Marking Optimization of Deterministic Timed Weighted Marked Graphs*

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Abstract

Timed marked graphs (TMGs), a special class of Petri nets, are used to model and analyze cyclic manufacturing systems. Weighted marked graphs (WMGs) are convenient to model systems with bulk services and arrivals. It is possible to apply analytical techniques for the marking optimization problem of such systems. The problem consists in minimizing the cost associated with an initial marking that ensures a given average cycle time. In this paper, we use simulation to compute the average cycle time and give a fast and effective heuristic solution.

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I. Introduction

Timed Petri nets are a well known tool to model discrete event systems and represent their dynamic behavior. In this paper, we study a particular class of timed Petri nets called *timed weighted marked graphs* (TWMGs). The main feature of this class of nets is that each place has only one input and one output transition. Moreover, the firing delay of each transition is *deterministic*.

TWMGs and TMGs are widely used in manufacturing applications to model complex assembly lines and solve cyclic scheduling problems. Workshop operations and products are usually modeled by transitions and tokens, respectively. Between two successive transformations, semi-finished products have to be stored or moved from a workshop to another one. The amount of products, also called Work In Process (WIP), that have to be stored or moved may have economical consequences. Therefore, the main problem for designers is to design a proper schedule of WIP that allows the system to reach a given productivity while the amount of WIP is the smallest.

Some results have been proven for this class of Petri nets. For instance, Teruel *et al*. [1] proposed serveral techniques for the analysis of WMGs. Nakamura and Silva [2] and Campos *et al*. [3] developed methods to compute the average cycle time for a given initial marking. Giua *et al*. [4] dealt with the firing rate optimization of cyclic timed event graphs by token allocations. However, in the literature, few works are found to deal with the marking optimization problem of TWMGs. Sauer [5] proposed a heuristic solution based on an iterative process to obtain an optimal solution. Touris and Sauer [6] presented an approach based on the branch and bound to solve the problem.

In this paper, we deal with the marking optimization problem of a TWMG. The problem consists in minimizing the cost associated with an initial marking that ensures a given average cycle time. We propose a fast and efficient iterative heuristic algorithm that starts with a live marking. At each step we select a place to which we add tokens until the average cycle time is lower than or equal to the desired value; the places are selected so as to minimally increase the marking cost. In several experiments, we carried out the obtained solution is either optimal or very close to the optimal one.

This paper is structured as follows. In the following section, we briefly recall some basic concepts and the main properties. In Section 3, we present the problem statement. In Section 4, we propose a heuristic solution for the marking optimization problem based on a live marking. Following the algorithm, some numerical examples are shown to illustrate it. And also we give details to compare our method with Sauer's. Conclusions and future work are finally drawn in Section 5.

II. Background

A. Generalities

We assume that the reader is familiar with the structure, firing rules, and basic properties of PNs (see [1], [7] and [8]). In this section, we will recall the formalism used in the paper. A *place/transition* net (P/T net) is a structure $N = (P, T, Pre, Post)$, where P is a set of n places; T is a set of m transitions; $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are the pre- and post-incidence functions that specify the arcs; $C = Post - Pre$ is the incidence matrix, where $\mathbb{N}$ is a set of non-negative integers. When $Pre(p,t) = k > 0$ we also say that the arc from place p to transition t has weight k. A similar terminology applies to function $Post$.

A vector $\boldsymbol{x} \in \mathbb{N}^{|T|}$ such that $\boldsymbol{x} \neq 0$ and $C \cdot \boldsymbol{x} = 0$ is a *T-semiflow*. A vector $\boldsymbol{y} \in \mathbb{N}^{|P|}$ is such that $\boldsymbol{y} \neq 0$ and $\boldsymbol{y}^T \cdot C = 0$ is a ***P-semiflow***. The supports of T-semiflows and of P-semiflows are defined by $\|\boldsymbol{x}\|$=$\{t \in T | \boldsymbol{x}(t) > 0\}$ and $\|\boldsymbol{y}\|$=$\{p \in P | \boldsymbol{y}(p) > 0\}$.

A marking is a vector $M : P \rightarrow \mathbb{N}$ that assigns to each place of a P/T net a non-negative integer number of tokens, represented by black dots; we denote the marking of place p as $M(p)$. A P/T system or net system $\langle N, M_0 \rangle$ is a net N with an initial marking M_0. A transition t is enabled at M if $M \geq Pre(\cdot, t)$ and may fire yielding the marking $M' = M + C(\cdot, t)$.

A P/T net is called *ordinary* when all of its arc weights are $1's$. A *marked graph* (MG, also called an event graph) is an ordinary Petri net such that satisfies the condition $|^\bullet p| = |p^\bullet| = 1$. A *weighted marked graph* (WMG, also called a weighted event graph) is a net that also satisfies this condition but may not be an ordinary net, i.e., the weight associated with each arc is a non-negative integer number.

A net is *strongly connected* if there exists a directed path from any node in $P \cup T$ to every other node. Let us define an *elementary circuit* γ (or elementary cycle) of a net as a directed path that goes from one node back to the same node without passing twice on the same node. In a strongly connected net, it is easy to show that each node belongs to an elementary circuit, and thus the name cyclic nets is also used to denote this class.

A deterministic timed P/T net is a pair N^δ = (N, δ), where $N = (P, T, Pre, Post)$ is a standard P/T net, and $\delta : T \rightarrow \mathbb{R}_0^+$, called firing delay, assigns a non-negative fixed firing duration to each transitions. A transition with a firing delay equal to 0 is said to be immediate. We consider a single server semantics, i.e., we assume that each transition can fire only once at each time even its enabling degree is greater than one.

Definition 1: (Campos *et al.* [3]) *Every elementary circuit* γ *of a WMG is neutral, if the following condition holds.*

$$\prod_{p \in \gamma} \frac{Pre(p, p^\bullet)}{Post(p, {}^\bullet p)} = 1$$

■

In other words, in a neutral circuit the product of the weights of all pre arcs is equal to the product of the weights of all post arcs. This means that if the circuit initially contains enough tokens, it is possible to fire all transitions along the path returning to the same initial marking. It is well known that a WMG whose circuits are all neutral has a unique T-semiflow $\boldsymbol{x}$ and it contains all transitions in its support [1].

In this paper, we limit our study to strongly connected WMG which are neutral.

The following notations will be used. Given a place p of a WMG, let t_i (resp. t_j) be its unique input (resp. output) transition as in Fig. 1. We denote $w(p) = Post(p, t_i)$ the weight of its input arc and $v(p) = Pre(p, t_j)$ the weight of its output arc. For any place $p \in P$, we denote by gcd_p (resp. lcm_p) the greatest common divisor (resp. the least common multiple) of the integers $w(p)$ and $\nu(p)$.

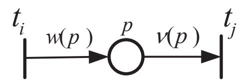

Fig. 1. A place p between two transitions t_i and t_j.

B. Liveness of a TWMG

A weighted circuit of a TWMG is live if each transition can be fired infinitely.

Proposition 1: (Teruel *et al*. [1]) *A TWMG is live iff each elementary circuit is live.* ■

In the case of a TMG, an elementary circuit is live if there exists at least one token in the circuit. However, determining the liveness of a weighted circuit is not so easy.

First, we review some sufficient conditions for the liveness of a TWMG presented in literature. Later, these conditions will be used in the proposed optimization approach.

Teruel *et al.* [1] and Chrzastowski-Wachtel and Raczunas [9] proposed a few methods to verify the liveness of a TWMG by characterizing the liveness of all its circuits. First they define a weighted function of the marking

$$W(M) = \boldsymbol{y}^T \cdot M$$

where $\boldsymbol{y}$ is a minimal P-semiflow. Furthermore, for each place p_i, they denote μ_i its output weight, i.e., $\mu_i = Pre(p_i, {p_i}^\bullet)$ and define the following marking:

$$M_D = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_n - 1).$$

The following result provides a sufficient, albeit restrictive, condition for liveness.

Proposition 2: (Teruel *et al.* [1]) *If $W(M_0) > W(M_D)$, then the weighted circuit is live.* ■

Less restrictive conditions for liveness also exist.

In [9] the *least live weight* of a weighted circuit with minimal P-semiflow y was defined as

$$W_L = W(M_D) - g(y_1, y_2, \cdots, y_n) \tag{1}$$

where g is the Frobenius number.[1] Note that a Frobenius number only exists if all its arguments are coprime and greater than one. The first condition is always verified in our case because we consider minimal P-semiflows. The second condition may not always be verified: when it is, the least live weight in Eq. (1) can be computed and the following proposition holds.

[1]Given positive integers y_1, y_2, $\cdots$, y_n such that $gcd(y_1, y_2, \cdots, y_n)$=1, the Frobenius number $g(y_1, y_2, \cdots, y_n)$ is the largest integer that cannot be expressed as an integer conical combination of these numbers, i.e., as a sum $k_1 y_1 + k_2 y_2 + \cdots + k_n y_n$, where k_1, k_2, $\cdots$, and k_n are non-negative integers.

Proposition 3: (Chrzastowski-Wachtel and Raczunas [9]) *If $W(M_0) = W_L$, then the weighted circuit is live.* ∎

In the case that there exists a 1 value in the minimal P-semiflow, then a least live weight cannot be computed by Eq. (1).

Proposition 4: (Teruel *et al.* [1]) *If the net is live then $W(M_0) \geq \lambda = \max_{j\in\{1,\ldots,n\}}\{\mu_j \cdot y_j\}$.* ∎

In case there is a 1 value in $\boldsymbol{x}$ this lower bound is also sufficient.

Corollary 1: (Teruel *et al.* [1]) *If $\exists i \in \{1, \cdots, m\}, x_i = 1$, where $\boldsymbol{x}$ is the minimal T-semiflow then, the least weight of a live marking is equal to*

$$W'_L = \lambda = \max_{j\in\{1,\ldots,n\}}\{\mu_j \cdot y_j\}$$

∎

If $W(M_0) < W'_L$, then the weighted circuit is not live; if $W(M_0) > W(M_D)$, then the weighted circuit is live.

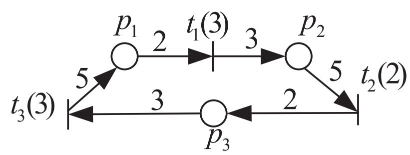

Fig. 2. A weighted circuit.

Let us consider the example in Fig. 2. We have $\boldsymbol{y}^T$ =(3, 2, 5), M_D=(1, 4, 2), $\boldsymbol{x}$=(5, 3, 2), W_L=$W(M_D) - g$= $21 - g$(3, 2, 5)=21−1=20.

It can be checked that every marking with a weight equal to 20 is live. For instance (0, 0, 4) as well as (0, 10, 0) is live. We use the approach above to select a live initial marking.

III. Problem statement

The *average cycle time* χ of a TWMG is the average time to fire once the T-semiflow under the earliest operational model (i.e., transitions are fired as soon as possible). Considering a net consisting only of one circuit, we define χ_γ as the average cycle time of circuit γ. Let Γ represent the set of elementary circuits of a cyclic TWMG.

It is well known that for a TMG the cycle time of the net is equal to the maximal cycle time over all circuits, i.e.,

$$\chi = \max_{\gamma\in\Gamma} \chi_\gamma$$

This result does not apply to TWMG as we will show in the examples of Figs. 4 and 5, but it holds that the average cycle time of each circuit is smaller than or equal to the one of the net. However, when the average cycle time of each circuit is lower than or equal to the upper bound, the average cycle time of the net will also converge to the upper bound.

The average cycle time of a TMG can be obtained by means of following LPP:

$$max\ \{\boldsymbol{y}^T \cdot Pre \cdot \boldsymbol{\delta} | \boldsymbol{y}^T \cdot C = 0, \boldsymbol{y}^T \cdot M_0 = 1, \boldsymbol{y} \geq 0\} \tag{2}$$

where $\boldsymbol{\delta} \in \mathbb{N}^m$ is the vector containing all firing delays of timed transitions (recall that $m = |T|$).

It is difficult to compute the average cycle time of a TWMG, and only bounds can be analytically computed. From Camposet *et al*. [3] a lower bound for the average cycle time of a live and bounded TWMG system can be computed by solving a linear programming problem (LPP). The analytical way to compute the average cycle time of TWMG is to convert a TWMG into a TMG and compute the average cycle time of TMG (see [2]and [10]). Otherwise, we can use simulation to compute the average cycle time which is faster and easier.

In this paper the marking optimization problems of a TWMG is considered. We aim to find an initial marking M_0 under which the average cycle time is less than or equal to a given value. Among all feasible solutions, we look for those that minimize a weighted function of the initial marking.

In other words we look for the optimal solution of the following optimization problem:

$$(P1) = \begin{cases} minimize\ f(M_0) = \boldsymbol{y}^T \cdot M_0 \\ s.t.\ \chi(M_0) \leq b \end{cases} \tag{3}$$

where

- $\chi(M_0)$ is the average cycle time of the TWMG associated with marking M_0,
- b is a given positive real value that represents the upper bound on the cycle time, and
- $\boldsymbol{y}^T = (y_1, \ldots, y_n)$ is a P-semiflow. In general, we choose the P-semiflow equal to the sum of all minimal P-semiflows.

We choose $\boldsymbol{y}$ as a P-semiflow since the value of $\boldsymbol{y}^T \cdot M_0$ for every reachable marking $M \in R(N, M_0)$ is invariant.

Proposition 5: (Sauer [5]) *Problem (3) has a solution iff* $b \geq \chi^* = max\{x_i \cdot \delta_i, t_i \in T\}$ (where $\boldsymbol{x}$ is the minimal T-semiflow).

IV. Marking optimization for TWMG: A heuristic solution

We propose here a fast and efficient heuristic solution based on an iterative process to solve the problem. We first start from a least live marking, then we compute the average cycle time of each circuit. If there exist some circuits whose average cycle time is greater than the cycle time upper bound, we add tokens to these circuits at each iteration until the average cycle time is lower than or equal to the cycle time upper bound. We select the places to which tokens should be added so as to increase the performance index $f(M_0)$ as little as possible.

A. The main idea of the heuristic solution

1) Selection of a proper initial marking: For each circuit, there exist some markings that satisfy the least weight condition. We prefer to make the net live while the following condition is satisfied.

$$
\begin{cases}
min\ f(M_0) = \boldsymbol{y}^T \cdot M_0 \\
s.t.\ \boldsymbol{y}_\gamma^T \cdot M_0 = W_L\ (W_L')\ or\ \boldsymbol{y}_\gamma^T \cdot M_0 > W(M_D), \forall \gamma \in \Gamma
\end{cases}
\tag{4}
$$

where

- $\boldsymbol{y}^T$ is a P-semiflow of a TWMG.
- $\boldsymbol{y}_\gamma^T$ is a P-semiflow of a circuit γ, i.e.,

$$y_\gamma(t_i) = y(t_i)\ if\ t_i \in \gamma,\ else\ y_\gamma(t_i) = 0$$

When there exists more than one feasible solution of the initial marking M_0, we choose one. If we start the iteration from a marking that satisfies the condition above, we can ensure that the the net is live and the value of performance index $f(M_0)$ is small. If the average cycle time of M_0 is greater than the cycle time upper bound, we add tokens to the net until the requirement on the average cycle time is satisfied. Otherwise, the initial marking M_0 is optimal.

2) Selection of the places to add tokens:

Useful tokens: The initial marking $M_0(p)$ of any place p can be replaced by $M_0^\star(p)$ tokens without any influence on the precedence constraints induced by p (see [11] and [12]), where

$$M_0^\star(p) = \left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p$$

As a result, we can deduce that the liveness of the net and the average cycle time under M_0 and $M_0^\star$ are the same. However, the value of $f(M_0^\star)$ is smaller than or equal to $f(M_0)$.

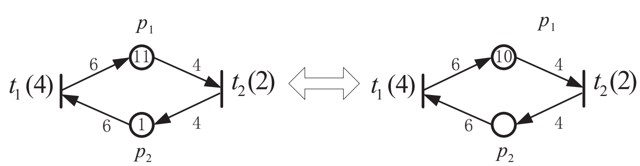

Fig. 3. Useful tokens.

Let us see the example in Fig. 3. The initial marking of the TWMG is $M_0 = (11, 1)$ and $gcd_{p_1} = gcd_{p_2} = 2$.

$$M_0^\star(p_1) = \left\lfloor \frac{M_0(p_1)}{gcd_{p_1}} \right\rfloor \cdot gcd_{p_1} = \left\lfloor \frac{11}{2} \right\rfloor \cdot 2 = 10$$

$$M_0^\star(p_2) = \left\lfloor \frac{M_0(p_2)}{gcd_{p_2}} \right\rfloor \cdot gcd_{p_2} = \left\lfloor \frac{1}{2} \right\rfloor \cdot 1 = 0$$

Then $M_0^\star = (10, 0)$ and we can check that $f(M_0^\star) = 10 < f(M_0) = 12$ and the average cycle time of M_0 and $M_0^\star$ are identical, i.e., $\chi(M_0) = \chi(M_0^\star)$.

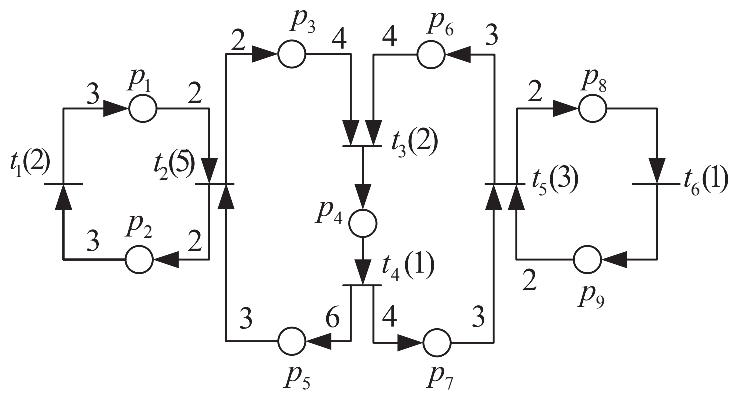

Fig. 4. The TWMG of Example 1.

After we select the initial marking, we can compute the average cycle time χ of the TWMG and χ_γ for every elementary circuit. We select the critical circuits to which tokens should be added. We denote Γ_c the set of these critical circuits, i.e., $\Gamma_c = \{\gamma \in \Gamma \| \chi_\gamma > \chi^*\}$. For each critical circuit, we select one place p_r and add gcd_{p_r} tokens to this place. We choose the one that increases $f(M_0)$ as little as possible, i.e., the increment of the criteria value $f(M_0)$ should be the least after adding gcd_{p_r} tokens. We define an n-dimensional vector $\boldsymbol{I}$ of zeros and ones.

$$\boldsymbol{I}^T = (I_{p_1}, I_{p_2}, \cdots, I_{p_n})$$

where

$$I_{p_r} = \begin{cases} 1, & add \ \ gcd_{p_r} \ \ tokens \ \ to \ \ place \ \ p_r \\ 0, & add \ \ 0 \ \ tokens \ \ to \ \ place \ \ p_r \end{cases} \tag{5}$$

In other words, we add tokens to places with the coefficient I_{p_r} = 1. Let P_a be the set of these places and $\boldsymbol{G_d}$ = ($gcd_{p_1} \cdot y_1$, $gcd_{p_2} \cdot y_2$, $\cdots$, $gcd_{p_n} \cdot y_n$), where $\boldsymbol{y}$ is a P-semiflow of the net and $gcd_{p_r} \cdot y_r$ represents the increment of $f(M_0)$ after adding gcd_{p_r} tokens to place p_r. We denote by $\Delta f(M_0)$ the total increment of $f(M_0)$, where

$$\Delta f(M_0) = \boldsymbol{I}^T \cdot G_d$$

Then, we can select places by solving the following problem:

$$\begin{cases} \min \quad \Delta f(M_0) \\ s.t. \ \sum_{p \in \gamma} I_p = 1, \ \forall \gamma \in \Gamma_c \end{cases} \tag{6}$$

The constrains in Eq. (6) will ensure that only one place should be selected for each critical circuit.

3) Heuristic solution: We can summarize the proposed procedure in the following algorithm.

In step 3, the average cycle time needs to be computed. This can be done using any of the techniques that we have mentioned in section 3. In this paper, we use simulation to compute the average cycle time.

Algorithm 1

1: Compute W_L, W'_L or W_D for every elementary circuit γ.

2: Compute a marking M which satisfies Eq. (4) then, compute $M^\star$ and initialize the initial marking $M_0 := M^\star$.

3: Compute the average cycle time χ

if $\chi \le \chi^*$

stop and the optimal solution is M_0;

else

compute χ_γ, $\forall\gamma \in \Gamma$.

end if

4: Compute Γ_c.

5: Compute I and P_a.

6: Add tokens to P_a.

7: Go to step 3.

TABLE I

THE ITERATION PROCESS FOR EXAMPLE 1.

M_0	χ_{γ_1}	χ_{γ_2}	χ_{γ_3}	χ_{γ_4}	χ	χ^*	$f(M_0)$	Γ_c	P_a
(4, 0, 4, 0, 0, 0, 6, 2, 0)	38	39	21	20	43	30	24	$\{\gamma_1, \gamma_2\}$	$\{p_1, p_5\}$
(5, 0, 4, 0, 3, 0, 6, 2, 0)	34	30	21	20	34	30	31	$\{\gamma_1\}$	$\{p_1\}$
(6, 0, 4, 0, 3, 0, 6, 2, 0)	30	30	21	20	30	30	32		

TABLE II

THE ITERATION PROCESS FOR EXAMPLE 2.

M_0	χ_{γ_1}	χ_{γ_2}	χ	χ^*	$f(M_0)$	Γ_c	P_a
(8, 3, 0, 0, 0)	29	26	30	21	20	$\{\gamma_1, \gamma_2\}$	$\{p_3\}$
(8, 3, 2, 0, 0)	25	26	26	21	24	$\{\gamma_1, \gamma_2\}$	$\{p_3\}$
(8, 3, 4, 0, 0)	21	21	21	21	28		

B. Examples

We consider the first example in Figure 4. There are four weighted circuits in the TWMG:

$$\begin{cases} \gamma_1 = \{p_1, t_2, p_2, t_1\} \\ \gamma_2 = \{p_3, t_3, p_4, t_4, p_5, t_2\} \\ \gamma_3 = \{p_6, t_3, p_4, t_4, p_7, t_5\} \\ \gamma_4 = \{p_8, t_6, p_9, t_5\} \end{cases}$$

The minimal T-semiflows of γ_1, γ_2, γ_3, and γ_4 are

$$\begin{cases} \boldsymbol{x}_1 = (2,3,0,0,0,0) \\ \boldsymbol{x}_2 = (0,2,1,1,0,0) \\ \boldsymbol{x}_3 = (0,0,3,3,4,0) \\ \boldsymbol{x}_4 = (0,0,0,0,1,2) \end{cases}$$

while the minimal P-semiflows of γ_1, γ_2, γ_3, and γ_4 are

$$\begin{cases} \boldsymbol{y}_1 = (1,1,0,0,0,0,0,0,0) \\ \boldsymbol{y}_2 = (0,0,3,12,2,0,0,0,0) \\ \boldsymbol{y}_3 = (0,0,0,4,0,1,1,0,0) \\ \boldsymbol{y}_4 = (0,0,0,0,0,0,0,1,1) \end{cases}$$

Therefore, the P-semiflow used in the criteria $f(M_0)$ is $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4$= (1, 1, 3, 16, 2, 1, 1, 1, 1), and the minimal T-semiflow of the net is $\boldsymbol{x}$=(4, 6, 3, 3, 4, 8).

$$\begin{cases} \gamma_1 : since\ y_1 = 1,\ x_i \neq 1\ (i = 1, 2),\ W(M_D) = \\ 1 \times 1 + 1 \times 2 = 3 \\ \gamma_2 : since\ x_3 = 1,\ W'_L = max\{4 \times 3, 1 \times 12, 3 \times 2\} = 12 \\ \gamma_3 : since\ y_6 = 1,\ x_i \neq 1\ (i = 3, 4, 5), W(M_D) = \\ 4 \times 0 + 1 \times 2 + 1 \times 3 = 5 \\ \gamma_4 : since\ x_5 = 1,\ W'_L = max\{1 \times 1, 2 \times 1\} = 2 \end{cases}$$

We have gcd_{p_1} = 1, gcd_{p_2} = 1, gcd_{p_3} = 2, gcd_{p_4} = 1, gcd_{p_5} = 3, gcd_{p_6} = 1, gcd_{p_7} = 1, gcd_{p_8} = 1, and gcd_{p_9} = 1.

$$\boldsymbol{G_d} = (1,1,6,16,6,1,1,1,1)$$

$$\min \quad f(M) = M(p_1) + M(p_2) + 3M(p_3) + 16M(p_4) + 2M(p_5) + M(p_6) + M(p_7) + M(p_8) + M(p_9)$$

$$s.t \begin{cases} M(p_1) + M(p_2) > 3 \\ 3M(p_3) + 12M(p_4) + 2M(p_5) = 12 \\ 4M(p_4) + M(p_6) + M(p_7) > 5 \\ M(p_8) + M(p_9) = 2 \end{cases}$$

We obtain a marking $M = (4,0,4,0,0,0,6,2,0)$ and the initial marking $M_0 = M^\star = M$. From Table I we can find that γ_1 and γ_2 are critical circuits at the initial marking M_0. Then, we compute $\boldsymbol{I}^T$ and P_a to add tokens.

$$\min \quad \Delta f(M_0) = I_{p_1} + I_{p_2} + 6I_{p_3} + 16I_{p_4} + 6I_{p_5} + I_{p_6} + I_{p_7} + I_{p_8} + I_{p_9}$$

$$s.t \begin{cases} I_{p_1} + I_{p_2} = 1 \\ I_{p_3} + I_{p_4} + I_{p_5} = 1 \end{cases}$$

We can find $\boldsymbol{I}^T$ = (1, 0, 0, 0, 1, 0, 0, 0, 0), and $P_a = \{p_1, p_5\}$. Then, we add 1 token and 3 tokens to place p_1 and p_5, respectively. We can observe from Table I that after the first iteration step, the net has 1 critical circuit γ_1. Then, we only need to add tokens to place p_1 to decrease the average cycle time. The optimal marking is M = (6, 0, 4, 0, 3, 0, 6, 2, 0) and the weight sum of tokens is $f(M)$ = 32.

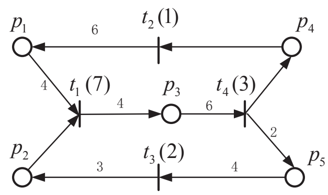

Fig. 5. An example from Sauer [5].

Let us consider another example in Fig. 5. There are two weighted circuits:

$$\begin{cases} \gamma_1 = \{p_1, t_1, p_3, t_4, p_4, t_2\} \\ \gamma_2 = \{p_2, t_1, p_3, t_4, p_5, t_3\} \end{cases}$$

The minimal T-semiflows of γ_1 and γ_2 are

$$\boldsymbol{x}_1 = (3, 2, 0, 2), \quad \boldsymbol{x}_2 = (3, 0, 1, 2)$$

while the minimal P-semiflows of γ_1 and γ_2 are

$$\boldsymbol{y}_1 = (1, 0, 1, 6, 0), \quad \boldsymbol{y}_2 = (0, 4, 1, 0, 3)$$

Therefore, the P-semiflow used in the criteria $f(M_0)$ is $\boldsymbol{y}$ = $\boldsymbol{y}_1$ + $\boldsymbol{y}_2$ = (1, 4, 2, 6, 3), and the minimal T-semiflow of the net is $\boldsymbol{x}$ = (3, 2, 1, 2).

For γ_1, since $y_1 = 1$ and $x_i \neq 1$ $(i = 1, 2, 4)$, we can compute $W(M_D) = 8$; for γ_2, $x_3 = 1$ and $W'_L = max\{\mu_j \cdot y_j\} = \{1 \times 4, 6 \times 1, 4 \times 3\} = 12$.

Thus we have gcd_{p_1} = 2, gcd_{p_2} = 1, gcd_{p_3} = 2, gcd_{p_4} = 1, and gcd_{p_5} = 2.

$$\boldsymbol{G_d} = (2, 4, 4, 6, 6)$$

min $f(M) = M(p_1) + 4M(p_2) + 2M(p_3) + 6M(p_4) + 3M(p_5)$

$$s.t. \begin{cases} M(p_1) + M(p_3) + 6M(p_4) > 8 \\ 4M(p_2) + M(p_3) + 3M(p_5) = 12 \end{cases}$$

We obtain a marking $M = (9, 3, 0, 0, 0)$ and

$$M^\star(p_1) = \left\lfloor \frac{M(p_1)}{gcd_{p_1}} \right\rfloor \cdot gcd_{p_1} = \left\lfloor \frac{9}{2} \right\rfloor \cdot 2 = 8$$

$$M^\star(p_2) = \left\lfloor \frac{M(p_2)}{gcd_{p_2}} \right\rfloor \cdot gcd_{p_2} = \left\lfloor \frac{3}{1} \right\rfloor \cdot 1 = 3$$

Then the initial marking is $M_0 = M^\star = (8, 3, 0, 0, 0)$. We can observe from Table II that both γ_1 and γ_2 are critical circuits at M_0. Accordingly, we compute $\boldsymbol{I}^T$ and P_a to select places which tokens should be added.

$$\min\ \Delta f(M_0) = 2I_{p_1} + 4I_{p_2} + 4I_{p_3} + 6I_{p_4} + 6I_{p_5}$$

$$s.t. \begin{cases} I_{p_1} + I_{p_3} + I_{p_4} = 1 \\ I_{p_2} + I_{p_3} + I_{p_5} = 1 \end{cases}$$

We find $\boldsymbol{I}^T$ = (0, 0, 1, 0, 0) and P_a = $\{p_3\}$ which means that we only need to add tokens to one place (p_3) to decrease the average cycle time of γ_1 and γ_2 simultaneously. The optimal marking is $(8, 3, 4, 0, 0)$.

C. Comparison with previous approaches

An approach dealing with the marking optimization problem of TWMG was presented by Sauer [5], where an iterative heuristic solution algorithm was proposed, which starts with a feasible marking and removes one token at a time, as long as the average cycle time is lower than or equal to the cycle time upper bound. In this approach, the initial marking M_0 that contains enough tokens to ensure it represents a feasible solution. Then, the value of the average cycle time is computed by using simulation. At each iteration, only one token is removed from a place and the liveness of the net needs to be checked. Since the number of tokens is large, the algorithm usually requires a large number of iterations to reach convergence and the time to verify the liveness is also long.

For the heuristic solution proposed in this paper, we start with a live marking that has a small value of performance index $f(M_0)$. Then we add tokens to some places until the average cycle time is lower than or equal to a given value. We use simulation to compute the average cycle time. The iteration steps are less and we do not need to check the liveness of the net for every iteration that save us a lot of time.

We can see the comparison between our method and Sauer's method from Table III. The number inside the parentheses are results of Sauer's approach.

TABLE III

THE COMPARISON OF TWO METHODS.

	Example 1	Example 2
iteration step	3 (66)	2 (17)
Times(s)	267 (4996)	75 (353)
Optimal solution	32 (32)	28 (28)

V. CONCLUSION

This paper deals with the marking optimization problem of deterministic TWMG. We present a new heuristic solution which is fast and efficient to solve the problem of minimizing a P-semiflow criteria while obtaining a given average cycle time. It is shown the proposed approach may provide an optimal solution in some cases.

Future work will pertains to the problem that finds an optimal allocation policy to minimize the average cycle time when a certain number of tokens is given.

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