

A STATE VARIABLE APPROACH FOR THE MODELING AND CONTROL OF FLEXIBLE MANUFACTURING SYSTEMS

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In this paper we present a new approach for the modeling and control of flexible manufacturing systems (FMS) characterized by unreliable machines, buffers of finite capacity, arbitrary service time distributions and deterministic sequencing and routing policies. Our main goal is the design of the FMS configuration embedded with its optimum control policy. The problem is addressed using first and second order fluid approximation obtained by splitting the process in two hierarchical layers and defining what we call micro and macro events. With an original approach this fluid approximation lends itself to a discrete time linear stochastic state variable model which offers average values and variances of both performance measures and of their gradients with respect to the most significant FMS parameters. Finally we investigate problems concerning the non-differentiability of the performance indices with respect to certain design parameters, specifically the control parameters as they appear as structural elements of the model.

1 Introduction

An FMS is a queueing network system where different classes of products are processed contemporaneously. Each product has to perform its own orderly sequence of operations, different for each class, in order to be completed. The same machine can perform operations on different product classes, eventually with different service times. The same operation can be performed on alternative machines. Flexibility is the capability of the FMS to cope in time with changing product class blend and production inconveniences such as buffer blockages and machine breakdowns, maintaining optimum production target, machine load balance and, if required, an assigned production mix.

The evolution in time of the FMS will be discussed within a framework that distinguishes two levels of aggregation. The lower layer represents the microscopic behavior of arrivals and departures of parts to/from each machine (*micro events*). It will be modeled in an aggregate view by using first and second order fluid approximation. At the higher layer a discrete event model will represent the transitions of the FMS through a sequence of operational states, that we call *macro states*, at the occurrence of the *macro events*. The degrees of freedom of an FMS are exploited by its control system by dynamically changing routing and priority of parts according to their class as a function of the current operational state of the shop-floor. The control is composed of two parts: a dispatcher and a monitoring system. During the permanence into each macro state, the dispatcher will perform myopically with heuristic rules based on the current state configuration, by sequencing parts waiting in the buffers and by routing them

at the exit of the machines according to their class. Similar ideas have been already adopted in their developments by Chen and Yao [2]. At the same time the monitoring system will detect macro events and will dictate the transition into a new macro state. The overall control behavior along with its heuristics will be accounted for in an aggregate view by assuming that at the occurrence of any macro event new reference values of all average flow rates will be computed for each product class as the solution of a linear programming (LP) problem. Then these reference values will be guaranteed by the control during the macro state.

The main contribution of our work is to represent both layers of the process by a discrete time linear stochastic state variable model that can be used to evaluate average values and variances of both performance measures and of their gradients with respect to the design parameters. A further result of this modeling approach consists in a neat presentation of the non-differentiability of performances with respect to certain parameters of the model, as already observed by other authors [10]. Finally this model avoids any other non-differentiability problem related to the perturbations made on structural parameters [5], such as routing coefficients and priority sequencing, as they are embedded along with the control law directly in the model, hence they are no more independent parameters.

1.1 Previous Results

A substantial body of literature about DEDS is concerning the analysis of production lines. Even so analytic results for the more general settings discussed here are available only for lines composed of 2 and 3 machines, while the analysis of longer lines involves the use of approximate

decomposition techniques and Markov models as suggested by Gershwin [6]. Since efficient analytical models are not available, discrete event simulation models along with gradient estimation, have been extensively used for analysis and design issues. Readers are referred to Ho [7] and Suri [10] for PA techniques and gradient estimation. Combining simulation and elementary queueing analysis Phillis and Kouikoglou [9] developed a discrete event continuous flow model more efficient than conventional simulators, to predict the system evolution. Then, by using PA techniques, they obtained the optimal distribution of repair rates among machines in order to maximize the throughput. Recently the fluid approximation theory has been deeply examined [3] and new results have been found.

2 Description of the Model

The queueing network considered in this work consists of a set of n single-server stations, denoted by M_i , for $i=1, \dots, n$, serving m_i classes of products, indexed by $r=1, \dots, m_i$.

Parts of different classes move generically from machine M_i to M_j according to the production cycle of their class and they are queued in buffers, one for each machine, with the initial one acting as a limited supply of products, thus representing the production target, and the final buffer acting as a limited storage area for collecting finished products.

The buffers in front of the machines have finite capacity C_i and the machines are unreliable. We consider failures as *operation-dependent* failures and we define for each machine the *Production Volume Before Breaking*, denoted by w_i , and the *Repairing Time*, denoted by d_i , both assumed as independent identically distributed random variables characterized by their mean values and variances. Machine service times are also assumed independent random variables with identical distribution with mean τ_i^r and finite variance.

We denote by $\mathbf{v}_{\text{MAX}} = [\dots, v_{\text{MAX},i}^1, \dots, v_{\text{MAX},i}^m, \dots]^T$ the maximum average machine production rates vector, i.e. $v_{\text{MAX},i}^r = 1/\tau_i^r$. The buffer capacity vector is indicated with $\mathbf{C} = [C_1, \dots, C_n]^T$ and the vectors of the average values of the reliability random variables are denoted by $\mathbf{w} = [\bar{w}_1, \dots, \bar{w}_n]^T$ and $\mathbf{d} = [\bar{d}_1, \dots, \bar{d}_n]^T$. Finally we represent by \tilde{w}_i and \tilde{d}_i the samples of the random variables w_i and d_i drawn during a simulation run.

For convenience in this presentation all design parameters are denoted by $\Theta = [\Theta_1 \ \Theta_2]$ and grouped according to $\Theta_1 = \mathbf{v}_{\text{MAX}}$ and $\Theta_2 = [\mathbf{C}^T \ \mathbf{w}^T \ \mathbf{d}^T]^T$.

2.1 The Microscopic Layer: Fluid and Diffusion Approximations

Fluid and diffusion approximations involve smoothing of discrete event processes [3] and they are often used to describe the asymptotic behavior of queueing systems via functional strong law of large numbers [4, 8]. We consider fluid models under conditions of balanced heavy loading and we adopt such approximations for modeling the microscopic behavior of an FMS during the permanence into the macro states, called *macro periods*, indicated with $\Delta_k = [t_k, t_{k+1})$ for $k=0, 1, 2, \dots$, where t_k denote the occurrences in time of the macro events. Further, within each macro state we assume that the stochastic processes which describe the queueing system are stationary. Then the global behavior of the FMS over the entire time horizon will be piecewise stationary.

Let $\{\delta_{i,j}^r(t), t \geq 0\}$ be the stochastic counting process accounting for the number of departures of parts of class r from machine M_i to M_j . We derive a second-order fluid approximation to describe these counting processes, assumed independent among different classes of products and different sources and destinations, by continuous processes denoted by $\{N_{i,j}^r(t), t \geq 0\}$, that have mean values given by first-order fluid quantities, the *flow rates* $v_{i,j}^r(t)$, and finite variances given by *Brownian motions* $\{\psi_{i,j}^r(t), t \geq 0\}$ which describe the fluctuations about those means. These Brownian processes are also assumed independent among different sources and destinations and different classes of products. Let us define

$$N_{in,i}^r(t_k) = \sum_h N_{h,i}^r(t_k), \quad N_{in,i}(t_k) = \sum_r N_{in,i}^r(t_k)$$

$$N_{out,i}^r(t_k) = \sum_j N_{i,j}^r(t_k), \quad N_{out,i}(t_k) = \sum_r N_{out,i}^r(t_k)$$

as the input and output fluid processes at the beginning of each macro period,

$$v_{in,i}^r(t_k) = \sum_h v_{h,i}^r(t_k), \quad v_{in,i}(t_k) = \sum_r v_{in,i}^r(t_k)$$

$$v_{out,i}^r(t_k) = \sum_j v_{i,j}^r(t_k), \quad v_{out,i}(t_k) = \sum_r v_{out,i}^r(t_k)$$

as the inflow and outflow rates of parts which are constant values in the interval $[t_k, t_{k+1})$, and

$$\tilde{\psi}_{i,j}^r(t_k, t_{k+1}) = \psi_{i,j}^r(t_{k+1}) - \psi_{i,j}^r(t_k)$$

$$\tilde{\psi}_{out,i}^r(t_k, t_{k+1}) = \sum_j \tilde{\psi}_{i,j}^r(t_k, t_{k+1}), \quad \tilde{\psi}_{in,i}^r(t_k, t_{k+1}) = \sum_h \tilde{\psi}_{h,i}^r(t_k, t_{k+1})$$

as the independent increments of the Brownian processes in the interval $[t_k, t_{k+1})$. We find that [4, 8]:

$$N_{out,i}^r(t_{k+1}) = N_{out,i}^r(t_k) + v_{out,i}^r(t_k) \cdot (t_{k+1} - t_k) + \tilde{\psi}_{out,i}^r(t_k, t_{k+1})$$

$$\text{var}[\tilde{\psi}_{out,i}^r(t_k, t_{k+1})] = \sigma_{out,i}^2 \cdot v_{out,i}^r \cdot (t_{k+1} - t_k) \quad (1)$$

$$N_{in,i}^r(t_{k+1}) = N_{in,i}^r(t_k) + v_{in,i}^r(t_k) \cdot (t_{k+1} - t_k) + \tilde{\psi}_{in,i}^r(t_k, t_{k+1})$$

$$\text{var}[\tilde{\psi}_{in,i}^r(t_k, t_{k+1})] = \sigma_{in,i}^2 \cdot v_{in,i}^r \cdot (t_{k+1} - t_k) \quad (2)$$

where $\sigma_{out_i}^2$ and $\sigma_{in_i}^2$ are the variances of interarrival and interdeparture times for parts of class r at the input and output of each machine. Then the buffer levels for parts of class r are indicated by $x_i^r(t)$ and they are simply given by the following equation:

$$x_i^r(t_{k+1}) = x_i^r(t_k) + [v_{in,i}^r(t_k) - v_{out,i}^r(t_k)] \cdot (t_{k+1} - t_k) + \tilde{\Psi}_{in,i}^r(t_k, t_{k+1}) - \tilde{\Psi}_{out,i}^r(t_k, t_{k+1}) \quad (3)$$

2.2 The Macroscopic Layer: Macro Events and Macro States

At the macroscopic level the FMS evolves through a sequence of *macro states*, characterized by the functional status of the physical components of all services: the machines (*operational* or *down*) and the buffers (*full*, *not full-not empty*, *empty*).

The set of macro event types is indicated with $E = \{F_i, R_i, RC_i, BE_{i,r}, BF_i\}$, which elements are defined as follows:

- F_i (*Failure of machine M_i*). After a machine is repaired, failures occur after the production volume \tilde{w}_i .
- R_i (*Repair of machine M_i*). When a machine fails, it will be repaired after \tilde{d}_i time units.
- BF_i (*Buffer Full at machine M_i*). The buffer level reaches its capacity C_i while $v_{in,i}(t) > v_{out,i}(t)$ ¹.
- $BE_{i,r}$ (*Buffer Empty for parts of class r at machine M_i*). The buffer level for parts of class r reaches 0 while $v_{in,i}^r(t) < v_{out,i}^r(t)$ ¹.
- RC_i (*outflow Rate Changed for machine M_i*). When the machine is operational, upon the occurrence of exogenous macro events, changes occur in the inflow and outflow rates of parts of machine M_i .

The set of admissible macro states is denoted by $S = \{(m_1, b_1), \dots, (m_n, b_n)\}$. Each element $(m_i, b_i) \in S$ combines the functional status of the physical components of each service, where symbols m_i and b_i represent the machine status and the buffer status during the macro period Δ_k , respectively. Then the machine status are:

- $M_{i,O}$ (*Machine Operational*). Machine M_i reaches this state at the occurrence of macro events R_i and then leaves it at the occurrence of macro events F_i .
- $M_{i,B}$ (*Machine Broken*). Machine M_i reaches this state at the occurrence of macro events F_i and then leave it at the occurrence of macro events R_i . The output process will be $N_{out,i}(t) = 0$ during that period.

The buffer status are:

- $B_{i,F}$ (*Buffer Full*). State reached at the occurrence of macro events BF_i . Services leave it at the occurrence of any macro event which will result in the condition $v_{in,i}(t) < v_{out,i}(t)$. The input process will follow the mean of the output process, even though we will assume independent random noises for the two processes.
- $B_{i,E}$ (*Buffer Empty for parts of class r*). State reached at the occurrence of macro events $BE_{i,r}$. Services leave it at the occurrence of any macro event which will result in the condition $v_{in,i}^r(t_k) > v_{out,i}^r(t_k)$. The input process associated with the inflow of parts of class r will follow the mean of the corresponding output process, even though we will assume independent random noises for the two processes.
- $B_{i,N}$ (*Buffer not-Full not-Empty*). State reached at the occurrence of exogenous macro events RC_i . Operational services will be considered operating under the heavy traffic conditions, and the buffer levels will be given by Eq. (3).

3 The Dynamic Control Problem

The dynamic control policy adopted in this work generates myopically a piecewise optimal solution obtained via a sequence of LP problems, one for each macro period, where each solution represents the average machine production rates $v_{i,j}^r(t_k)$ to be expected for the interval of time $[t_k, t_{k+1})$. During the permanence into a macro state the control system will perform its task by sequencing parts waiting in the buffers and by routing them at the exit of the machines according to their class, aimed at maintaining these average flow rates as reference values. First we apply our approach to a tandem production line in order to show that a general FMS can be treated as a simple multi variable extension of this elementary case.

3.1 Single-Machine Single-Class Tandem Line

As shown by several authors, e.g. Phillis and Kouikoglou [9], the fluid model simulation of a tandem line is achieved by adjusting the flow rates at the occurrence of macro events by means of a set of simple rules. We give a formal description to such logic rules as an LP problem. In fact when the system enters a new macro state, the average flow rates as reference values for the next macro period, are determined as the solution of the following LP problem:

$$J(\Theta) = \max_{v_{out,i}} \sum_i v_{out,i}(t_k) \quad \text{subject to:} \quad (4)$$

$$v_{out,i}(t_k) \leq v_{MAX,i} \quad \text{for any operational machine}$$

$$v_{in,i}(t_k) \leq v_{out,i}(t_k) \quad \text{for machines that have full buffer}$$

$$v_{out,i}(t_k) \leq v_{in,i}(t_k) \quad \text{for machines that have empty buffer}$$

¹ This specification that seems obvious is done to distinguish macro from micro events.

The performance index defined in (4) states that any operational machine will be working at its maximum production rate allowed by the line.

Proposition 3.1 *Let us consider tandem lines composed of n machines and $n+1$ buffers. We assert that a basis for the LP problem (4) consists of at most $2n-1$ linearly independent vectors and that all basis inverses contain only the entries 0, 1, -1.*

Remarks. The average machine production rates are continuous and piecewise linear (slope 0, 1, -1) functions in the maximum machine production rates. Thus the outflow rates vector is everywhere differentiable with respect to the vector Θ_1 , except for a finite number of points corresponding to the boundary points of the basis characteristic intervals, whereas we can instead compute left and right derivatives.

3.2 Multi-Machine Multi-Class Production Line

With the same logic adopted in the previous section, the multi-machine multi-class case is approached here as a simple extension of the tandem line by building up a similar LP problem.

$$0 \leq \sum_r \sum_h v_{i,h}^r(t_k) \leq v_{MAX,i}^r(\Theta_1) \quad (5a)$$

$$0 \leq \sum_r \sum_h \frac{1}{v_{MAX,i}^r(\Theta_1)} v_{i,h}^r(t_k) \leq 1 \quad (5b)$$

Constraint (5a) will apply whenever service times on the same machine are identical for all classes while constraint (5b) whenever service times are different.

$$\sum_r \sum_j v_{j,i}^r(t_k) \leq \sum_r \sum_h v_{i,h}^r(t_k) \quad \text{buffer full} \quad (6)$$

$$\sum_h v_{i,h}^r(t_k) \leq \sum_j v_{j,i}^r(t_k) \quad \text{buffer empty for parts of class } r \quad (7)$$

Constraint (6) has to be satisfied for machine M_i whenever its buffer is full and constraint (7) whenever the size of parts of class r is 0. Finally we define the constraint that the production, denoted by P_d , has to satisfy a certain mix:

$$\sum_i v_{out,i}^r(t_k) = m^r \cdot P_d \quad (8)$$

The goal is to maximize machines utilization, therefore we define the following performance index:

$$J(\Theta) = \max_{v_{out,i}^r} \sum_r \sum_i v_{out,i}^r(t_k) \quad (9)$$

Let $\mathbf{v}_{out}(t_k) = [\dots, v_{i,j}^r(t_k, \Theta_1), \dots]^T$ be the optimal outflow rates vector solution of the LP problem and that will represent the input data for the controller. It should be

noted that this solution implicitly resolves all routing and sequencing issues in an optimal manner.

4 Sensitivity Analysis of the LP problem

We perform sensitivity analysis of the solutions of the LP problem stated in the previous section with respect to the parameters Θ_1 . For simplicity in this presentation, we will only undertake the analysis of the parametric right-hand side problem, assuming identical service times for each class processed at the same machine and we make only use of constraint (5a).

Let \mathbf{B} be the basis heading of an optimal basic solution of (5-9). Then \mathbf{A}_B is the basis matrix and \mathbf{b} denotes the right-hand side vector. The optimal solution is obtained as $\mathbf{V}_B(t_k) = \mathbf{A}_B^{-1} \cdot \mathbf{b}(\Theta_1)$ and its gradient matrix with respect to \mathbf{b} can be easily computed as:

$$\nabla_{\mathbf{b}(\Theta_1)} \mathbf{V}_B(t_k) = \mathbf{A}_B^{-1} \quad (10)$$

Furthermore the gradient matrix of \mathbf{v}_{out} with respect to Θ_1 , denoted by $\mathbf{M} = \nabla_{\Theta_1} \mathbf{v}_{out}(t_k)$, will be given by entries of \mathbf{A}_B^{-1} for those rates $v_{i,j}^r(t_k)$ which belongs to the optimal basis and zero elsewhere.

Proposition 4.1 *We assert that the solution of the LP problem (5-9) is continuous in the design parameter vector Θ_1 even when the perturbed parameter determines changes in the optimal basis, and it is non-differentiable in a finite number of points of the real axis corresponding to the boundary points of the characteristic intervals of the optimal basis.*

Hence the derivatives of the optimum outflow rates with respect to the maximum machine production rates are not defined only at selected points. We can instead provide right and left derivatives that can be easily obtained from the solution of the LP problem itself. Finally performance measures and production rates are piecewise linear and continuous functions in the design parameter vector Θ_1 . This non-differentiability behavior has been already observed in tandem lines by Suri [10] when the average production rates of two machines are equal. With our approach the result is immediately generalized to arbitrary queuing network systems.

5 A Discrete Time State Variable Model

We now develop a discrete time linear stochastic state variable model, which samples are the occurrence of the macro events at time t_k , for $k=0,1,2,\dots$. Then this model embeds all the information required to evaluate performance measures, their gradients with respect to the

system parameters and the uncertainty introduced by the fluid approximation. We define for each service the following set of equations:

$$x_i^r(t_{k+1}) = x_i^r(t_k) + [v_{in,i}^r(t_k) - v_{out,i}^r(t_k)] \cdot u(k) + \quad (11)$$

$$+ \tilde{\psi}_{in,i}^r(t_{k+1}, t_k) - \tilde{\psi}_{out,i}^r(t_{k+1}, t_k), \quad \text{for } r = 1, \dots, m_i$$

$$\chi_i(t_{k+1}) = \alpha_i(k) \cdot \chi_i(t_k) + v_{out,i}(t_k) \cdot u(k) + \quad (12)$$

$$+ \sum_{r=1}^{m_i} \tilde{\psi}_{out,i}^r(t_k, t_{k+1}), \quad \alpha_i(k) = 0,1$$

$$s_i(t_{k+1}) = \beta_i(k) \cdot s_i(t_k) + \beta_i(k) \cdot u(k), \quad \beta_i(k) = 0,1 \quad (13)$$

where $t_{k+1} = t_k + u(k)$. At the occurrence of any macro event Eq. (11) account for the current class-specific buffer levels while the current cumulative buffer levels are simply obtained as $x_i(t_k) = \sum_r x_i^r(t_k)$. Equations (12) and (13) represent the partial production volume currently processed by the machine next to fail since the last repair, and the time spent by the machine under repairing since the last failure, respectively. Indicators α_i and β_i are switched from 0 to 1 at the occurrence of the proper macro events. Then the state of the system is represented by the vector $\mathbf{x}(k)$ defined as:

$$\mathbf{x}(k) = [t_k, x_1^1(t_k), \dots, x_1^{m_1}(t_k), \chi_1(t_k), s_1(t_k), \dots, \dots, x_n^1(t_k), \dots, x_n^{m_n}(t_k), \chi_n(t_k), s_n(t_k), x_{n+1}(t_k)]^T$$

The total number of classes processed at all machines is $d = m_1 + \dots + m_n$, therefore the dimension of the state vector is $(d+2n)+2$. In fact there are $(d+2n)$ equations as defined in (11-13) plus the two extra equations:

$$t_{k+1} = t_k + u(k)$$

$$x_{n+1}(t_{k+1}) = x_{n+1}(t_k) + v_{out,n}(t_k) \cdot u(k) + \sum_{r=1}^{m_n} \tilde{\psi}_{out,n}^r(t_{k+1}, t_k) \quad (14)$$

where state variables t_k and $x_{n+1}(t_k)$ occupy the first and the last position within vector $\mathbf{x}(k)$, respectively. These equations account for the current time of the occurring macro event and for the final storage area (total production).

We can now formally described the behavior of the system by defining a discrete time linear stochastic state variable model expressed in matrix notation by:

$$\mathbf{x}(k+1) = \mathbf{A}(k) \cdot \mathbf{x}(k) + \mathbf{B}(k, \mathbf{v}_{out}(\Theta_1)) \cdot u(k) + \mathbf{G}(k) \cdot \mathbf{n}(k)$$

$$\mathbf{Q}(k) = \text{cov}[\mathbf{n}(k)] \quad (15)$$

where

$$\mathbf{n}(k) = [\dots, \tilde{\psi}_{i,j}^1(t_{k+1}, t_k), \dots, \tilde{\psi}_{i,j}^{m_i}(t_{k+1}, t_k), \dots]^T$$

$$\mathbf{Q}(k) = \text{diag}(\dots, \text{var}[\tilde{\psi}_{i,j}^1(t_{k+1}, t_k)], \dots, \text{var}[\tilde{\psi}_{i,j}^{m_i}(t_{k+1}, t_k)], \dots)$$

Matrix $\mathbf{A}(k)$ is diagonal with entries 0 and 1. Vector $\mathbf{B}(k, \mathbf{v}_{out}(\Theta_1))$ has elements that depend on the average machine outflow rates. Vectors $\mathbf{n}(k)$ are independent random vectors as made of increments of Brownian

motions, accounting for the approximation introduced in the fluid model. Matrix $\mathbf{G}(k)$ has entries 0 and ± 1 accounting for the inflows and outflows of parts at any machine. Finally, assuming at each source-destination and for each class independent noise processes, matrix $\mathbf{Q}(k)$ will be simply diagonal with entries of the form $q_{l,l} = \tilde{\sigma} \cdot (t_{k+1} - t_k)$, where in general $\tilde{\sigma} = \sigma^2 \cdot v^{-3}$ as given in (1) and (2) or $\tilde{\sigma} = v_{i,j}^r(t_k, \Theta_1)$ if Poisson processes are assumed.

Note that with this model any macro event does occur when a certain state variable, or the linear combination $x_i(t_k) = \sum_r x_i^r(t_k)$, reaches a specified value. Specifically when a machine breaks down or gets repaired it must result $\chi_i(t_{k+1}) = \tilde{w}_i$ or $s_i(t_{k+1}) = \tilde{d}_i$. When a buffer gets full or empty for parts of class r the condition $\sum_r x_i^r(t_{k+1}) = C_i$ or $x_i^r(t_{k+1}) = 0$ will be satisfied. We indicate with $r(k+1)$ these values reached by the appropriate state variables at the occurrence of the macro events. It does result:

$$r(k+1) = \mathbf{e}_i^T(k) \cdot \mathbf{x}(k+1) = \quad (16)$$

$$= \mathbf{H}(k) \cdot \mathbf{x}(k) + b(k, \mathbf{v}_{out}) \cdot u(k) + \mathbf{e}_i^T(k) \cdot \mathbf{G}(k) \cdot \mathbf{n}(k)$$

$$u(k) = K(k) \cdot [r(k+1) - \mathbf{H}(k) \cdot \mathbf{x}(k) + \quad (17)$$

$$- \mathbf{e}_i^T(k) \cdot \mathbf{G}(k) \cdot \mathbf{n}(k)]$$

where $\mathbf{e}_i(k)$ is a vector with entries 0 and 1 which selects a state variable or the linear combination $\sum_r x_i^r(t_k)$ within $\mathbf{x}(k)$ that is leading to the new transition, and

$$\mathbf{H}(k) = \mathbf{e}_i^T(k) \cdot \mathbf{A}(k)$$

$$b(k, \mathbf{v}_{out}(\Theta_1)) = \mathbf{e}_i^T(k) \cdot \mathbf{B}(k) \quad \text{and} \quad K(k) = \frac{1}{b(k, \mathbf{v}_{out})}$$

Equation (15) can be straightforward rewritten as a closed-loop system:

$$\mathbf{x}(k+1) = \mathbf{A}_H(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{x}(k) + \quad (18)$$

$$+ K(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{B}(k, \mathbf{v}_{out}(\Theta_1)) \cdot r(k+1, \Theta_2) +$$

$$+ [\mathbf{I} - K(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{B}(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{e}_i^T(k)] \cdot \mathbf{G}(k) \cdot \mathbf{n}(k)$$

where

$$\mathbf{A}_H(k, \mathbf{v}_{out}(\Theta_1)) = \mathbf{A}(k) - K(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{B}(k, \mathbf{v}_{out}(\Theta_1)) \cdot \mathbf{H}(k)$$

5.1 Propagation of Mean and Covariance of the State of the System

From Eq. (18) the mean $\bar{\mathbf{x}}(k)$ of the state of the system may be calculated by propagating the following equation:

$$\bar{\mathbf{x}}(k+1) = \mathbf{A}_H(k, \mathbf{v}_{out}(\Theta_1)) \cdot \bar{\mathbf{x}}(k) + \quad (19)$$

$$+ \mathbf{B}(k, \mathbf{v}_{out}(\Theta_1)) \cdot K(k, \mathbf{v}_{out}(\Theta_1)) \cdot r(k+1, \Theta_2)$$

starting from the initial value $\bar{\mathbf{x}}(0)$. Eq. (19) allows us to evaluate performance measure of the form $J = f(\bar{\mathbf{x}}, \Theta)$,

gradients and their uncertainty, by the completion time of a certain production target or by the productivity achieved by the lowest product class.

The behavior of the covariance matrix $\mathbf{P}_{11}(k)$ of the state of the system is given by equations:

$$\begin{aligned}\mathbf{P}_{11}(k+1) &= \mathbf{A}_H(k) \cdot \mathbf{P}_{11}(k) \cdot \mathbf{A}_H^T(k) + \tilde{\mathbf{G}}(k) \cdot \mathbf{Q}(k) \cdot \tilde{\mathbf{G}}(k)^T, \quad k \geq 1 \\ \mathbf{P}_{11}(0) &= \text{cov}[\mathbf{x}(0)]\end{aligned}\quad (20)$$

where

$$\tilde{\mathbf{G}}(k) = [\mathbf{I} - \mathbf{K}(k, \mathbf{v}_{\text{out}})] \cdot \mathbf{B}(k, \mathbf{v}_{\text{out}}) \cdot \mathbf{e}_i^T(k) \cdot \mathbf{G}(k)$$

We indicate with Φ the state transition matrix of equation (18):

$$\Phi(k, j) = \mathbf{A}_H(k-1) \cdot \dots \cdot \mathbf{A}_H(j-1) \cdot \mathbf{A}_H(j), \quad k > j$$

Proposition 5.1 *We assert that $\Phi(k, j)$ is element-wise bounded, i.e. $-\mathbf{B} \leq \Phi(k, j) \leq \mathbf{B}$. Hence there are finite matrices \mathbf{S} and \mathbf{N} of appropriate dimension such that the covariance matrix $\mathbf{P}_{11}(k)$ is asymptotic to $\mathbf{N} + k \cdot \mathbf{S}$ as $k \rightarrow \infty$.*

5.2 Sensitivity Analysis of the State of the System

We can now evaluate the sensitivity of the state of the system with respect to the design parameters Θ . Recalling that $\mathbf{M} = \nabla_{\Theta_1} \mathbf{v}_{\text{out}}(t_k)$, the gradient of $\mathbf{x}(k)$ with respect to Θ_1 ($\nabla_{\mathbf{v}_{\text{out}}} \mathbf{x} \cdot \nabla_{\Theta_1} \mathbf{v}_{\text{out}}$) is given by:

$$\begin{aligned}\nabla_{\Theta_1} \mathbf{x}(k+1) &= \mathbf{A}_H(k) \cdot \nabla_{\Theta_1} \mathbf{x}(k) + \\ &+ [r(k+1) - \mathbf{H}(k) \cdot \mathbf{x}(k)] \cdot \nabla_{\mathbf{v}_{\text{out}}} [\mathbf{B}(k) \cdot \mathbf{K}(k)] \cdot \mathbf{M} + \\ &- \mathbf{e}_i^T(k) \cdot \mathbf{G}(k) \cdot \mathbf{n}(k) \cdot \nabla_{\mathbf{v}_{\text{out}}} [\mathbf{B}(k) \cdot \mathbf{K}(k)] \cdot \mathbf{M}\end{aligned}\quad (21)$$

We observe that the gradient matrix (21) does depend on the current state and the input noise, hence its entries are affected by noise. The gradient of $\mathbf{x}(k)$ with respect to Θ_2 is computed as:

$$\begin{aligned}\nabla_{\Theta_2} \mathbf{x}(k+1) &= \mathbf{A}_H(k) \cdot \nabla_{\Theta_2} \mathbf{x}(k) + \\ &+ \mathbf{B}(k, \mathbf{v}_{\text{out}}) \cdot \mathbf{K}(k, \mathbf{v}_{\text{out}}) \cdot \nabla_{\Theta_2} r(k+1, \Theta_2)\end{aligned}\quad (22)$$

The gradient matrix of $\mathbf{x}(k)$ with respect to Θ is obtained by propagating equations (21) and (22). In order to evaluate the variance of each gradient entry with respect to Θ_1 , the covariance matrices of each column of the gradient matrix for each $v_{\text{MAX},i}$ have to be propagated separately by using the following Lyapunov equation:

$$\begin{aligned}\begin{bmatrix} \mathbf{P}_{11}(k+1) & \mathbf{P}_{12,i}(k+1) \\ \mathbf{P}_{12,i}^T(k+1) & \mathbf{P}_{22,i}(k+1) \end{bmatrix} &= \tilde{\mathbf{A}}_i(k) \cdot \begin{bmatrix} \mathbf{P}_{11}(k) & \mathbf{P}_{12,i}(k) \\ \mathbf{P}_{12,i}^T(k) & \mathbf{P}_{22,i}(k) \end{bmatrix} \cdot \tilde{\mathbf{A}}_i^T(k) + \\ &+ \mathbf{G}_H(k) \cdot \mathbf{Q}(k) \cdot \mathbf{G}_H^T(k)\end{aligned}\quad (23)$$

where

$$\tilde{\mathbf{A}}_i(k) = \begin{bmatrix} \mathbf{A}_H(k) & \mathbf{0} \\ -\frac{\partial(\mathbf{B} \cdot \mathbf{K})}{\partial \Theta_{1,i}} \cdot \mathbf{H}(k) & \mathbf{A}_H(k) \end{bmatrix}, \quad \mathbf{G}_H(k) = \begin{bmatrix} \mathbf{I} \\ -\frac{\partial(\mathbf{B} \cdot \mathbf{K})}{\partial \Theta_{1,i}} \cdot \mathbf{e}_i^T(k) \end{bmatrix} \cdot \mathbf{G}(k)$$

The diagonal entries of matrices $\mathbf{P}_{22,i}(k)$ represent the variances of the gradient of $\mathbf{x}(k)$ with respect to the maximum production rates of machine M_i .

6 Conclusions

The main contribution of this paper has been to offer a model of an FMS where each design configuration is embedded with its optimal control strategy, so that while performing the simulation for a certain configuration, this is done under optimal control law. We achieve this result by solving a sequence of LP problems which emulate in the fluid model the behavior of the discrete event system in presence of a real-time dispatcher that makes use of a myopic strategy for implementing a multi-class queueing policy and routing, in order to continuously maximize machines utilization and guaranteeing mix balance. Embedding the optimal control law into the system model we avoid non-differentiability problems related to the perturbations made on structural parameters, such as routing coefficients and priority sequencing [1, 5]. Those parameters are no more independent variable.

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