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Marking Estimation in Labelled Petri nets by the Representative Marking Graph

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Abstract: In this paper a method to recognize the set of consistent markings in labelled Petri nets is proposed. In this method, the set of unobservable transitions are partitioned into pseudo-observable and strictly unobservable ones, and the subnet induced by the latter is acyclic. The unobservable reach of a marking can be characterized by the union of the strictly unobservable reach of several basis markings, called *representative markings*, in the unobservable subnet. The set of consistent markings can be characterized by a linear algebraic system based on those representative markings. Based on the representative marking graph, the current marking estimation problem for a labelled Petri net can be efficiently solved. This method does not require the assumption that the unobservable subnet is acyclic.

Keywords: Supervisory Control, Petri Net, Discrete Event System, Basis Marking

1. INTRODUCTION

Petri nets have been proposed as a fundamental model for *Discrete Event Systems* in a wide variety of applications and have been an asset to reduce the computational complexity involved in solving control problems. In this manuscript, we focus on the marking estimation problem in a special Petri net model called *labelled Petri nets*. In a labelled Petri net some transitions are *unobservable*, i.e., their firing cannot be detected by an external agent, and some transitions are not *distinguishable*, i.e., the agent cannot determine which one has fired among all those sharing the same label. Due to the presence of these transitions, to determine the current *marking* (i.e., *state*) of the plant net becomes difficult. The observability of labelled Petri nets, i.e., a property ensuring that the current marking can be precisely determined, is studied in (Ru and Hadjicostis, 2010), where a sensor deployment method is proposed to estimate the current marking in the modified net. However, in general cases where the observation structure cannot be modified, it is not possible to determine the exact current marking but only a set of possible markings called *consistent markings*. The marking estimation problem plays an important role in Petri net theory since it is relevant to many problems, including *supervisory control* (Huang et al., 2006; Moody and Antsaklis, 2000; Park and Reveliotis, 2002; Ru et al., 2014), *observation* (Corona et al., 2007; Giua and Seatzu, 2002), *diagnosis* (Cabasino et al., 2010; Lefebvre and Delherm, 2007; Lefebvre and Moudni, 2001; Ramirez-Trevino et al., 2007), and *opacity* (Tong et al., 2017).

In particular, marking estimation problem in Petri nets has received much attention, and several approaches have been developed for its solution. If all transitions are observable, Giua and Seatzu (2002) proposed a method to estimate the lower bound of the current marking in case that the information of the initial marking is uncomplete. Moreover, several efficient methods based on *minimal explanations* are proposed by Cabasino *et al.* Cabasino et al. (2010) and by Jiroveanu *et al.* (Jiroveanu and Boel, 2010; Jiroveanu et al., 2008) for fault diagnosis in Petri nets. It was shown in (Cabasino et al., 2010; Tong et al., 2017) that only a subset of the reachability space, consisting of the so-called *basis markings*, needs to be enumerated, while all other markings reachable from them by firing only unobservable

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transitions can be characterized by a linear algebraic system. The drawback of the method relies on the assumption that the *unobservable subnet* does not contain cycles. However, such assumption (which is common in automata) is unnecessary in Petri net models, since unobservable cycles in Petri nets do not necessarily imply a divergent behavior. Moreover, people may encounter unobservable cycles when modeling many physical systems by Petri nets (see Example 1 in Section 3).

To handle labelled Petri nets with unobservable cycles, Ru *et al.* (Ru et al., 2014) and Cabasino *et al.* (Cabasino et al., 2015) developed methods based on the notion of *reduced consistent markings* (RCMs), which can be used to for marking avoidance and probabilistic marking estimation in some classes of labelled Petri nets. However, although the set of consistent markings is the union of the unobservable reach of all RCMs, there is no efficient method to recognize the set of consistent markings from RCMs except to enumerate all reachable markings from each RCM in the unobservable subnet.

In this paper, we relax the structural assumption concerning the acyclicity of unobservable subnets considered in (Cabasino et al., 2010), thus generalizing the class of nets that the approach can handle. The key feature of this approach is to treat a subset of unobservable transitions as *pseudo-observable* so that the remaining transitions form an acyclic subnet, and hence it can be applied to Petri nets with arbitrary structures of the unobservable subnet. In such a case the unobservable reach of a marking can be characterized by the union of the strictly unobservable reach of several basis markings, called *representative markings*. By computing an *representative marking graph*, consistent markings can be characterized by a linear algebraic system parameterized by the representative markings. The proposed approach requires a very low online computational effort since the most burdensome part of the observer design is done offline.

This paper is organized in five sections. The basics of Petri nets are recalled in Section 2. Section 3 introduce several notions, based on which properties of unobservable reach are studied. In Section 4 an algorithm is proposed to construct the representative marking graph that can be used for marking estimation. Conclusions are given in Section 5.

2. PRELIMINARIES

2.1 Petri Net

A Petri net is a four-tuple $N = (P, T, Pre, Post)$, where P is a set of m *places* represented by circles; T is a set of n *transitions* represented by bars; $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are the *pre-* and *post-incidence functions*, respectively, which specify the arcs in the net and are represented as matrices in $\mathbb{N}^{m\times n}$ (here $\mathbb{N} = \{0, 1, 2, \ldots\}$). The *incidence matrix* of a net is defined by $C = Post - Pre \in \mathbb{Z}^{m\times n}$ (here $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$). A net is said to be *acyclic* if there does not exist a sequence $v_1 v_2 \cdots v_k$ where $v_i \in P \cup T$ such that $v_i \in^\bullet v_{i+1}$ for $i \in \{1, \ldots, k_1\}$ and $v_k \in^\bullet v_1$.

A *marking* is a vector $M : P \to \mathbb{N}$ that assigns to each place of a Petri net a non-negative integer number of tokens, represented by black dots and can also be represented as an m-component vector. We denote by $M(p)$ the marking of place p. A *marked net* $\langle N, M_0\rangle$ is a net N with an initial marking M_0. We denote by $R(N, M_0)$ the set of all markings reachable from the initial one. We also use $x_1 p_1 + \cdots + x_n p_n$ to denote the marking $[x_1, \ldots, x_n]^T$ for simplicity.

A transition t is *enabled* at M if $M \geq Pre(\cdot, t)$ and may fire reaching a new marking $M' = M_0 + C(\cdot, t)$. We write $M[\sigma\rangle$ to denote that the sequence of transitions σ is enabled at M, and we write $M[\sigma\rangle M'$ to denote that the firing of σ yields M'.

We use $\mathbf{y}_\sigma$ to denote the *firing vector* (also called the *Parikh vector*) of $\sigma \in T^*$, i.e., $y_\sigma(t) = k$ if transition t appears k times in σ.

A Petri net $\langle N, M_0\rangle$ is said to be *bounded* if there exists an integer $K \in \mathbb{N}$ such that for all $M \in R(N, M_0)$, $M(p) \leq K$ for all $p \in P$. A net N is *structurally bounded* if for any $M_0 \in \mathbb{N}^m$, the marked net $\langle N, M_0\rangle$ is bounded.

Given a net $N = (P, T, Pre, Post)$ we say that $\hat{N} = (\hat{P}, \hat{T}, \hat{Pre}, \hat{Post})$ is a subnet of N if $\hat{P} \subseteq P$, $\hat{T} \subseteq T$ and $\hat{Pre}$ (resp., $\hat{Post}$) is the restriction of Pre (resp., $Post$) to $\hat{P} \times \hat{T}$.

Proposition 1. (Murata, 1989) Given a Petri net $N = (P, T, Pre, Post)$ that is acyclic and two markings M and M', if $\exists \mathbf{y} \in \mathbb{N}^n, \mathbf{y} \geq \mathbf{0}$ such that $M + C \cdot \mathbf{y} = M' \geq 0$, then there exists a sequence $\sigma \in T^*$ whose firing vector is $\mathbf{y}$ such that $M[\sigma\rangle M'$.

2.2 Labelled Petri Net

A *labeled Petri net* (LPN) is a 4-tuple $G = (N, M_0, E, \ell)$, where $\langle N, M_0\rangle$ is a marked net, E is the *alphabet* (a set of labels), and $\ell : T \to E \cup \{\varepsilon\}$ is the *labeling function* that assigns to each transition $t \in T$ either a symbol from E or the empty word ε. Therefore, the set of transitions can be partitioned into two disjoint sets $T = T_o \cup T_{uo}$, where $T_o = \{t \in T \mid \ell(t) \in E\}$ is the set of observable transitions and $T_{uo} = T \setminus T_o = \{t \in T \mid \ell(t) = \varepsilon\}$ is the set of unobservable transitions. We use $\ell(t) = e$ to denote that the label of the transition t is e. The labeling function can be extended to sequences $\ell : T^* \to E^*$, i.e., $\ell(\sigma t) = \ell(\sigma)\ell(t)$ with $\sigma \in T^*$ and $t \in T$. The cardinality of T_o and T_{uo} are denoted as n_o and n_{uo}, respectively.

We use w to denote the *word* that is observed from σ, i.e., $w = \ell(\sigma)$. The *language* of the labelled net G is denoted as $\mathcal{L}(G) = \{w \in E^* \mid (\exists \sigma, M_0[\sigma\rangle)\ell(\sigma) = w\}$. We use $M_1[w\rangle M_2$ to denote that $\exists \sigma \in T^*, \ell(\sigma) = w$ and the firing of σ at M_1 yields M_2.

2.3 Basis Marking and Basis Reachability Graph

In this subsection we revise the main definitions concerning basis markings presented in (Cabasino et al., 2010), since the original definitions are tailored for diagnosis purpose.

Given a labelled Petri net $G = (N, M_0, E, \ell)$, $N = (P, T, Pre, Post)$ where $T = T_o \cup T_{uo}$ and the subnet induced by T_{uo} is acyclic, for a marking M and a transition $t \in T_o$, the set of *explanations* of t at M is the set:

$$\Sigma(M,t) = \{\sigma \in T_{uo}^* \mid M[\sigma\rangle M', M' \geq Pre(\cdot,t)\},$$

and the set of *explanation vectors* is the set:

$$Y(M,t) = \{\mathbf{y}_\sigma \in \mathbb{N}^{n_{uo}} \mid \sigma \in \Sigma(M,t)\}.$$

Moreover, the set of *minimal explanation vectors* is:

$$Y_{min}(M,t) = \{\mathbf{y} \in \mathbb{N}^{n_{uo}} \mid \nexists \mathbf{y}' \in Y(M,t), \mathbf{y}' \lneq \mathbf{y}\}$$

which consists of all minimal elements in $Y(M,t)$.

Given a labelled Petri net $G = (N, M_0, E, \ell)$, $N = (P,T,Pre,Post)$ where $T = T_o \cup T_{uo}$ and the subnet induced by T_{uo} is acyclic, its *basis marking set* $\mathcal{M}(G, M_0)$ is iteratively defined as follows:

- $M_0 \in \mathcal{M}(G, M_0)$;
- If $M \in \mathcal{M}(G, M_0)$, then $\forall t \in T_o$, $\forall \mathbf{y} \in Y_{min}(M,t)$,
  $(M' = M + C_{uo} \cdot \mathbf{y} + C(\cdot,t)) \Rightarrow (M' \in \mathcal{M}(G, M_0))$.

A marking M in $\mathcal{M}(G, M_0)$ is called a *basis marking* of G. The *basis reachability graph* of G, denoted as $\mathcal{B}(G, M_0)$, can also be iteratively defined as follows:

- M_0 is the root node in $\mathcal{B}(G, M_0)$;
- If $M \in \mathcal{B}(G, M_0)$, then $\forall t \in T_o$, $\forall \mathbf{y} \in Y_{min}(M,t)$, $\forall M' = M + C_{uo} \cdot \mathbf{y} + C(\cdot,t)$, $M' \in \mathcal{B}(G, M_0)$ holds, and there is an arc from M to M' with a label $(t, \mathbf{y})$.

The work of (Cabasino et al., 2010) provided a tabular algorithm to compute $Y_{min}(M,t)$ in Petri nets with acyclic unobservable subnet [1] , and also an algorithm to compute the corresponding BRG. However, if the unobservable subnet contains cycles, basis markings cannot be used for the purpose of marking estimation (this will be shown in Example 1 shortly), since the state equation does not provide a sufficient condition for the marking reachability in nets that contains cycles.

3. BASIS MARKINGS AND UNOBSERVABLE REACHES

In this paper we propose a different strategy to solve this problem. The key feature of this approach is that some unobservable transitions that create cycles in the unobservable nets are now treated as "observable" so that the remaining part of the unobservable subnet is acyclic.

3.1 Partition of Unobservable Transitions

Given an unobservable subnet that is not acyclic, we can partition the unobservable transition set T_{uo} into two new sets: $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$ such that the subnet induced by $\hat{T}_{uo}$ is acyclic. This partition is always possible and can be efficiently constructed, e.g., $\hat{T}_{uo}$ can be obtained by recursively removing some transitions from T_{uo} until an acyclic subnet is obtained. The transitions in $\hat{T}_o$ are called *pseudo-observable transitions* and the transitions in $\hat{T}_{uo}$ are called *strictly unobservable transitions*. Note that the partition of T_{uo} into $\hat{T}_o$ and $\hat{T}_{uo}$ does not necessarily have a physical meaning, and such partition is not unique in general.

[1] If the unobservable subnet contains cycles, a different algorithm in (Jiroveanu et al., 2008) can be used to compute $Y_{min}(M,t)$.

Definition 1. Given a labelled Petri net $G = (N, M_0, E, \ell)$, $N = (P,T,Pre,Post)$ where $T = T_o \cup T_{uo}$, $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$, and the subnet induced by $\hat{T}_{uo}$ is acyclic, the net $N_{uo} = (P, T_{uo}, Pre_{uo}, Post_{uo})$ and the net $\hat{N}_{uo} = (P, \hat{T}_{uo}, \hat{P}re_{uo}, \hat{P}ost_{uo})$ are called the *unobservable subnet* and the *strictly unobservable subnet*, respectively, and their incidence matrices are denoted as C_{uo} and $\hat{C}_{uo}$, respectively. We denote $|T_o| = n_o$, $|T_{uo}| = n_{uo}$, $|\hat{T}_o| = \hat{n}_o$, and $|\hat{T}_{uo}| = \hat{n}_{uo}$. □

In the following we give a series of definitions on *strict explanations* and *strictly minimal explanations*. We remind that if $\hat{T}_o = \emptyset$ and $\hat{T}_{uo} = T_{uo}$ then these definitions reduce to classical definitions of *explanations* and *explanation vectors* in (Cabasino et al., 2010).

Definition 2. Given a labelled Petri net $G = (N, M_0, E, \ell)$ in which $T = T_o \cup T_{uo}$, $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$, a marking M, and a transition $t \in T_o \cup \hat{T}_o$, we define

$$\hat{\Sigma}(M,t) = \{\sigma \in \hat{T}_{uo}^* \mid M[\sigma\rangle M', M' \geq Pre(\cdot,t)\}$$

the set of *strict explanations* of t at M, and we define

$$\hat{Y}(M,t) = \{\mathbf{y}_\sigma \in \mathbb{N}^{\hat{n}_{uo}} \mid \sigma \in \hat{\Sigma}(M,t)\}$$

the set of *strict explanation vectors*. □

The physical meaning of $\hat{\Sigma}(M,t)$ is the following: from M if we want to enable $t \in T_o \cup \hat{T}_o$ by firing only strictly unobservable transitions, then some sequence $\sigma \in \hat{\Sigma}(M,t)$ must fire. The set $\hat{Y}(M,t)$ is composed of the firing vectors associated to the firing sequences in $\hat{\Sigma}(M,t)$.

Definition 3. Given a labelled Petri net $G = (N, M_0, E, \ell)$ in which $T = T_o \cup T_{uo}$, $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$, a marking M, and a transition $t \in T_o \cup \hat{T}_o$, we define

$$\hat{\Sigma}_{min}(M,t) = \{\sigma \in \hat{\Sigma}(M,t) \mid \nexists \sigma' \in \hat{\Sigma}(M,t) : \mathbf{y}_{\sigma'} \lneq \mathbf{y}_\sigma\}$$

the set of *strict minimal explanations* of t at M, and we define

$$\hat{Y}_{min}(M,t) = \{\mathbf{y}_\sigma \in \mathbb{N}^{\hat{n}_{uo}} \mid \sigma \in \hat{\Sigma}_{min}(M,t)\}$$

the corresponding set of *strict minimal explanation vectors*. □

In plain words, $\hat{\Sigma}_{min}(M,t)$ is the set of sequences in $\hat{\Sigma}(M,t)$ with minimal firing sequences and $\hat{Y}_{min}(M,t)$ is the set of these minimal firing vectors.

Typically $\hat{\Sigma}_{min}(M,t)$ and $\hat{Y}_{min}(M,t)$ are not singletons, since there are possibly multiple minimal sequences $\sigma \in \hat{T}_{uo}^*$ that can enable a transition $t \in T_o \cup \hat{T}_o$. If the $\hat{T}_o$-induced subnet is acyclic and backward-conflict-free (i.e., each place has at most one input transition), then $\hat{\Sigma}_{min}(M,t)$ and $\hat{Y}_{min}(M,t)$ are always singletons (Corona et al., 2007). If $\hat{\Sigma}(M,t) = \hat{\Sigma}_{min}(M,t) = \emptyset$ (which implies that $\hat{Y}(M,t) = \hat{Y}_{min}(M,t) = \emptyset$), then from M one cannot enable $t \in T_o \cup \hat{T}_o$ by firing only strictly unobservable transitions.

Since the strictly unobservable subnet is acyclic, the algorithm based on algebraic manipulations can be used to efficiently compute $\hat{Y}_{min}(M,t)$ from a given marking M and a transition $t \in T_o \cup \hat{T}_o$, if the net is bounded Cabasino et al. (2010). Moreover, a more general approach to compute $\hat{Y}_{min}(M,t)$ which can be applied for unbounded nets has been presented in (Ma et al., 2017).

3.2 Unobservable Reach

Next we give the definitions of *unobservable reach* and *strictly unobservable reach* of a given marking.

Definition 4. Given a labelled Petri net G in which $T = T_o \cup T_{uo}, T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$, and a marking M, its *unobservable reach* is defined as:

$$R_{uo}(G, M) = \{M' \in \mathbb{N}^m \mid \exists \sigma \in T^*_{uo}, M[\sigma\rangle M'\},$$

its *strictly unobservable reach* w.r.t. $\hat{T}_{uo}$ is defined as:

$$\hat{R}_{uo}(G, M, \hat{T}_{uo}) = \{M' \in \mathbb{N}^m \mid \exists \sigma \in \hat{T}^*_{uo}, M[\sigma\rangle M'\}.$$

□

The physical meaning of the unobservable reach of M is the set of markings that are reachable by firing only unobservable transitions, and the physical meaning of its strictly unobservable reach is the set of markings that are reachable by firing only strictly unobservable transitions. Since the strictly unobservable subnet is acyclic, $\hat{R}_{uo}(G, M, \hat{T}_{uo})$ consists of markings that satisfy the state equation of the strictly unobservable subnet $\hat{G}_{uo}$.

Proposition 2. Given a labelled Petri net G and a marking M, its strictly unobservable reach w.r.t. $\hat{T}_{uo}$ is:

$$\hat{R}_{uo}(G, M, \hat{T}_{uo}) = \{M' \in \mathbb{N}^m \mid (\exists \mathbf{y} \in \mathbb{N}^{\hat{n}_{uo}}) M' = M + \hat{C}_{uo} \cdot \mathbf{y}\}. \quad (1)$$

Proof: This result directly follows from Proposition 1 since the $\hat{T}_{uo}$-induced subnet is acyclic. □

An analogous result does not hold for $R_{uo}(G, M)$ since the unobservable subnet is not assumed to be acyclic. However, the following proposition (which is Theorem 3.8 in (Cabasino et al., 2010)) shows that the unobservable reach of a marking M can be written as a union of strictly unobservable reaches of basis markings in the unobservable subnet.

Theorem 1. Given a labelled net $G = (N, M_0, E, \ell)$ whose unobservable subnet is acyclic. There exists a sequence $\sigma \in T^*$ such that $M_0[\sigma\rangle M$ if and only if there also exist a basis marking M_b such that $M_0[\ell(\sigma)\rangle M_b$ and an unobservable sequence $\sigma \in T^*_{uo}$ such that $M_b[\sigma_u\rangle M$.

Proposition 3. Given a labelled Petri net $G = (N, M_0, E, \ell)$, let $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$ such that the subnet induced by $\hat{T}_{uo}$ is acyclic. It holds:

$$R_{uo}(G, M) = \bigcup_{M_b \in \mathcal{M}(G_{uo}, M)} \hat{R}_{uo}(G, M_b, \hat{T}_{uo}) \quad (2)$$

where $G_{uo} = (N_{uo}, M_0, \{\hat{e}\}, \ell')$ in which N_{uo} is the unobservable subnet, and $\ell'(t) = \hat{e}$ which assigns a unique label $\hat{e}$ to all pseudo-observable transition $t \in \hat{T}_o$ while $\ell'(t) = \varepsilon$ for all $t \in \hat{T}_{uo}$,

Proof: By Definition 4, $R_{uo}(G, M) = R(N_{uo}, M)$ holds, i.e., the unobservable reach of M in G consists of all markings that are reachable from M in the unobservable subnet N_{uo}. Since in N_{uo} the transition set T_{uo} can be partitioned into $\hat{T}_o$ and $\hat{T}_{uo}$ while the $\hat{T}_{uo}$-induced subnet is acyclic, by Theorem 3.8 in (Cabasino et al., 2010), $R(N_{uo}, M) = \bigcup_{M_b \in \mathcal{M}(G_{uo}, M)} R_{uo}(G_{uo}, M_b)$ holds. Since $R_{uo}(G_{uo}, M_b) = \hat{R}_{uo}(G, M_b, \hat{T}_{uo})$, hence

$$R_{uo}(G, M) = R(N_{uo}, M) = \bigcup_{M_b \in \mathcal{M}(G_{uo}, M)} R_{uo}(G, M_b, \hat{T}_{uo})$$

holds, which concludes the proof. □

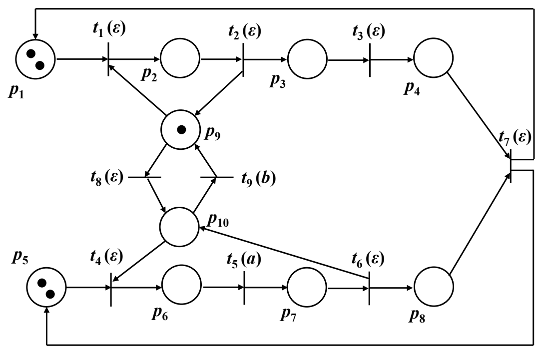

Fig. 1. A labelled Petri net plant for Example 1 in which $T_o = \{t_5, t_9\}, T_{uo} = \{t_1, t_2, t_3, t_4, t_6, t_7, t_8\}$.

By Proposition 3, the unobservable reach of a marking M in G is the union of the strictly unobservable reaches of those basis markings $M_b \in \mathcal{M}(G_{uo}, M)$. Moreover, Proposition 2 indicates that the unobservable reach of an arbitrary marking M can be characterized by a linear system of basis markings in the unobservable subnet G_{uo}, as stated in the following corollary.

Corollary 1. Given a labelled Petri net $G = (N, M_0, E, \ell)$ in which $T = T_o \cup T_{uo}$, $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$ such that $\hat{N}_{uo}$ (i.e., the subnet induced by $\hat{T}_{uo}$) is acyclic and given a marking M, the following condition holds:

$$R_{uo}(G,M) = \bigcup_{M \in \mathcal{M}(G_{uo}, M)} \{M' \mid (\exists \mathbf{y} \in \mathbb{N}^{\hat{n}_{uo}}) M' = M + \hat{C}_{uo} \cdot \mathbf{y}\}. \quad (3)$$

By Proposition 3 and Corollary 1, we can use the markings $\mathcal{M}(G_{uo}, M)$ to represent $R_{uo}(G, M)$ since all markings in $R_{uo}(G, M)$ can be characterized by the linear algebraic system of markings in $\mathcal{M}(G_{uo}, M)$. This allows us to use relatively few markings to represent the unobservable reach of a given marking, which helps us to build our algorithm in the next section. Although for different marking M the set of basis markings $\mathcal{M}(G_{uo}, M)$ is different, the net structure is always the same and hence some intermediate results can be reused during the computation. Computing $\mathcal{M}(G_{uo}, M)$ can be done by a simplified basis marking enumeration by Algorithm 1, whose correctness can be analogously derived from the construction of BRG in (Cabasino et al., 2010). In particular, we note that for bounded nets $\mathcal{M}(G_{uo}, M) \subseteq R(N, M)$, and hence $\mathcal{M}_{new}$ eventually become empty in a finite number of steps and Algorithm 1 terminates.

Example 1. Consider the labelled Petri net G in Figure 1. It models a system that contains two workflows ($p_1 t_1 p_2 t_2 p_3 t_3 p_4$ and $p_5 t_4 p_6 t_5 p_7 t_6 p_8$) that machine two types of parts that are assembled later (transition t_7). There is a robot that can machine parts on one workflow (p_2 on workflow 1 or p_6, p_7 on workflow

Algorithm 1 Basis Unobservable Representation

Input: A labelled Petri net $G = (N, M, E, \ell)$ where $T = T_o \cup T_{uo}$, $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$
Output: The basis unobservable representation $\mathcal{M}(G_{uo}, M)$

```
Let 𝓜 = ∅, let 𝓜_new = {M};
while 𝓜_new ≠ ∅, do
    Select a marking M' ∈ 𝓜_new;
    for all t ∈ T̂_o, do
        for all y ∈ Ŷ_min(M', t), do
            Let M̂' = M' + Ĉ_uo · y + C(·, t);
            if M̂' ∉ 𝓜 ∪ 𝓜_new then
                Let 𝓜_new = 𝓜_new ∪ {M̂'};
            end if
        end for
    end for
    Let 𝓜 = 𝓜 ∪ {M'}, let 𝓜_new = 𝓜_new \ {M'};
end while
Output 𝓜(G_uo, M) = 𝓜.
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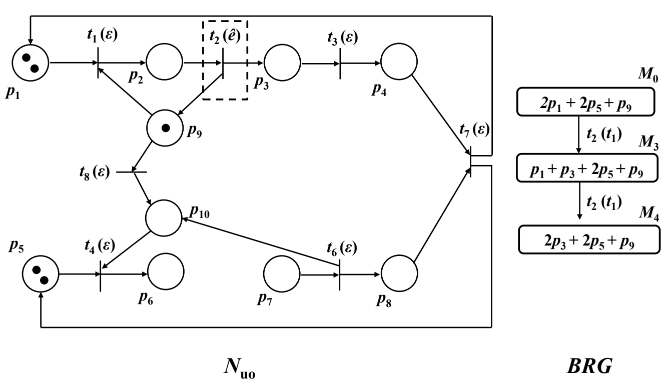

Fig. 2. The unobservable subnet G_{uo} of the net G in Figure 1. The transition t_2 is treated as pseudo-observable while other transitions are strictly unobservable. In the BRG of this net, the notion $(\cdot)$ in $t(\cdot)$ on arcs denotes the (strict) minimal explanation of t.

2) at the same time. Suppose that two sensors are deployed on t_5 and t_9, respectively, i.e., $\ell(t_5) = a$, $\ell(t_9) = b$, and $\ell(t) = \varepsilon$ for all other transitions. The reachability graph of the net has 69 reachable markings, which is too complex to be graphically presented here.

Since the unobservable subnet contains cycles ($t_1 p_2 t_2 p_9$), the classical BRG approach in (Cabasino et al., 2010) cannot be applied for marking estimation, since the unobservable reach $\hat{R}_{uo}(G, M_b)$ of a basis marking M_b cannot be characterized by the linear expression $M_b + C_{uo} \cdot \mathbf{y}$. For example, at a basis marking $M_1 = 2p_1 + p_5 + p_7$, there is a marking $M_2 = p_1 + p_3 + p_5 + p_7$ such that $M_2 = M_1 + C_{uo} \cdot \mathbf{y}$ where $\mathbf{y} = [1, 1, 0, 0, 0, 0, 0, 0, 0]^T$, but one can readily verify that M_2 is not reachable from M_1 since the robot is occupied by workflow 2 at M_1 such that workflow 1 cannot proceed. Hence the classical basis markings cannot be used to estimate the current markings.

On the other hand, let us consider a further partition $T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$ where $\hat{T}_o = \{t_2\}$, i.e., t_2 is treated as a pseudo-observable transition. One can verify that the subnet induced by $\hat{T}_o = \{t_1, t_3, t_4, t_6, t_7, t_8\}$ is acyclic. For the initial marking $M_0 = 2p_1 + 2p_5 + p_9$, the basis markings $\mathcal{M}(G_{uo}, M_0)$ (the structure of G_{uo} is shown in Figure 2 in which t_2 is the only observable transition) consists of three markings: $\mathcal{M}(G_{uo}, M_0) = \{M_0, M_3, M_4\}$ where $M_3 = p_1 + p_3 + 2p_5 + p_9$, $M_4 = 2p_3 + 2p_5 + p_9$. One can verify that $R_{uo}(G, M_0) = \bigcup_{M_b \in \{M_0, M_3, M_4\}} \{M' \mid (\exists \mathbf{y} \in \mathbb{N}^{\hat{n}_{uo}}) M' = M_b + \hat{C}_{uo} \cdot \mathbf{y}\}$, according to Proposition 3. □

As we have mentioned, given a labelled net G, the possible partition of T_{uo} into $\hat{T}_o$ and $\hat{T}_{uo}$ is not unique. However, to characterize $R_{uo}(G, M)$ by the basis markings $\mathcal{M}(G_{uo}, M)$, it is preferable to select a set of pseudo-observable transitions with a minimal cardinality, since $|\mathcal{M}(G_{uo}, M)|$ is non-decreasing with the increase of the set $\hat{T}_o$ (Ma et al., 2017).

4. REPRESENTATIVE MARKINGS AND THE REPRESENTATIVE MARKING GRAPH

Definition 5. Given a labelled Petri net $G = (N, M_0, E, \ell)$, the consistent marking set of a word $w \in \mathcal{L}(G)$ is defined as:

$$\mathcal{C}(w) = \{M \mid M_0[w\rangle M\}.$$

A marking $M \in \mathcal{C}(w)$ is called a *consistent marking* of w. □

The consistent marking set $\mathcal{C}(w)$ consists of all markings that are reachable from M_0 by firing some sequences σ whose observation $\ell(\sigma)$ is w. In the following we propose an algorithm to construct a current marking estimator called the *representative marking graph* (RMG).

Definition 6. Given a labelled Petri net $G = (N, M_0, E, \ell)$, its *representative marking graph* (RMG) is a deterministic finite state automaton constructed by Algorithm 2. The RMG $\mathcal{B}$ is a quadruple $(\mathcal{X}, E, \delta, X_0)$, where:

- each state X in the state set $\mathcal{X}$ is a set of markings called *representative markings*;
- the event set E is the set of labels;
- δ is the transition relation;
- the initial state is $X_0 \in \mathcal{X}$.

□

Algorithm 2 works in the following way. Initially, the set $\mathcal{X}_{new}$ consists of an initial state X_0 which contains $\mathcal{M}(G_{uo}, M_0)$ and X_0 is not checked. In the iteration cycle, if $\mathcal{X}_{new}$ is not empty, then a state $X \in \mathcal{X}_{new}$ is selected. For each event $e \in E$, for each pair (t, M) where $t \in T(e)$ and $M \in X$, the set $\hat{Y}_{min}(M, t)$ is calculated. Then for each $\mathbf{y} \in \hat{Y}_{min}(M, t)$, a new marking $\hat{M} = M + \hat{C}_{uo} \cdot \mathbf{y} + C(\cdot, t)$ is computed. By Step 9 its representation set $\mathcal{M}(G_{uo}, \hat{M})$ is computed by Algorithm 1 and all representative markings in it are added to X_{temp}. Finally we have X_{temp} that consists of all markings that can be reached from some marking in X by firing a transition t labelled e and with one of its strict minimal explanations. If X_{temp} does not exist in $\mathcal{X} \cup \mathcal{X}_{new}$, this means that X_{temp} is

Algorithm 2 Representative Marking Graph

Input: A labelled Petri net $G = (N, M_0, E, \ell)$, $T = T_o \cup T_{uo}, T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$
Output: The RMG $\mathcal{B} = (\mathcal{X}, E, \delta, X_0)$

```
Let 𝒳 = ∅, 𝒳_new = {X_0} = {ℳ(G_uo, M_0)};
while 𝒳_new ≠ ∅, do
    Select a state X ∈ 𝒳_new;
    Let X_temp = ∅;
    for all e ∈ E, do
        for all t ∈ T(e), M ∈ X, do
            for all y ∈ Ŷ_min(M, t), do
                Let M̂ = M + Ĉ_uo · y + C(·, t);
                X_temp = X_temp ∪ ℳ(G_uo, M̂);
            end for
        end for
        if ∄X' ∈ 𝒳 ∪ 𝒳_new, X' = X_temp, then
            Let 𝒳_new = 𝒳_new ∪ {X'};
            Let δ(X, e) = X';
        else
            Let δ(X, e) = X';
        end if
    end for
    Let 𝒳 = 𝒳 ∪ {X}, let 𝒳_new = 𝒳_new \ {X}.
end while
```

a new node, and hence X_{temp} is added to $\mathcal{X}_{new}$, and $\delta(X, e)$ is defined accordingly. At the end of this iteration cycle, X is moved from $\mathcal{X}_{new}$ to $\mathcal{X}$ to denote that X has been checked. This procedure runs iteratively until there is no unchecked state in $\mathcal{X}_{new}$. Since $\mathcal{M}(G_{uo}, M) \subseteq R(N, M_0)$, we can conclude that $\mathcal{X} \subseteq 2^{R(N,M_0)}$, i.e., Algorithm 2 terminates in a finite number of steps.

Definition 7. Given a labelled Petri net G in which $T = T_o \cup \hat{T}_o \cup \hat{T}_{uo}$ where the $\hat{T}_{uo}$-induced subnet is acyclic, the marking set $X = \delta(X_0, w)$ is called the *representative marking set* of w in G, denoted as $\mathcal{C}_{rep}(w)$. □

The following theorem shows that the RMG $\mathcal{B}$ can be used to characterize the consistent marking set $\mathcal{C}(w)$ for a given observation w. In short, $\mathcal{C}(w)$ can be characterized by a linear system parameterized by the corresponding representative markings $\mathcal{C}_{rep}(w)$.

Theorem 2. Given a labelled Petri net G in which $T = T_o \cup T_{uo}, T_{uo} = \hat{T}_o \cup \hat{T}_{uo}$, and an arbitrary word $w \in \mathcal{L}(G)$, it holds:

$$\mathcal{C}(w) = \bigcup_{M \in \delta(X_0, w)} \hat{R}_{uo}(G, M, \hat{T}_o).$$

Proof: We prove this theorem by induction.

(Base) If $w = \lambda$, i.e., the empty observation, then $C(\lambda) = \bigcup_{M\in\delta(X_0,w)} \hat{R}_{uo}(G, M, \hat{T}_o)$ holds by Proposition 3.

(Induction) Suppose that the statement holds for a word w, i.e., $\mathcal{C}(w) = \bigcup_{M\in\delta(X_0,w)} \hat{R}_{uo}(G, M, \hat{T}_o)$.

Consider a word we and an arbitrary marking M that belongs to the consistent marking set $\mathcal{C}(we)$. By definition, there exist a marking $M_1 \in \mathcal{C}(w)$, a transition t with $\ell(t) = e$, and $\sigma_1 \in T_{uo}^*$ such that $M_1[t\rangle M_2[\sigma_1\rangle M$. Since $M_1 \in \mathcal{C}(w) = \bigcup_{M\in\delta(X_0,w)}$, there exists a representative marking $M_{rep} \in \delta(X_0, w)$ such that $M_1 \in \hat{R}_{uo}(G, M_{rep}, \hat{T}_o)$. Then $\exists \sigma_2 \in \hat{T}_{uo}^*$ such that $M_{rep}[\sigma_2\rangle M_1[t\rangle M_2[\sigma_1\rangle M$.

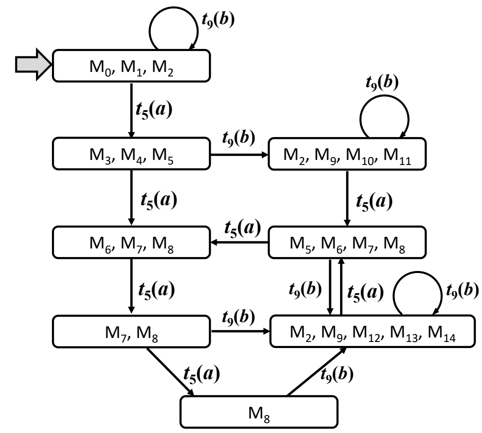

Fig. 3. The RMG $\mathcal{B}$ of the net in Figure 1 in which $T_o = \{t_5, t_9\}$, $\hat{T}_o = \{t_2\}$, and $\hat{T}_{uo} = \{t_1, t_3, t_4, t_6, t_7, t_8\}$.

If σ_2 is not the strict minimal explanation of t at M_{rep}, then there must exist a firing sequence $\sigma_3 \in \hat{T}_{uo}^*$ that is the strict minimal explanation such that $M_{rep}[\sigma_3 t\rangle M_3$, and $\mathbf{y}_{\sigma_3} \lneq \mathbf{y}_{\sigma_2}$. Let $\sigma_4 \in \hat{T}_{uo}^*$ be a firing sequence such that $\mathbf{y}_{\sigma_2} = \mathbf{y}_{\sigma_3} + \mathbf{y}_{\sigma_4}$. Then we claim that $M_{rep}[\sigma_3 t\rangle M_3[\sigma_4\rangle M_2$ holds. This is due to the fact that $M_3 + \hat{C}_{uo} \cdot \mathbf{y}_{\sigma_4} = M_4 \geq \mathbf{0}$, and hence, since the strictly unobservable net is acyclic, by Proposition 2 the state equation is a sufficient condition for the firing of a sequence σ_4 with a firing vector $\mathbf{y}_{\sigma_4}$. Since $M_{rep} \in \delta(X_0, w)$ and $M_3 = M_{rep} + \hat{C}_{uo} \cdot \mathbf{y}_{\sigma_4} + C(\cdot, t)$, it holds $M_3 \in \delta(\delta(X_0, w), e) = \delta(X, we)$ by Algorithm 2.

Since $M_3[\sigma_4\rangle M_2[\sigma_1\rangle M$, we have $M \in \hat{R}_{uo}(G, M_3, \hat{T}_{uo})$ and $M \in \hat{R}_{uo}(G, M_3, \hat{T}_o) \subseteq \bigcup_{M\in\delta(X_0,we)} \hat{R}_{uo}(G, M, \hat{T}_o)$. Since the marking $M \in \mathcal{C}(we)$ is arbitrarily chosen, $\mathcal{C}(we) \subseteq \bigcup_{M\in\delta(X_0,we)} \hat{R}_{uo}(G, M, \hat{T}_o)$ holds.

On the other hand, by the computation of $\delta(X_0, we)$ from $\delta(X_0, w)$ in Algorithm 2 as discussed before, it is trivial to prove that $\mathcal{C}(we) \supseteq \bigcup_{M\in\delta(X_0,we)} \hat{R}_{uo}(G, M, \hat{T}_o)$, which concludes the proof. □

By Theorem 2, to compute the consistent marking set of a given observation w, one just need to check its representative markings in $\mathcal{C}_{rep}(w) = \delta(X_0, w)$, as stated in the following corollary:

Corollary 2. Given a labelled Petri net $G = (N, M_0, E, \ell)$, its consistent marking set $\mathcal{C}(w)$ of a word w satisfies:

$$\mathcal{C}(w) = \{M' \in \mathbb{N}^m \mid (\exists M \in \mathcal{C}_{rep}(w)) \quad M' = M + \hat{C}_{uo} \cdot \mathbf{y}\}.$$

Example 2. Still consider the net in Figure 1. By letting $\hat{T}_o = \{t_2\}$ and $\hat{T}_{uo} = T_{uo} \setminus \{t_2\}$ and applying Algorithm 2, its RMG is shown in Figure 3 and the representative markings are

Table 1. The list of representative markings in Figure 3

M_0	[2 0 0 0 2 0 0 0 1 0]	M_8	[2 0 0 0 0 0 1 1 0 0]
M_1	[1 0 1 0 2 0 0 0 1 0]	M_9	[0 0 2 0 1 0 0 1 1 0]
M_2	[0 0 2 0 2 0 0 0 1 0]	M_{10}	[2 0 0 0 1 0 0 1 1 0]
M_3	[2 0 0 0 1 0 1 0 0 0]	M_{11}	[1 0 1 0 1 0 0 1 1 0]
M_4	[1 0 1 0 1 0 1 0 0 0]	M_{12}	[2 0 0 0 0 0 0 2 1 0]
M_5	[0 0 2 0 1 0 1 0 0 0]	M_{13}	[1 0 1 0 0 0 0 2 1 0]
M_6	[0 0 2 0 0 0 1 1 0 0]	M_{14}	[0 0 2 0 0 0 0 2 1 0]
M_7	[1 0 1 0 0 0 1 1 0 0]		

listed in Table 1. This RMG, consisting of only 8 states and 15 representative markings in all, can be used as the current marking estimator of the original net (which has 69 reachable markings). By observing a word w, the consistent marking set $\mathcal{C}(w)$ are such marking M satisfying the following IPP:

$$\begin{cases} M = M_{rep} + \hat{C}_{uo} \cdot \mathbf{y} \geq \mathbf{0} \\ \mathbf{y} \geq \mathbf{0} \\ M_{rep} \in \delta(X_0, w) \end{cases} \tag{4}$$

For instance, given an observation $w = a$, the representative marking set $\mathcal{C}_{rep}(a)$ contains 3 representative markings (i.e., $\delta(X_0, a) = \{M_3, M_4, M_5\}$) that characterizes the consistent marking set $\mathcal{C}(a)$ by the linear algebraic system Eq. (4). For the observation $w = ab$, there are 4 representative markings in $\mathcal{C}_{rep}(ab)$ (i.e., M_2, M_9, M_{10}, and M_{11}). Moreover, it is very easy to compute $\mathcal{C}_{rep}(we)$ from $\mathcal{C}_{rep}(w)$ by looking for the state $X = \delta(\mathcal{C}_{rep}(w), e)$ in the RMG. Since nearly all computation is done offline, the online computational effort of this method is negligible. □

By Theorem 2 and Corollary 2 the consistent markings of a Petri net can be efficiently described by a linear algebraic system parameterized by a set of representative markings. Since this representative marking analysis approach by-passes the need of enumeration and on-line maintenance/updating of a large list of consistent markings, it brings significant advantages from the point of view of the computational effort.

5. CONCLUSION

In this paper a method to estimate the consistent markings in labelled Petri nets is proposed, which is based on the representative marking analysis. This method does not require the assumption that the unobservable subnet is acyclic. The set of consistent markings can be described by a linear algebraic system parameterized by the representative markings that can be efficiently computed from the representative marking graph.

REFERENCES

Cabasino, M., Giua, A., and Seatzu., C. (2010). Fault detection for discrete event systems using Petri nets with unobservable transitions. *Automatica*, 46(9), 1531–1539.

Cabasino, M., Hadjicostis, C.N., and Seatzu, C. (2015). Probabilistic marking estimation in labeled Petri nets. *IEEE Transactions on Automatic Control*, 60, 528–533.

Corona, D., Giua, A., and Seatzu, C. (2007). Marking estimation of Petri nets with silent transitions. *IEEE Transactions on Automatic Control*, 52(9), 1695–1699.

Giua, A. and Seatzu, C. (2002). Observability of place/transition nets. *IEEE Transactions on Automatic Control*, 47(9), 1424–1437.

Huang, Y.S., Jeng, M., Xie, X., and Chung, D.H. (2006). Siphon-based deadlock prevention policy for flexible manufacturing systems. *IEEE Transactions on Systems, Man and Cybernetics, Part A*, 36(6), 1248–1256.

Jiroveanu, G. and Boel, R.K. (2010). The diagnosability of Petri net models using minimal explanations. *IEEE Transactions on Automatic Control*, 55(7), 1663–1668.

Jiroveanu, G., Boel, R.K., and Bordbar, B. (2008). On-line monitoring of large Petri net models under partial observation. *Discrete Event Dynamic Systems*, 18(3), 323–354.

Lefebvre, D. and Delherm, C. (2007). Diagnosis of des with Petri net models. *IEEE Transactions on Automation Science and Engineering*, 4(1), 114–118.

Lefebvre, D. and Moudni, A.E. (2001). Firing and enabling sequences estimation for timed Petri nets. *IEEE Transactions on Systems, Man and Cybernetics, Part A*, 31(3), 153–162.

Ma, Z.Y., Tong, Y., Li, Z.W., and Giua, A. (2017). Basis marking representation of Petri net reachability spaces and its application to the reachability problem. *IEEE Transactions on Automatic Control*, 62(3), 1078–1093.

Moody, J. and Antsaklis, P. (2000). Petri net supervisors for DES with uncontrollable and unobservable transitions. *IEEE Transactions on Automatic Control*, 45(3), 462–476.

Murata, G.T. (1989). Petri nets: properties, analysis and applications. *Proceedings of the IEEE*, 77(4), 541–580.

Park, J. and Reveliotis, S.A. (2002). Liveness-enforcing supervision for resource allocation systems with uncontrollable behavior and forbidden states. *IEEE Transactions on Robotics and Automation*, 18(2), 234–240.

Ramirez-Trevino, A., Ruiz-Beltran, E., Rivera-Rangel, I., and López-Mellado, E. (2007). Online fault diagnosis of discrete event systems: A Petri net-based approach. *IEEE Transactions on Automation Science and Engineering*, 4(1), 31–39.

Ru, Y., Cabasino, M.P., Giua, A., and Hadjicostis, C.N. (2014). Supervisor synthesis for discrete event systems under partial observation and arbitrary forbidden state specifications. *Discrete Event Dynamic Systems*, 24(3), 275–307.

Ru, Y. and Hadjicostis, C.N. (2010). Sensor selection for structural observability in discrete event systems modeled by Petri nets. *IEEE Transactions on Automatic Control*, 55(8), 1751–1764.

Tong, Y., Li, Z.W., Seatzu, C., and Giua, A. (2017). Verification of state-based opacity using Petri nets. *IEEE Transactions on Automatic Control*, 62(6). DOI: 10.1109/TAC.2016.2620429.