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# Weak (approximate) detectability of labeled Petri net systems with inhibitor arcs ⋆

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**Abstract:** Weak (approximate) detectability of a labeled Petri net (LPN) system (with inhibitor arcs) is a property such that if the property is satisfied then there exists an infinite label sequence generated by the system such that all markings after a time step can determined (in a prescribed subset of reachable markings) by the label sequence. Specifically, we prove that the problems of deciding weak detectability of LPN systems with inhibitor arcs and weak approximate detectability of LPN systems are both undecidable.



*Keywords:* Petri net with inhibitor arcs, weak detectability, weak approximate detectability, decidability

## 1. INTRODUCTION

*Detectability* is a property that describes whether the current and all subsequent states of a system can be determined by observing output sequences (Giua and Seatzu, 2002; Shu et al., 2007; Shu and Lin, 2011, 2013; Fornasini and Valcher, 2013; Xu and Hong, 2013; Zhang et al., 2016). It plays a fundamental role in many related control problems such as observer design and controller synthesis. Hence for different applications, it is meaningful to characterize detectability for control systems in different frameworks.

For *discrete event systems* (DESs), the verification problem for detectability in the framework of *finite automata* has been widely studied (Shu et al., 2007; Shu and Lin, 2011, 2013; Zhang, 2017; Masopust, 2017; Yin and Lafortune, 2017; Yin, 2017; Keroglou and Hadjicostis, 2015). For different uses, detectability is formulated as *strong detectability* and *weak detectability* (Shu et al., 2007), where the former describes whether any sufficiently long output sequence (corresponding to observable event sequence) can determine the current and all subsequent states, while the latter means whether some sufficiently long output sequence can do that. Strong detectability can be verified in polynomial time but weak detectability can only be verified in exponential time currently (Shu et al., 2007; Shu and Lin, 2011). It is proved that the problem of deciding weak detectability of DESs in the framework of finite automata is PSPACE-complete even for only deterministic DESs whose events are all observable (Zhang, 2017), hence it is unlikely that there exists a polynomial time algorithm for verifying weak detectability. Then what if the framework of *Petri nets* is considered? Different from the finiteness of states and events of finite-automaton-based DESs, although Petri-net-based DESs have finitely many transitions (i.e., events), they may have at most countably infinitely many markings (i.e., states). Hence the weak detectability for Petri-net-based DESs may be more complex than that for finite-automaton-based DESs.

Taking opacity for example, where opacity is a property that describes whether an intruder (outside a system) can never determine whether some states of the system prior to the current time step are secret, although the problems of verifying different types of opacity of finite-automaton-based DESs are at least NP-hard, they are decidable (Saboori and Hadjicostis, 2012, 2011, 2007, 2013) (stochastic finite automata excluded (Saboori and Hadjicostis, 2014)). However, the opacity verification problems are generally undecidable (Bryans et al., 2008; Jacob et al., 2016; Tong et al., 2017). Then it is interesting to study whether from the perspective of detectability, whether Petri-net-based DESs are more complex than finite-automaton-based DESs. In this paper, we obtain related preliminary results.

The contributions of this paper are as follows. We prove that 1) the problem of deciding weak detectability of *labeled Petri net* (LPN) systems with *inhibitor arcs* is undecidable, and 2) the problem of deciding *weak approximate detectability* of LPN systems is also undecidable, where weak approximate detectability means that whether some sufficiently long label sequence (corresponding to output sequence) can determine the current and all subsequent markings in one of some prescribed subsets of reachable markings.

The remainder of the paper is arranged as follows. Section 2 introduces necessary preliminaries, Section 3 shows the main results, and Section 4 ends up with some remarks.

## 2. PRELIMINARIES

For a finite set $S$, $S^*$ and $S^\omega$ are used to denote the sets of finite sequences (called *words*) of elements of $S$ including the empty word $\epsilon$ and infinite sequences (called *configurations*) of elements of $S$, respectively. For a word $s \in S^*$, $|s|$ stands

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for its length, and we set $|s'| = +\infty$ for all $s' \in S^\omega$. For $s \in S$ and natural number $k$, $s^k$ and $s^\omega$ denote the $k$-length word and configuration consisting of copies of $s$'s, respectively. For a word (configuration) $s \in S^*(S^\omega)$, a word $s' \in S^*$ is called a *prefix* of $s$, denoted as $s' \sqsubseteq s$, if there exists another word (configuration) $s'' \in S^*(S^\omega)$ such that $s = s's''$. For two natural numbers $i \le j$, $[i, j]$ denotes the set of all integers between $i$ and $j$; and for a set $S$, $|S|$ its cardinality.

A Petri net is a quadruple $N = (P, T, Pre, Post)$, where $P$ is a finite set of *places* graphically represented by circles; $T$ is a finite set of *transitions* graphically represented by bars; $P \cup T \neq \emptyset$, $P \cap T = \emptyset$; $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are the *pre-* and *post-incidence functions* that specify the arcs directed from places to transitions, and vice versa, where $\mathbb{N}$ stands for the set of natural numbers. The value of $Pre$ or $Post$ at an arc is graphically represented as the weight of the arc. The *incidence function* is defined as $C = Post - Pre$.

A *marking* is a mapping $M : P \to \mathbb{N}$ that assigns to each place of a Petri net a natural number of tokens, graphically represented by black dots. For a marking $M \in \mathbb{N}^P$, a transition $t \in T$ is called *enabled* at $M$ if $M(p) \ge Pre(p, t)$ for any $p \in P$, and is denoted by $M[t\rangle$, where as usual $\mathbb{N}^P$ denotes the set of mappings from $P$ to $\mathbb{N}$. An enabled transition $t$ at $M$ may *fire* and yield a new making $M'(p) = M(p) + C(p, t)$ for all $p \in P$, written as $M[t\rangle M'$. As usual, we assume that at each marking and each time step, at most one transition fires. For a marking $M$, a sequence $t_1 \dots t_n$ of transitions is called enabled at $M$ if $t_1$ is enabled at $M$, $t_2$ is enabled at the unique $M_2$ satisfying $M[t_1\rangle M_2, \dots, t_n$ is enabled at the unique $M_{n-1}$ satisfying $M[t_1\rangle \cdots [t_{n-1}\rangle M_{n-1}$. We write the firing of $t_1 \dots t_n$ at $M$ as $M[t_1 \dots t_n\rangle$ for short, and similarly denote the firing of $t_1 \dots t_n$ at $M$ yielding $M'$ by $M[t_1 \dots t_n\rangle M'$. $\mathcal{T}(N, M_0) := \{s \in T^* | M_0[s\rangle\}$ is used to denote the set of transition sequences enabled at $M_0$. Particularly we have $M_0[\epsilon\rangle M_0$. A pair $(N, M_0)$ is called a *Petri net system*, where $N = (P, T, Pre, Post)$ is a Petri net, $M_0 : P \to \mathbb{N}$ is called the *initial marking*, and the system evolves initially at $M_0$ as transition sequences fire. Denote the set of *reachable markings* of the system by $\mathcal{R}(N, M_0) := \{M \in \mathbb{N}^P | \exists s \in T^*, M_0[s\rangle M'\}$. For a Petri net system $(N, M_0)$, $\mathcal{R}(N, M_0)$ is at most countably infinite.

An LPN system is a quadruple $(N, M_0, \Sigma, \ell)$, where $N$ is a Petri net, $M_0$ is an initial marking, $\Sigma$ is an *alphabet* (a finite set of labels), and $\ell : T \to \Sigma$ is a *labeling function* that assigns to each transition $t \in T$ a symbol of $\Sigma$ (In (Hack, 1976), such a labeling function is called "$\lambda$-free"). The labeling function $\ell : T \to \Sigma$ can be recursively extended to $\ell : T^* \to \Sigma^*$ as $\ell(st) = \ell(s)\ell(t)$ with $s \in T^*$ and $t \in T$. Particularly we let $\ell(\epsilon) = \epsilon$. For an LPN system $G = (N, M_0, \Sigma, \ell)$, the *language* generated by $G$ is denoted by $\mathcal{L}(G) := \{\sigma \in \Sigma^* | \exists s \in T^*, M_0[s\rangle, \ell(s) = \sigma\}$, i.e., the set of labels of transition sequences enabled at the initial marking $M_0$. We also say for each $\sigma \in \mathcal{L}(G)$, system $G$ generates $\sigma$. For $\sigma \in \Sigma^\omega$, we say $G$ generates $\sigma$ if $G$ generates each prefix of $\sigma$.

Note that for an LPN system $G = (N, M_0, \Sigma, \ell)$, when we observe a label sequence $\sigma \in \Sigma^*$, there may exist at most finitely many firing transition sequences $s \in T^*$ such that $\ell(s) = \sigma$. Denote the set of markings in which the system can be when observing $\sigma$ by $\mathcal{M}(G, \sigma) := \{M \in \mathbb{N}^P | \exists s \in T^*, M_0[s\rangle M, \ell(s) = \sigma\}$, then for each $\sigma \in \Sigma^*$, $\mathcal{M}(G, \sigma)$ is finite.

For an LPN system, for arcs from places to transitions, normally positive numbers of tokens of places make a transition fire. However, when the situation that no token of places makes a transition fire also occurs, the generalized LPN system is called an LPN system with *inhibitor arcs*. Formally a Petri net with inhibitor arcs is a quintuple $N' = (P, T, Pre, Pre', Post)$, where $P$ and $T$ are also finite sets of places and transitions such that $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$, $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are still the pre- and post-incidence functions, $Pre' : P \times T \to \{0, 1\}$ is the *inhibitor pre-incidence function* such that $Pre(p, t) \cdot Pre'(p, t) = 0$ for all $p \in P$ and $t \in T$, guaranteeing that there exists at most one of a normal arc and an inhibitor arc from $p$ to $t$. Here a transition $t \in T$ is enabled at a marking $M \in \mathbb{N}^P$ if and only if $M(p) \ge Pre(p, t)$ for any $p \in P$ satisfying $Pre(p, t) > 0$ and $M(p) = 0$ for any $p \in P$ satisfying $Pre'(p, t) > 0$. The firing of a transition $t \in T$ at a marking $M \in \mathbb{N}^P$ yields a marking $M'(p) = M(p) + Post(p, t) - Pre(p, t)$ if $Pre'(p, t) = 0$ and $M'(p) = Post(p, t)$ if $Pre'(p, t) > 0$, where $p \in P$ and $t \in T$. Similarly, an LPN system with inhibitor arcs is a quadruple $G' = (N', M_0, \Sigma, \ell)$, where $N' = (P, T, Pre, Pre', Post)$ is a Petri net with inhibitor arcs, $M_0 \in \mathbb{N}^P$ is an initial marking, $\Sigma$ is again an alphabet, and $\ell : T \to \Sigma$ is again a labeling function. The set $\mathcal{T}(N', M_0)$ of transition sequences enabled at $M_0$, the $\mathcal{R}(N', M_0)$ of reachable markings, the language $\mathcal{L}(G')$ generated by $G'$, and the set $\mathcal{M}(G', \sigma)$ of markings in which the system can be when observing $\sigma \in \Sigma^*$, are defined in an analogue way as those for LPN systems.

In what follows, we will prove some undecidable results for detectability-related problems for LPN systems (with inhibitor arcs). We obtain these results by reducing a classical undecidable problem on the *language equivalence* of LPN systems as shown below to the problems under consideration.

*Proposition 2.1.* (Hack, 1976, Theorem 8.2) It is undecidable to verify for two given LPN systems $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, $i = 1, 2$, whether one has $\mathcal{L}(G_1) = \mathcal{L}(G_2)$.

## 3. MAIN RESULTS

Consider an LPN system $G' = (N', M_0, \Sigma, \ell)$ with inhibitor arcs, where $N' = (P, T, Pre, Pre', Post)$ is a Petri net with inhibitor arcs. The concept of weak detectability can be intuitively described as whether there exists a label sequence such that when observing the label sequence, the markings of $G'$ are uniquely determined after a time step. It is formulated as below.

*Definition 1.* Consider an LPN system $G' = (N' = (P, T, Pre, Pre', Post), M_0, \Sigma, \ell)$ with inhibitor arcs. System $G'$ is called *weakly detectable* if there exists a label sequence $\sigma \in \Sigma^\omega$ such that for some positive integer $k$, $|\mathcal{M}(G', \sigma')| = 1$ for any prefix $\sigma'$ of $\sigma$ satisfying $|\sigma'| \ge k$.

Sometimes, we do not need to determine the marking of an LPN system, but only need to know whether the marking belongs to some prescribed subset of reachable markings, we call such a property weak approximate detectability, which is formulated as below.

*Definition 2.* Consider an LPN system $G' = (N' = (P, T, Pre, Pre', Post), M_0, \Sigma, \ell)$ with inhibitor arcs. Given a positive integer $n > 1$ and a partition $\{R_1, \dots, R_n\}$ of the set $\mathcal{R}(N', M_0)$ of reachable markings, system $G'$ is called *weakly approximately detectable* with respect to partition $\{R_1, \dots, R_n\}$ if there exists a label sequence $\sigma \in \Sigma^\omega$ such that for some positive integer $k$, for any prefix $\sigma'$ of $\sigma$ satisfying $|\sigma'| \geq k$, $\emptyset \neq \mathcal{M}(G', \sigma') \subseteq R_i$ for some $i \in [1, n]$.

For weak detectability of LPN systems with inhibitor arcs, we have the following result.

*Theorem 3.1.* It is undecidable to verify whether a given LPN system with inhibitor arcs is weakly detectable.

**Proof.** We prove this result by reducing the language equivalence problem of LPN systems (Proposition 2.1) to the weak detectability problem of LPN systems with inhibitor arcs.

Arbitrarily given two LPN systems $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, where $N_i = (P_i, T_i, Pre_i, Post_i)$, $i = 1, 2$, $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, we next construct a new LPN system $G$ with inhibitor arcs from $G_1$ and $G_2$ by adding additional places, transitions, and arcs, and prove that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ if and only if $G$ is not weakly detectable.

$G$ is specified as follows: (1) Add 5 new places $p_0, p_1^1, p_1^2, p_2^1, p_2^2$ to $G_1$ and $G_2$, and initially $p_0$ has 1 token, and the other 4 places have no token. (2) Add 6 new transitions $t_0^1, t_0^2, t_1^1, t_1^2, t_1, t_2$. (3) Add new arcs $p_0 \to t_0^i \to p_1^i \to t_1^i \to p_2^i \to t_i \to p_2^i$, $i = 1, 2$. (4) For each transition $t \in T_i$, add arcs $p_1^i \to t \to p_1^i$, $i = 1, 2$. (5) For each place $p \in P_i$, add inhibitor arc $p \dashv t_i$, $i = 1, 2$. (6) For each place $p \in P_i$, add transition $t_p$ and arcs $p \to t_p \to p_2^i \to t_p$, $i = 1, 2$. (7) All new added transitions are labeled by $\sigma_G \notin \Sigma$. All new added arcs and inhibitor arcs are with weight 1. See Fig. 1 as an example.

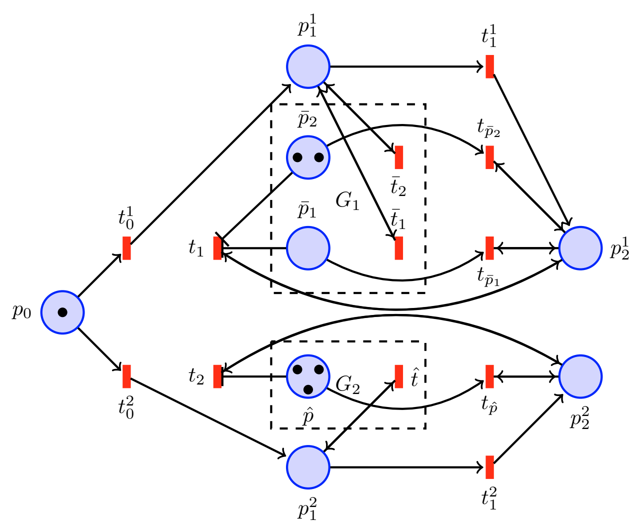


Fig. 1. Example for the reduction in the proof of Theorem 3.1.

For LPN system $G$ with inhibitor arcs, initially only transition $t_0^1$ or $t_0^2$ can fire. After $t_0^1$ ($t_0^2$) fires, the unique token in place $p_0$ moves to place $p_1^1$ ($p_1^2$), initializing system $G_1$ ($G_2$). While $G_1$ ($G_2$) is running, only transition $t_1^1$ ($t_1^2$) outside $T_1 \cup T_2$ can fire. The firing of $t_1^1$ ($t_1^2$) moves the token in place $p_1^1$ ($p_1^2$) to place $p_2^1$ ($p_2^2$), and stops system $G_1$ ($G_2$) from running, yielding that $G_1$ ($G_2$) will never run again, and for each $p \in P_1$ ($p \in P_2$), transition $t_p$ fires repetitively until there exists no token in place $p$. After all places in $P_1$ ($P_2$) become empty, only transition $t_1$ ($t_2$) can fire, and can fire repetitively forever. All in all, all possible infinite transition sequences fired by system $G$ are of the form $t_0^1 s t_1^1 s' t_1^\omega$, $t_0^1 s''$, $t_0^2 r t_1^2 r' t_2^\omega$, or $t_0^2 r''$, where $s \in (T_1)^*$, $s' \in \{t_p | p \in P_1\}^*$, $s'' \in (T_1)^\omega$, $r \in (T_2)^*$, $r' \in \{t_p | p \in P_2\}^*$, $r'' \in (T_2)^\omega$. Note that for some systems $G_1$ and $G_2$, the corresponding system $G$ never fires $t_0^1 s''$ or $t_0^2 r''$ as above, e.g., when $\mathcal{L}(G_1) \cup \mathcal{L}(G_2)$ is finite; but for all $G_1$ and $G_2$, the corresponding $G$ fires $t_0^1 s t_1^1 s' t_1^\omega$ and $t_0^2 r t_1^2 r' t_2^\omega$ as above.

If $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$, without loss of generality, we assume that there exists $\sigma \in \mathcal{L}(G_1) \setminus \mathcal{L}(G_2)$. Then when system $G$ generates $\sigma_G \sigma (\sigma_G)^\omega$, it only fires $t_0^1 s t_1^1 s' (t_1)^\omega$, where $s \in (T_1)^*$, $\ell_G(s) = \sigma$, $s' \in \{t_p | p \in P_1\}^*$, $|s'| = \sum_{p \in P_1} M(p)$, $M \in \mathbb{N}^{P_1}$ is the marking satisfying $M_0^1[s\rangle M$ uniquely determined by $s$. When we observe prefix $\sigma_G \sigma (\sigma_G)^k$ of $\sigma_G \sigma (\sigma_G)^\omega$ for any integer $k > K := \max\{\sum_{p \in P_1} M'(p) | \exists \tilde{s} \in (T_1)^*, \ell_G(\tilde{s}) = \sigma, M_0^1[\tilde{s}\rangle M'\}$ (note that $\{\tilde{s} \in (T_1)^* | \ell_G(\tilde{s}) = \sigma, M_0^1[\tilde{s}\rangle\}$ is a finite set, hence $K$ is a natural number), the set $\mathcal{M}(G, \sigma_G \sigma (\sigma_G)^k)$ of reachable markings of system $G$ after observing $\sigma_G \sigma (\sigma_G)^k$ is a singleton, and its unique element $M_G \in \mathbb{N}^{P_G}$ satisfies that $M_G(p_0) = M_G(p_1^1) = M_G(p_1^2) = M_G(p_2^2) = M_G(p) = 0$ for any $p \in P_1$, $M_G(p_2^1) = 1$, $M_G|_{P_2} = M_0^2$. Hence system $G$ is weakly detectable.

If $\mathcal{L}(G_1) = \mathcal{L}(G_2)$, then system $G$ may generate only configurations $\sigma_G \sigma'$ or $\sigma_G \sigma (\sigma_G)^\omega$, where $\sigma' \in \Sigma^\omega$, $\sigma \in \mathcal{L}(G_1)$. For the former case, for any positive integer $k$ and any $k$ length prefix $\sigma''$ of $\sigma'$, there exist firing sequences $s \in (T_1)^*$ of system $G_1$ and $r \in (T_2)^*$ of system $G_2$ such that $\ell_G(s) = \ell_G(r) = \sigma''$. Then $\mathcal{M}(G, \sigma'')$ includes a marking $M_G \in \mathbb{N}^{P_G}$ satisfying $M_G(p_1^1) = 1$ and $M_G(p_1^2) = 0$ and also a marking $M'_G \in \mathbb{N}^{P_G}$ satisfying $M'_G(p_1^1) = 0$ and $M'_G(p_1^2) = 1$. That is, $\mathcal{M}(G, \sigma'')$ is not a singleton. For the latter case, when we observe $\sigma_G \sigma (\sigma_G)^k$, where $k$ is a sufficiently large natural number, we have $G$ may fire both $t_0^1 s t_1^1 s' (t_1)^{k-1-|s'|}$ and $t_0^2 r t_1^2 r' (t_2)^{k-1-|r'|}$, where $s \in (T_1)^*$, $r \in (T_2)^*$, $\ell_G(s) = \ell_G(r) = \sigma$, $s' \in \{t_p | p \in P_1\}^*$, $r' \in \{t_p | p \in P_2\}^*$, $|s'| \leq k-1$, $|r'| \leq k-1$. Then we obtain two markings $M_G, M'_G \in \mathbb{N}^{P_G}$ satisfying that $M_0^G[t_0^1 s t_1^1 s' (t_1)^{k-1-|s'|}\rangle M_G$ and $M_0^G[t_0^2 r t_1^2 r' (t_2)^{k-1-|r'|}\rangle M'_G$, $M_G(p_2^1) = 1$, $M_G(p_2^2) = 0$, $M'_G(p_2^1) = 0$, $M'_G(p_2^2) = 1$. That is, $\mathcal{M}(G, \sigma_G \sigma (\sigma_G)^k)$ is not a singleton for any sufficiently large $k$. We have checked all label sequences generated by system $G$, hence $G$ is not weakly detectable, which completes the proof. ■

For weakly approximate detectability of LPN systems, the following result holds.

*Theorem 3.2.* Let $n > 1$ be a positive integer. It is undecidable to verify for an LPN system and a partition $\{R_1, \ldots, R_n\}$ of the set of its reachable markings, whether the system is weakly approximately detectable with respect to $\{R_1, \ldots, R_n\}$.

**Proof.** We prove this result also by reducing the language equivalence problem of LPN systems (Proposition 2.1) to the weak approximate detectability problem of LPN systems.

Let $l \geq 3$ be an integer. Arbitrarily given two LPN systems $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, where $N_i = (P_i, T_i, Pre_i, Post_i)$, $i = 1, 2$, $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, we next construct a new LPN system $G = (N_G, M_0^G, \Sigma \cup \{\sigma_G\}, \ell_G)$ from $G_1$ and $G_2$. $G$ is specified as follows: (1) Add $l + 2$ places $p_0, p_1^1, p_1^2, p_2, \ldots, p_l$ to $G_1$ and $G_2$, where initially $p_0$ has one token, and all the other places have no token. (2) Add $l + 3$ transitions $t_0^1, t_0^2, t_1^1, t_1^2, t_2, \ldots, t_l$, and arcs $p_0 \to t_0^1 \to p_1^1 \to t_1^1 \to p_2 \to t_2 \to \cdots \to p_l \to t_l \to p_2$, and $p_0 \to t_0^2 \to p_1^2 \to t_1^2 \to p_l$, where these transitions are labeled by $\sigma_G \notin \Sigma$. (3) For each transition $t \in T_i$, add arcs $p_1^i \to t \to p_1^i$, $i = 1, 2$.

(4) All these new added arcs are with weight 1. See Fig. 2 as an example.

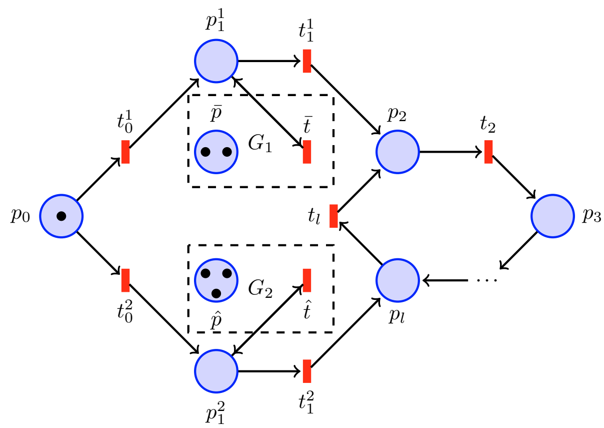


Fig. 2. Example for the reduction in the proof of Theorem 3.2.

For LPN system $G$, initially only transition $t_0^1$ or $t_0^2$ can fire. After $t_0^1$ ($t_0^2$) fires, the unique token in place $p_0$ moves to place $p_1^1$ ($p_1^2$), initializing system $G_1$ ($G_2$). While $G_1$ ($G_2$) is running, only transition $t_1^1$ ($t_1^2$) outside $T_1 \cup T_2$ can fire. The firing of $t_1^1$ ($t_1^2$) moves the token in place $p_1^1$ ($p_1^2$) to place $p_2$ ($p_l$), and terminates the running of system $G_1$ ($G_2$), yielding that the token in $p_2$ ($p_l$) can move along the direction $p_2 \to \cdots \to p_l \to p_2$ periodically forever, but $G_1$ ($G_2$) will never run again. Hence system $G$ may fire only infinite transition sequences $t_0^1 s t_1^1 t_2 \dots t_l t_2 \dots t_l \dots$, $t_0^1 s'$, $t_0^2 r t_1^2 t_l t_2 \dots t_l t_2 \dots$, or $t_0^2 r'$, where $s \in (T_1)^*$, $s' \in (T_1)^\omega$, $r \in (T_2)^*$, $r' \in (T_2)^\omega$. So system $G$ can generate only configurations $\sigma_G \sigma(\sigma_G)^\omega$ or $\sigma_G \sigma'$ where $\sigma \in \Sigma^*$, $\sigma' \in \Sigma^\omega$. Note that for some systems $G_1$ and $G_2$, the corresponding system $G$ never fires $t_0^1 s'$ or $t_0^2 r'$ as above, e.g., when $\mathcal{L}(G_1) \cup \mathcal{L}(G_2)$ is finite; but for all $G_1$ and $G_2$, the corresponding $G$ fires $t_0^1 s t_1^1 t_2 \dots t_l t_2 \dots t_l \dots$ and $t_0^2 r t_1^2 t_l t_2 \dots t_l t_2 \dots$ as above.

$n > 3$:

Let $l = n - 1$. We partition the set $\mathcal{R}(N_G, M_0^G)$ of reachable markings of system $G$ as follows:

$$\begin{aligned} R_1 =& \{M \in \mathbb{N}^{P_G} | M(p_0) \text{ or } M(p_1^1) = 1, \\ & M(p_1^2) = M(p_j) = 0, j \in [2, l]\} \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_i =& \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^1) = M(p_1^2) = 0, \\ & M(p_i) = 1, M(p_j) = 0, j \in [2, l] \setminus \{i\}\} \\ & \cap \mathcal{R}(N_G, M_0^G), \quad i \in [2, l], \\ R_{l+1} =& \{M \in \mathbb{N}^{P_G} | M(p_1^2) = 1, \\ & M(p_0) = M(p_1^1) = M(p_j) = 0, j \in [2, l]\} \\ & \cap \mathcal{R}(N_G, M_0^G). \end{aligned} \tag{1}$$

That is, $\cup_{i=1}^{l+1} R_i = \mathcal{R}(N_G, M_0^G)$, and $R_i \cap R_j = \emptyset$ for all different $i, j \in [1, l+1]$.

If $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$, without loss of generality, we assume that there exists $\sigma \in \mathcal{L}(G_1) \setminus \mathcal{L}(G_2)$. Then when system $G$ generates configuration $\sigma_G \sigma(\sigma_G)^\omega$, it can fire only transition sequences $t_0^1 s t_1^1 t_2 \dots t_l t_2 \dots t_l \dots$, where $s \in (T_1)^*$, $\ell_G(s) = \sigma$. It can be directly seen for each positive integer $k$, $\emptyset \neq \mathcal{M}(G, \sigma_G \sigma(\sigma_G)^k) \subseteq R_{(k-1) \mod (l-1)+2}$, where $(k-1) \mod (l-1)$ means the remainder of $k-1$ divided by $l-1$. That is, system $G$ is weakly approximately detectable with respect to partition (1).

Next we assume that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$. Note that system $G$ generates only configurations $\sigma_G \sigma'$ or $\sigma_G \sigma(\sigma_G)^\omega$, where $\sigma' \in \Sigma^\omega$, $\sigma \in \Sigma^*$. For the former case, for each prefix $\sigma''$ of $\sigma'$, there exist firing sequences $s \in (T_1)^*$ of system $G_1$ and $r \in (T_2)^*$ of system $G_2$ such that $\ell_G(s) = \ell_G(r) = \sigma''$, and markings $M_G, M_G' \in \mathbb{N}^{P_G}$ such that $M_0^G[t_0^1 s\rangle M_G$, $M_0^G[t_0^2 r\rangle M_G'$, $M_G(p_1^1) = 1$, $M_G(p_1^2) = 0$, $M_G'(p_1^1) = 0$, and $M_G'(p_1^2) = 1$, then we have $\mathcal{M}(G, \sigma'') \cap R_1 \neq \emptyset$ and $\mathcal{M}(G, \sigma'') \cap R_{l+1} \neq \emptyset$. For the latter case, arbitrarily chosen a prefix $\sigma_G \sigma(\sigma_G)^k$ of $\sigma_G \sigma(\sigma_G)^\omega$, where $k$ is an arbitrary positive integer, we have there exist firing sequences $s \in (T_1)^*$ of system $G_1$ and $r \in (T_2)^*$ of system $G_2$ such that $\ell_G(s) = \ell_G(r) = \sigma$ and system $G$ can fire both $t_0^1 s s'$ and $t_0^2 r r'$, where $s'$ and $r'$ are $k$ length prefixes of $t_2 \dots t_l t_2 \dots t_l \dots$ and $t_l t_2 \dots t_l t_2 \dots$, respectively. Since $G$ will fire both $t_0^1 s s'$ and $t_0^2 r r'$, we have $\mathcal{M}(G, \sigma_G \sigma(\sigma_G)^k) \cap R_{(k-1) \mod (l-1)+2} \neq \emptyset$ and $\mathcal{M}(G, \sigma_G \sigma(\sigma_G)^k) \cap R_{(k-2) \mod (l-1)+2} \neq \emptyset$. Hence for each positive integer $k$, $\mathcal{M}(G, \sigma_G \sigma(\sigma_G)^k)$ intersects both $R_{(k-1) \mod (l-1)+2}$ and $R_{(k-2) \mod (l-1)+2}$, where $(k-1) \mod (l-1) \neq (k-2) \mod (l-1)$. We have checked all label sequences generated by $G$, hence $G$ is not weakly approximately detectable with respect to partition (1).

Hence $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$ if and only if $G$ is weakly approximately detectable with respect to partition (1).

$n = 3$:

Let $l = 3$. We partition the set $\mathcal{R}(N_G, M_0^G)$ of reachable markings of system $G$ as follows:

$$\begin{aligned} R_1 =& \{M \in \mathbb{N}^{P_G} | M(p_0) \text{ or } M(p_1^1) = 1, \\ & M(p_1^2) = M(p_2) = M(p_3) = 0\} \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_2 =& \{M \in \mathbb{N}^{P_G} | M(p_2) = 1, \\ & M(p_0) = M(p_1^1) = M(p_1^2) = M(p_3) = 0\} \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_3 =& \{M \in \mathbb{N}^{P_G} | M(p_1^2) \text{ or } M(p_3) = 1, \\ & M(p_0) = M(p_1^1) = M(p_2) = 0\} \\ & \cap \mathcal{R}(N_G, M_0^G). \end{aligned} \tag{2}$$

Similar to the case $n > 3$, we also have that $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$ if and only if system $G$ is weakly approximately detectable with respect to partition (2).

$n = 2$:

Let $l = 3$. We partition the set $\mathcal{R}(N_G, M_0^G)$ of reachable markings of system $G$ as follows:

$$\begin{aligned} R_1 =& \{M \in \mathbb{N}^{P_G} | M(p_0) \text{ or } M(p_1^1) \text{ or } M(p_2) = 1, \\ & M(p_1^2) = M(p_3) = 0\} \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_2 =& \{M \in \mathbb{N}^{P_G} | M(p_1^2) \text{ or } M(p_3) = 1, \\ & M(p_0) = M(p_1^1) = M(p_2) = 0\} \\ & \cap \mathcal{R}(N_G, M_0^G). \end{aligned} \tag{3}$$

Similarly we also have that $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$ if and only if system $G$ is weakly approximately detectable with respect to partition (3), which completes the proof.

■

## 4. CONCLUSION

In this paper, we proved that the problems of deciding weak detectability of LPN systems with inhibitor arcs and weak approximate detectability of LPN systems are both undecidable. There are many related problems that are worthy of further study, e.g., decidability of weak detectability of LPN systems, formulation and verification of strong detectability of LPN systems. It is not difficult to obtain that the problem of deciding weak detectability of bounded LPN systems is decidable, so it is interesting to design fast verification algorithms.

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