$ where $j_i = [l_i, u_i]$ or $j_i = id\}$. Here we consider the relation $[l_i, u_i] = \{(\boldsymbol{x}, \boldsymbol{x}') \mid \boldsymbol{x}' \in [l_i, u_i]\}$ and id is the identity relation.* ■

Note that, following [1, 9], we are assuming that the behaviour of this class of HA is not constrained by any invariant function.

4 Hybrid Petri nets and hybrid automata

In this section we explore the relations between hybrid automata and hybrid Petri nets. Table 1 summarizes the differences existing between the two hybrid models.

In both models the state consists of a discrete part (location, discrete marking) and a continuous part (continuous state, continuous marking). In the hybrid automaton only the location $l \in L$ is represented in the transition structure, while the continuous state $\boldsymbol{x} \in \mathbb{R}^n$ is given using an algebraic formalism. In the HPN the net marking $\boldsymbol{m} = (\boldsymbol{m}^d, \boldsymbol{m}^c) \in \mathbb{N}^{n_d} \times (\mathbb{R}_0^+)^{n_c}$ represents with a single formalism both discrete and continuous state. Another important difference is the fact that a Petri net may have an infinite number of discrete markings (i.e., the discrete state space may be infinite), while the locations of an automaton may only vary within the finite set L.

The activity function which constrains $\dot{\boldsymbol{x}} \in act_l(\boldsymbol{x})$ finds its counterpart in hybrid Petri nets. In fact, the continuous marking of a HPN varies with $\dot{\boldsymbol{m}}^c = \boldsymbol{C}_{cc}\boldsymbol{v}$, where $\boldsymbol{v} \in \mathcal{S}(N, \boldsymbol{m})$ is the IFS at $\boldsymbol{m}$. Clearly, in the case of HPNs this set has a special structure (and in particular it can be shown that it is a linear convex set, i.e., properly speaking a HPN is a *linear* HA [2]), while a hybrid automaton admits more general activities.

Similarly, while in a hybrid automaton the continuous state at each location l may be constrained by an arbitrary invariant function inv_l, in a hybrid Petri net the only constraint to continuous marking is that it must be non–negative and it is the same for all discrete markings. Note, however, that this invariant function is not explicitly given in the model, i.e., in the definition of HPNs we did not mention any constraint on the continuous markings. This constraint follows from the definition of initial marking (that must be non negative) and from the fact that the set of admissible IFS vectors is defined so that for each empty continuous place the output flow can never exceed the input flow. Finally, we also observe that while no upper bound of the continuous marking — this is an invariant function that may be useful in many cases — can be directly imposed, it may be possible to bound a continuous place adding to the net a new complementary continuous place, a technique also used in discrete nets.

The guard $g \subset \mathbb{R}^n$ associated to an edge of a HA corresponds to the enabling condition $\boldsymbol{m} \geq Pre(\cdot, t)$ associated to a discrete transition t of a HPN. Note first of all that while each edge of a HA represents a single event, in a HPN a single transition may represent different events. Thus, the fact that the transition enabling depends on the discrete marking $\boldsymbol{m}^d$ is used to specify that a given transition corresponds to an event that may be enabled only by a subset of locations. On the other hand, the fact that the transition enabling depends also on the continuous marking $\boldsymbol{m}^c$ is the HPN counterpart of the guard g of an hybrid automaton. Note also that the enabling of a HPN is a guard with special structure: it is a "right closed set", i.e., if $(\boldsymbol{m}^d, \boldsymbol{m}^c) \in g$, and $\tilde{\boldsymbol{m}}^c \geq \boldsymbol{m}^c \Longrightarrow (\boldsymbol{m}^d, \tilde{\boldsymbol{m}}^c) \in g$.

Finally, in hybrid automata the jump relation $j \subset \mathbb{R}^n \times \mathbb{R}^n$ defines for each edge the updated value that the continuous state assumes when the location varies, i.e., when an event occurs. The updated value may always be the same each time the event occurs, may depend on the value that the continuous state has before the event occurs, or may also be non deterministic, in the sense that the relation j may be one–to–many. In a HPN, on the other hand, the firing of a discrete transition produces a *constant variation* on the continuous marking, i.e., if the continuous marking *before* the transition firing is any vector $\boldsymbol{m}^c$ the updated marking $\tilde{\boldsymbol{m}}^c = \boldsymbol{m}^c + \boldsymbol{C}_{cd}(\cdot, t)$ will differ from it by an additive quantity $\boldsymbol{C}_{cd}(\cdot, t)$. Thus, while the jump relation of a HA may be used to associate to an event firing a variable variations of the continuous state, a HPN can only produce constant discrete marking variations. In particular, in HPNs the reset of the continuous marking is not possible. Furthermore, we remark that in HPNs a single transition may represent different events. Thus each transition firing updates not only the continuous marking but the discrete marking as well. The discrete marking updating is used to specify the updated discrete state (location) reached after the occurrence of the event from a given discrete marking (location).

To summarize, HPNs can be seen as a restriction of HA with the only exception that a HPN may have an infinite number of locations. Note, however, that the generality of HA has as a consequence the fact that most properties are undecidable unless very strong restrictions are added to the basic model. These restrictions sensibly reduce the gap between the interesting (i.e., decidable) classes of HA and HPNs. The strongest restriction of HPNs is the fact that the content of a continuous place cannot be reset to zero.

5 Single–rate hybrid Petri nets

In this section we define a special class of hybrid Petri nets called *single–rate* HPNs that can be seen as the net counterpart of timed automata with skewed clocks. It consists of a HPN where the continuous dynamics is such that the marking of each continuous place constantly increases with an integer slope.

Definition 9. *A single–rate hybrid Petri net (SRHPN) is a HPN where:*

- $T_c = \{t_c\}$,
- $^\bullet t_c = \emptyset$,
- $\mathcal{C}(t_c) = (v, v)$ *where* $v \in \mathbb{N}^+$,
- $\forall i \mid p_i \in P_c \,:\, Post(p_i, t_c) = w_i \in \mathbb{N}^+$ *and* $\{w_i\}$ *is a prime set, i.e., the* w_i*'s do not have a factor common to all of them,*
- $Pre, Post \in \mathbb{N}^{n \times q}$. ■

Thus a single–rate hybrid Petri net has a *single* continuous transition t_c that is always enabled — because it has no input places — and whose firing speed is constant. The marking of all continuous places increases with constant rate during a time step. Discontinuous variations of continuous markings may only follow the firing of discrete transitions.

The special structure of this net is such that at each step $\mathcal{S}(N, \boldsymbol{m}) = \{v\}$ is a singleton set and this set is always the same regardless of $\boldsymbol{m}$. It is important to note that all results presented in this paper still hold if we consider $\mathcal{C}(t_c) = (V', V)$. In this case the set of admissible IFSs $\mathcal{S}(N, \boldsymbol{m})$ is a segment and the marking of all continuous places may increase with different rates during a time step but the rates associated to different places always have the same ratio.

We have assumed without loss of generality that the set of all w_i — the weights of the arcs from the continuous transition t_c to the continuous places p_i — is a prime set. In fact, if this is not the case, we can always consider an equivalent net N' with the same structure as the original one but with different values of both v and w_i. The new firing speed would be $v' = v \cdot GCD(w_1, \cdots, w_{n_c})$ and the new weights would be $w'_i = w_i / GCD(w_1, \cdots, w_{n_c})$, where GCD denotes the greatest common divisor.

Furthermore, we assume that all arcs have integer weights. Such an assumption has been introduced for simplicity. In fact, whenever $Pre, Post \in \mathbb{Q}^{n \times q}$ all the weights could be multiplied by the least common multiple of the denominators of all the constants appearing in $Pre, Post$ to get a new hybrid net that is isomorphic with a new one where $Pre, Post \in \mathbb{N}^{n \times q}$. Even if $Pre, Post \in \mathbb{R}^{n \times q}$ but each weight has the same irrational numbers as common factors, an isomorphism with a net where $Pre, Post \in \mathbb{N}^{n \times q}$ can be determined.

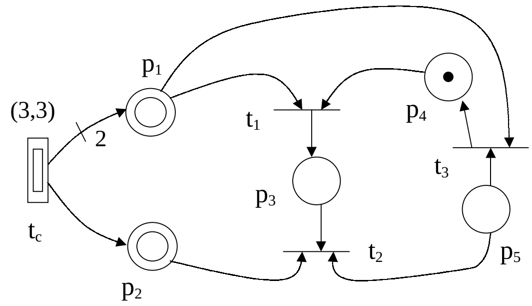

Figure 1: *A single-rate hybrid Petri net.*

The evolution of SRHPNs can be related to that of timed HA with skewed clocks. In fact, the continuous evolution (due to the firing of the transition t_c) is such that each continuous variable m_{p_i}, i.e., the marking of each continuous place p_i, has a constant derivative equal to vw_i during a time interval in which no discrete transition fires. Thus the derivative of each continuous variable is constant, but non necessarily equal to 1. Furthermore, different variables can have different derivatives. However, all the differences outlined in the previous section still hold. In particular, in SRHPNs the firing of a discrete transition may only produce constant variations on the continuous marking. On the other hand, SRHPNs can assume an infinite number of discrete states.

Example 10. The HPN in figure 1 is a SRHPN. It represents a production system with two continuous flows of parts (type 1 and type 2) that are put into two buffers (places p_1 and p_2). The batch processing of parts, represented by the cycle of discrete transitions, requires first a unit of part type 1, then a unit of part type 2 and then again a unit of part type 1. ■

Now, we prove that the reachability problem for SRHPNs is decidable.

Let us first define an equivalence relation on $(\mathbb{R}_0^+)^m$.

Definition 11. *Given a vector $\boldsymbol{w} = (w_1, \cdots, w_m) \in (\mathbb{N}^+)^m$ where $\{w_i\}$ is a prime set, we say that a vector $\boldsymbol{x} \in (\mathbb{R}_0^+)^m$ is w-consistent with $\boldsymbol{y} \in (\mathbb{R}_0^+)^m$ if:*

$$\exists\, b \in [0, 1) \;:\; \forall i = 1, \cdots, m,\; \langle y_i \rangle = \langle x_i + w_i\, b \rangle$$

where $\langle \cdot \rangle$ denotes the fractional part and we write
$\boldsymbol{x} \sim_w \boldsymbol{y}$. The equivalence classes of this relation are denoted $[\boldsymbol{x}]_w$. ■

Example 12. Let $\boldsymbol{x} = (0, 0.3)$ and $\boldsymbol{w} = (2 \;\; 1)^T$. In figure 2 the set of vectors w-consistent with $\boldsymbol{x}$ are represented in the plane (x_1, x_2) and lie on a family of parallel lines. All lines are equally spaced and are characterized by a constant slope equal to 2. ■

Definition 13. *Given a vector $\boldsymbol{x} \in (\mathbb{R}_0^+)^m$ and a vector $\boldsymbol{y} \in [\boldsymbol{x}]_w$, we define the vector $\boldsymbol{\gamma}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) = \boldsymbol{x} + \hat{\tau}\boldsymbol{w}$ where*

$$\hat{\tau} = \min\{\tau \geq 0 \mid \forall i = 1, \cdots, m,\; \langle x_i + \tau w_i \rangle = \langle y_i \rangle\}$$

and we call $\boldsymbol{\gamma}$ the w–cover of $\boldsymbol{x}$ with the same fractional part of $\boldsymbol{y}$. Note that $\hat{\tau} \in [0, 1)$ because $\{w_i\}$ is a prime set. ■

In plain words, if we consider a point $\boldsymbol{x}$ and if we move along the direction corresponding to the vector $\boldsymbol{w}$ — i.e., we move along the unique line in $[\boldsymbol{x}]_w$ passing through $\boldsymbol{x}$ — the w-cover $\boldsymbol{\gamma}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})$ is the first point we reach with the same fractional part of $\boldsymbol{y}$.

Now, let us provide a constructive algorithm to determine the numerical value of $\hat{\tau}$ and thus the vector $\boldsymbol{\gamma}$.

Algorithm 14. Observe that, since $\hat{\tau} \in [0,1)$, the integer part of each component of $\boldsymbol{\gamma}$ may differ from the integer part of the corresponding component of $\boldsymbol{x}$ by a quantity that belongs to the set:

$$\mathcal{I}_i = \begin{cases} \{0,1,\cdots,w_i-1\} & \text{if } \langle x_i \rangle \leq \langle y_i \rangle \\ \{1,2,\cdots,w_i\} & \text{otherwise.} \end{cases}$$

Compute $g_i = \langle \gamma_i \rangle - \langle x_i \rangle \equiv \langle y_i \rangle - \langle x_i \rangle$ (this is known) and define $k_i = \lfloor \gamma_i \rfloor - \lfloor x_i \rfloor$. Thus we need to solve for $\tau \in [0,1)$ the following system of equations

$$\gamma_i = x_i + \tau w_i, \quad (i = 1, \cdots, m),$$

that can be rewritten as

$$k_i + g_i = \tau w_i, \quad (i = 1, \cdots, m),$$

where the unknown terms are $\tau \in [0,1)$ and $k_i \in \mathcal{I}_i$.

If there exists one value of $\hat{\tau} \in [0,1)$ such that $\langle \boldsymbol{x} + \hat{\tau}\boldsymbol{w} \rangle = \langle \boldsymbol{y} \rangle$, then this value is unique and can be computed as follows.

for all $i = 1, 2, \cdots, m$
  begin
    $\mathcal{J}_i := \emptyset$;
    for all $k_i \in \mathcal{I}_i$, $\quad \mathcal{J}_i := \mathcal{J}_i \cup \{\dfrac{k_i + g_i}{w_i}\}$;
  end
$\hat{\tau} := \cap_{i=1}^{m} \mathcal{J}_i$;
$\boldsymbol{\gamma} := \boldsymbol{x} + \hat{\tau}\boldsymbol{w}$. □

Now, let us provide a necessary condition for a marking $\tilde{\boldsymbol{m}}$ to be reachable.

Lemma 15. *Let $(N, \boldsymbol{m})$ be a SRHPN system with $\boldsymbol{w} = \boldsymbol{C}_{cc}$, i.e., $Post(p_i, t_c) = w_i$ for $i = 1, \cdots, n_c$. If $\tilde{\boldsymbol{m}} \in R(N, \boldsymbol{m})$ then $\tilde{\boldsymbol{m}}^c \in [\boldsymbol{m}^c]_w$.*

Proof. If $\tilde{\boldsymbol{m}} \in R(N, \boldsymbol{m})$, then there exists a firing sequence $\sigma = \alpha_1, \alpha_2, \cdots, \alpha_k$ such that $\boldsymbol{m}[\alpha_1\rangle \boldsymbol{m}_1[\alpha_2\rangle \boldsymbol{m}_2 \cdots [\alpha_k\rangle \tilde{\boldsymbol{m}}$. It is enough to show that $\boldsymbol{m}_i^c \in [\boldsymbol{m}_{i-1}^c]_w$ and the result follows from the transitivity of the equivalence relation.

(Event step) Since all the arc weights are integers, the firing of a discrete transition produces no variation on the fractional parts of a continuous marking. Thus, if $\boldsymbol{m}_{i-1}[\alpha_i\rangle \boldsymbol{m}_i$ and $\alpha_i \in T_d$, then $\langle \boldsymbol{m}_{i-1} \rangle = \langle \boldsymbol{m}_i \rangle$ and $\boldsymbol{m}_i^c \in [\boldsymbol{m}_{i-1}^c]_w$.

(Time step) The firing of the continuous transition may produce a variation on the fractional parts of the continuous markings. A part from the simplest case of $\boldsymbol{w} = \boldsymbol{1}$, these variations have different magnitude. However, their ratio is always the same since the arc weights are constant. Thus, if $\alpha_i = \overline{\tau} \in \mathbb{R}^+$, then $\boldsymbol{m}_i^c = \int_0^{\overline{\tau}} \boldsymbol{C}_{cc} v(\tau) d\tau + \boldsymbol{m}_{i-1}^c$. However, $v(\tau)$ is constant and equal to v and $\boldsymbol{C}_{cc} = (w_1, \cdots, w_{n_c})^T$ by hypothesis, hence

$$\begin{cases} m_{i,p_1} = w_1\, v\, \overline{\tau} + m_{i-1,p_1} \\ \vdots \\ m_{i,p_{n_c}} = w_{n_c}\, v\, \overline{\tau} + m_{i-1,p_{n_c}}. \end{cases}$$

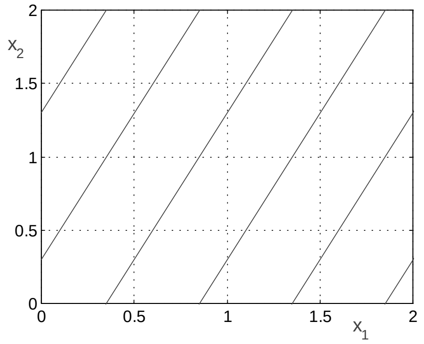

Figure 2: *The equivalence class* $[(0,0.3)]_w$, $\boldsymbol{w} = (2\ 1)^T$.

Now, let $b = \langle v\overline{\tau}\rangle$, then $\forall p \in P_c, \langle m_{i,p}\rangle = \langle m_{i-1,p} + w_i\, b\rangle$. Thus, $\boldsymbol{m}_i^c \in [\boldsymbol{m}_{i-1}^c]_w$. This completes the proof. □

Now, let us define a transformation on a hybrid Petri net.

Definition 16. *Let* $N = (P, T, Pre, Post, \mathcal{C})$ *be a HPN. We define* discretized PN associated *to* N *the P/T net* $\lfloor N\rfloor = (P', T', Pre', Post')$ *with:* $P' = P$*, i.e.,* $\lfloor N\rfloor$ *has as many places as* N*, but they are all discrete;* $T' = T$*, i.e.,* $\lfloor N\rfloor$ *has as many transitions as* N*, but they are all discrete;* $Pre'(p,t) = \lfloor Pre(p,t)\rfloor$ *and* $Post'(p,t) = \lfloor Post(p,t)\rfloor$*, where* $\lfloor\cdot\rfloor$ *denotes the integer part.* ■

Example 17. In figure 3 the discretized PN corresponding to the HPN in figure 1 is shown. ■

The following proposition shows that the discretized net can be used to determine if a marking $\boldsymbol{m}$ is reachable from $\boldsymbol{m}_0$ if the two makings have the same fractional part.

Proposition 18. *Let* $(N, \boldsymbol{m}_0)$ *be a SRHPN system and consider the discrete PN system* $(\lfloor N\rfloor, \lfloor \boldsymbol{m}_0\rfloor)$ *associated to* N*. Given any marking* $\boldsymbol{m}$ *with* $\langle \boldsymbol{m}\rangle = \langle \boldsymbol{m}_0\rangle$ *it holds* $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ *iff* $\lfloor \boldsymbol{m}\rfloor \in R(\lfloor N\rfloor, \lfloor \boldsymbol{m}_0\rfloor)$.

Proof. Let us denote $\bar{t}$ the discrete transition of $\lfloor N\rfloor$ corresponding to the continuous transition t_c in N and let v be the constant firing speed associated to t_c.

First, let us observe that $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ iff $\exists\, \sigma$ such that $\boldsymbol{m}_0[\sigma\rangle\boldsymbol{m}$. Since the continuous transition in $(N, \boldsymbol{m}_0)$ is always enabled, this implies that $\exists\, \tilde{\sigma} = \sigma_\tau\sigma_T$ such that $\boldsymbol{m}_0[\tilde{\sigma}\rangle\boldsymbol{m}$, where $\sigma_\tau \in \mathbb{R}_0^+$ and $\sigma_T \in T_d^*$, i.e., if $\boldsymbol{m}$ is reachable, then it may also be reached by a "normalized sequence" where a single time step occurs first, and all the event steps occur only at the end.

Since $\langle \boldsymbol{m}\rangle = \langle \boldsymbol{m}_0\rangle$, then $\boldsymbol{m}$ is reached from $\boldsymbol{m}_0$ by firing t_c for a time interval whose length is a multiple of $1/v$, i.e., $\sigma_\tau = k/v$.

Finally, the result follows from the fact that the firing of each discrete transition in N finds its counterpart in $\lfloor N\rfloor$ and the firing of t_c for a time interval of length $1/v$ corresponds to the firing of $\bar{t}$ in $\lfloor N\rfloor$. □

Now, we provide a necessary and sufficient condition for a marking $\boldsymbol{m}$ in a SRHPN to be reachable.

Theorem 19. *Let* $(N, \boldsymbol{m}_0)$ *be a SRHPN system with* $Post(p_i, t_c) = w_i$*, for* $i = 1, \cdots, n_c$*. Then,*

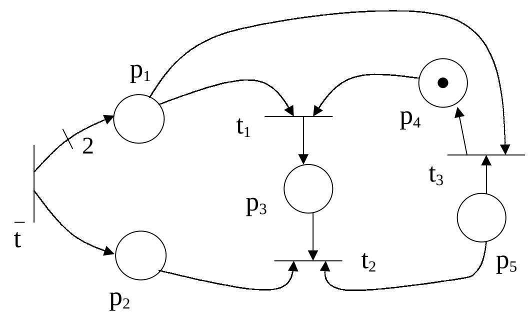

Figure 3: *The discretized PN corresponding to the SRHPN in figure 1.*

$\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ iff $\boldsymbol{m}^c \in [\boldsymbol{m}_0^c]_w$ and $\lfloor \boldsymbol{m} \rfloor \in R(\lfloor N \rfloor, \tilde{\boldsymbol{m}})$ where

$$\begin{cases} \tilde{\boldsymbol{m}}^c = \lfloor \boldsymbol{\gamma}(\boldsymbol{m}_0^c, \boldsymbol{m}^c, \boldsymbol{w}) \rfloor \\ \tilde{\boldsymbol{m}}^d = \boldsymbol{m}_0^d, \end{cases}$$

and $\lfloor N \rfloor$ is the discretized net associated to N.

Proof. As in the proof of the previous proposition, we observe that $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ iff there exists a normalized sequence $\sigma = \sigma_\tau \sigma_T$ such that $\boldsymbol{m}_0[\sigma\rangle\boldsymbol{m}$.

The firing sequence σ_τ can be written as $\sigma_\tau = \sigma'_\tau \sigma''_\tau$, where $\sigma'_\tau \in [0, 1/v)$, and $\sigma''_\tau = k/v$, with $k \in \mathbb{N}_0^+$. Therefore, $\boldsymbol{m}_0[\sigma'_\tau\rangle\boldsymbol{m}'_0[\sigma''_\tau\rangle\boldsymbol{m}'[\sigma_T\rangle\boldsymbol{m}$. Obviously, $\langle \boldsymbol{m}'_0 \rangle = \langle \boldsymbol{m}' \rangle = \langle \boldsymbol{m} \rangle$.

We further observe that the difference in the fractional part between $\boldsymbol{m}_0$ and $\boldsymbol{m}$ is due to the time step σ'_τ, that has a length less than $1/v$ and whose firing yields $\boldsymbol{m}'_0$ from $\boldsymbol{m}_0$. Moreover $\boldsymbol{m}'_0 = \int_0^{\sigma'_\tau} v\boldsymbol{C}_{cc} d\tau + \boldsymbol{m}_0 = v\sigma'_\tau \boldsymbol{C}_{cc} + \boldsymbol{m}_0 = b\boldsymbol{w} + \boldsymbol{m}_0$, where $b = v\sigma'_\tau \in [0, 1)$ and $\boldsymbol{w} = \boldsymbol{C}_{cc}$. Therefore, $\boldsymbol{m}'^c_0$ is exactly the w-cover of $\boldsymbol{m}_0^c$ with the same fractional part of $\boldsymbol{m}^c$, i.e., $\boldsymbol{m}'^c_0 = \boldsymbol{\gamma}(\boldsymbol{m}_0^c, \boldsymbol{m}^c, \boldsymbol{w})$ and the integer part of $\boldsymbol{m}'_0$ is exactly the marking $\tilde{\boldsymbol{m}}$ defined in the theorem statement.

Finally, by virtue of proposition 18, since $\boldsymbol{m}'_0$ and $\boldsymbol{m}$ have the same fractional part, then $\boldsymbol{m} \in R(N, \boldsymbol{m}'_0)$ if and only if $\lfloor \boldsymbol{m} \rfloor \in R(\lfloor N \rfloor, \lfloor \boldsymbol{m}'_0 \rfloor)$. □

Example 20. Let us consider the SRHPN system (N, m_0) in example 10 with initial marking $\boldsymbol{m}_0 = (1.3, 0.5, 0, 1, 0)^T$. Here $\boldsymbol{w} = (2, 1)^T$. We want to determine whether $\boldsymbol{m} = (5.1, 1.9, 0, 0, 1)^T \in R(N, \boldsymbol{m}_0)$ by applying theorem 19.

Clearly $\boldsymbol{m}^c \in [\boldsymbol{m}_0^c]_w$ because if we take $b = 0.4$, then $\forall p_i \in P_c$, $\langle m_{p_i} \rangle = \langle m_{0,p_i} + bw_i \rangle$.

By applying algorithm 14, we compute the marking $\boldsymbol{\gamma}(\boldsymbol{m}_0^c, \boldsymbol{m}^c, \boldsymbol{w}) = (2.1, 0.9)^T$ and we define $\tilde{\boldsymbol{m}} = (2.1, 0.9, 0, 1, 0)^T$ with $\tilde{\boldsymbol{m}}^c = \boldsymbol{\gamma}(\boldsymbol{m}_0^c, \boldsymbol{m}^c, \boldsymbol{w})$ and $\tilde{\boldsymbol{m}}^d = \boldsymbol{m}_0^d$.

If we consider the discretized PN in figure 3 we see that $\lfloor \boldsymbol{m} \rfloor = (5, 1, 0, 0, 1)^T$ is reachable from $(\lfloor \tilde{\boldsymbol{m}} \rfloor = (2, 0, 0, 1, 0)^T$. In fact, the firing sequence, say, $\overline{\sigma} = t_1 \; \overline{t} \; \overline{t} \; t_2$ is such that $\lfloor \tilde{\boldsymbol{m}} \rfloor \, [\overline{\sigma}\rangle \, \lfloor \boldsymbol{m} \rfloor$. Therefore, we can conclude that $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$. ■

By virtue of the above theorem 19, the results on the reachability of discrete Petri nets can be extended to SRHPNs, thus proving the validity of the following corollary.

Corollary 21. *The reachability problem is decidable for SRHPNs.*

Proof. Follows from theorem 19 and from the fact that the reachability problem is decidable for discrete PNs [10]. □

6 Conclusions

This work is part of a research activity aimed at exploring the properties of Hybrid Petri Nets. In particular we have considered here a special class of nets called called Single–Rate Hybrid Petri Nets.

The continuous dynamics of these nets is such that the vector of the marking derivatives of the continuous places is constant but for a scalar factor, thus these nets can be seen as the counterpart of timed automata with skewed clocks. We have proved that the reachability problem for this class can be reduced to the reachability problem of an equivalent discrete net and thus it is decidable.

Our future work will explore the properties of more general Hybrid Petri Net models.

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