Finite-Time Consensus with Disturbance Attenuation for Directed Switching Network Topologies by Discontinuous Local Interactions

Mauro Franceschelli, Alessandro Giua, Alessandro Pilloni, Alessandro Pisano and Elio Usai

Abstract

In this paper we investigate the properties of a decentralized consensus algorithm for a network of continuous-time integrators subject to unknown-but-bounded persistent disturbances. The proposed consensus algorithm is is based on a discontinuous local interaction rule. Under certain restrictions on the directed switching topology of the communication graph, it is proven that after a finite transient time the agents achieve an approximated consensus condition by attenuating the destabilizing effect of the disturbances. Lyapunov analysis is carried out to for characterizing the performance of the suggested algorithm. Simulative analysis are illustrated and commented to validate the developed result.

Draft version, published as: M. Franceschelli, A. Pilloni, A. Pisano, A. Giua, E. Usai, "Finite-Time Consensus with Disturbance Attenuation for Directed Switching Network Topologies by Discontinuous Local Interactions," 52nd IEEE Conf. on Decision and Control (Florence, Italy), Dec. 10-13, 2013.

The research leading to these results has been partially supported by the European Community's Seventh Framework Programme FP7/2007-2013 under Grant Agreement n°257462 [Research Project "HYCON2-Network of excellence"] and by Region Sardinia project n°CRP-24709 ["SIAR-Sistemi Interconnessi per l'Automazione su Reti"] (L.R.7/2007).

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I. INTRODUCTION

The problem of reaching consensus, i.e., driving the state of a set of interconnected dynamical systems towards the same value, has received much attention due to its many applications in, both, the modeling of natural phenomena such as flocking (see e.g. [1], [?], [2], [3]) and in the solution of several control problems involving synchronization or agreement between dynamical systems (see [4], [5], [6]).

In this paper we discuss an approach to reach consensus in a network of interacting agents whose dynamics are modeled by first order continuous time integrators subject to unknown-but-bounded persistent perturbations. The approach is based on a local interaction rule which combines linear and nonlinear terms. The linear terms, as usual, feed each agent with the difference between the current agent's state and the states of its neighbors, while the nonlinear terms consider the sign of those differences yielding a discontinuous local interaction rule involving sliding mode control concepts (see [7]).

Discontinuous local interaction rules have been used in the framework of consensus or agreement algorithms to exploit the underlying finite-time convergence and robustness against disturbances and unmodeled dynamics. Several examples of applications to flocking or synchronization problems can be found in the literature (see e.g. [8]). Discontinuous local interactions were studied in [9], within a general framework of gradient flows, and several examples of discontinuous consensus protocols were analyzed.

In [10], a finite-time consensus algorithm is proposed to address the leader-follower tracking problem in multi-robot systems with static topology but varying leader. In [11], [12] and [13], finite-time consensus algorithms are provided for networks of unperturbed integrators by exploiting discontinuous local interaction rules under time varying (both undirected and directed) network topologies.

The consensus problem in presence of measurement errors is studied in [14], in a discrete-time setting, with reference to linear consensus protocols with constant or vanishing weights. The authors derive explicit upper bounds to the maximum disagreement error as function of the bounds on the noise magnitude and of the smallest non-zero singular value of the network's state update matrix.

In [15] the authors suggest a class of non-linear continuous protocols that are able to achieve the so-called "ε-consensus", namely an approximate agreement condition where all agents converge towards a set, in spite of the presence of additive disturbances. Our work differs from [15] in that we consider a discontinuous protocol, as opposed to continuous, that is able to achieve almost complete disturbance rejection up to an arbitrarily small error if the network is always connected.

An approach that shares some technical issues with the protocol proposed in this paper is the continuous-time consensus problem in presence of quantization errors. In [16] the continuous-time consensus problem is studied in the case of quantized information exchange between agents, and this leads to an instance of discontinuous protocol where the effect of quantization can be regarded as a disturbance.

The approach illustrated in this paper further differs from the above mentioned literature works in that we address the analysis of the practical stability and disturbance attenuation properties of finite-time consensus under the effect of unknown perturbations and, additionally, with a switching and directed communication topology. In the present work the finite time transient to reach consensus can be made arbitrarily small by properly selecting the algorithm parameters. The disturbance rejection performance will primarily depend on the time-varying network connectivity properties. To the best of our knowledge, the above aspects were never simultaneously addressed and characterized in the existing literature.

The main result of the present work, outlined in Theorem 1, consists in proposing a feasible local interaction rule which provides finite time convergence of the network to a condition of approximate agreement, by attenuating the effect of the disturbances. This result is subject to the requirement that the time varying graph defining the network switching interaction topology stays weakly connected during, at least, a certain "minimal percentage" of time.

This paper generalizes the preliminary results presented in [17] by extending the analysis to cover directed switching topologies that were not dealt with in the original paper. The key factor enabling such an extension is a modification of the underlying Lyapunov analysis, which, in the present paper, involves a max function considering the maximal difference between the agents' states. This new approach considerably relaxes the conservativeness of the tuning inequalities guaranteeing convergence to the approximate consensus condition using lower values of the control gains. Additionally, we consider here continuous and discontinuous terms in the local interaction rule in such a way that the convergence to consensus can be accelerated by increasing the weight of the linear continuous terms, rather than those of the nonlinear discontinuous terms, thereby mitigating the chattering effect.

The structure of the paper is as follows. In Section II we recall some basic definitions and formulate the problem under investigation. In Section III we describe the proposed local interaction rule and we investigate the associated convergence properties by stating the main result of this paper. In Section IV some simulation results are presented, and, finally, in Section V conclusions are drawn and possible future research directions are discussed.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let us consider a network consisting of N interacting agents whose communication topology, is modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V}^2$ denote, respectively, the collection of agents and the edge set. An edge, denoted as (i,j), belongs to $\mathcal{E}$ if the agent j is able to obtain information from its neighbor i. As a consequence, the set of neighbors of the agent i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V}/\{i\} : (j,i) \in \mathcal{E}\}$. By assumption the presence of self-loops in $\mathcal{G}$ is not allowed. Each agent is modeled as a continuous-time perturbed integrator

$$\dot{x}_i(t) = \vartheta_i(t) + u_i(t), \quad x_i(0) = x_{i0}, \quad i \in \mathcal{V} \tag{1}$$

where $x_i(t) \in \mathbb{R}$ and x_{i0} are respectively the state of the i-th agent and its initial value, $u_i(t) \in \mathbb{R}$ is the local control input, and $\vartheta_i(t)$ is a bounded unknown perturbation.

The only assumption made on the unknown perturbations $\vartheta_i(t)$ is:

$$\exists\ \Pi \in \mathbb{R}^+\ :\ \forall\ i \in \mathcal{V}, \quad |\vartheta_i(t)| \leq \Pi \tag{2}$$

Assuming that at each time instant, only a subset of the available communication edges in $\mathcal{G}$ is active for information exchange, we define $\hat{\mathcal{G}}(t) = (\mathcal{V}, \mathcal{E}(t))$ as a time-varying graph representative of the active instantaneous topology, where $\mathcal{E}(t) \subseteq \mathcal{E}$ is the subset of active edges at time t. Accordingly, we can define the instantaneous neighbors set of the i-th agent as follows:

$$\mathcal{N}_i(t) = \{j \in \mathcal{V}\ :\ (j,i) \in \mathcal{E}(t)\} \subseteq \mathcal{N}_i \tag{3}$$

Let Γ and t_r be two positive constants, the task of the present paper is to design a local interaction rule $u_i(t)$, compatible with $\hat{\mathcal{G}}(t)$, which can guarantee, under suitable assumptions on the time-varying topology, the achievement of the next *practical finite-time consensus* condition

$$\exists\ \Gamma,\ t_r \in \mathbb{R}^+\ :\ \forall\ t > t_r,\ \forall\ i,j \in \mathcal{V}, \quad |x_i(t) - x_j(t)| \leq \Gamma \tag{4}$$

III. MAIN RESULT AND CONVERGENCE ANALYSIS

The proposed local interaction protocol is defined as follows:

$$u_i(t) = u_{i,1}(t) + u_{i,2}(t), \quad i \in \mathcal{V} \tag{5}$$

with

$$u_{i,1}(t) = -\lambda_1 \sum_{k \in \mathcal{N}_i(t)} (x_i(t) - x_k(t)), \tag{6}$$

$$u_{i,2}(t) = -\lambda_2 \sum_{k \in \mathcal{N}_i(t)} \text{sign}\,(x_i(t) - x_k(t)), \tag{7}$$

where λ_1 and λ_2 are the nonnegative tuning constants of the algorithm and the sign$(\cdot)$ function is defined as follows

$$\text{sign}\,(\mathfrak{S}) = \begin{cases} 1 & \text{if} \quad \mathfrak{S} > 0 \\ 0 & \text{if} \quad \mathfrak{S} = 0 \\ -1 & \text{if} \quad \mathfrak{S} < 0 \end{cases} \tag{8}$$

Let $r_{ik}(t)$ be a binary variable, representative of the presence or not of a directed communication channel coming from agent i to agent k at time t, denoted as:

$$r_{ik}(t) = \begin{cases} 1 & \text{if } k \in \mathcal{N}_i(t) \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Then, we can rewrite the linear and nonlinear control components $u_{i,1}(t)$ and $u_{i,2}(t)$ in (6) and (7) as follows:

$$u_{i,1}(t) = -\lambda_1 \sum_{k \in \mathcal{V}, k \neq i} r_{ik}(t) \cdot (x_i(t) - x_k(t)), \quad \lambda_1 \geq 0. \tag{10}$$

$$u_{i,2}(t) = -\lambda_2 \sum_{k \in \mathcal{V}, k \neq i} r_{ik}(t) \cdot \text{sign}(x_i(t) - x_k(t)), \quad \lambda_2 > 0. \tag{11}$$

Remark 1: *Due to the concurrent effect of the suggested discontinuous local interaction rule (11), the switching network topology $\hat{\mathcal{G}}(t)$, and the possibly discontinuous nature of the external disturbances (supposed to be only uniformly bounded), the closed loop network dynamics (1) will be discontinuous and the resulting solution notion needs to be discussed and clarified. For a differential equation with discontinuous right-hand side, following [18], we understand the resulting solution in the so-called Filippov sense as the solution of an appropriate differential inclusion, the existence of which is guaranteed (owing on certain properties of the associated set-valued map) and for which noticeable properties, such as absolute continuity, are in force. The reader is referred to [19] for a comprehensive account of the notions of solution for discontinuous dynamical systems.*

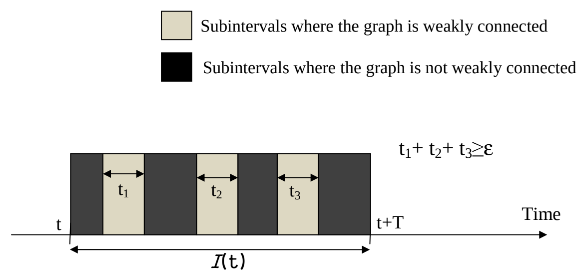

Fig. 1. Changes in network topology and communication constraints.

From now on we investigate the conditions under which the local interaction protocol (5)-(7) can achieve the approximate consensus conditions (4). Define a set of error variables for each edge in the network as follows

$$\delta_{ij}(t) = x_i(t) - x_j(t) \quad \text{with} \quad (i,j) \in \boldsymbol{\mathcal{E}}. \tag{12}$$

The dynamics of $\delta_{ij}(t)$ are easily obtained by differentiating (12), and considering the closed loop dynamics of each agents

$$\dot{x}_i = \vartheta_i - \lambda_1 \sum_{k\in\boldsymbol{\mathcal{V}},k\neq i} r_{ik}\delta_{ik} - \lambda_2 \sum_{k\in\boldsymbol{\mathcal{V}},k\neq i} r_{ik}\cdot \text{sign}(\delta_{ik}) \tag{13}$$

Trivial manipulations yield

$$\begin{aligned}\dot{\delta}_{ij} = \vartheta_i - \vartheta_j - \lambda_1 \left[\sum_{k\in\boldsymbol{\mathcal{V}},k\neq i} r_{ik}\delta_{ik} - \sum_{k\in\boldsymbol{\mathcal{V}},k\neq j} r_{jk}\delta_{jk}\right] + \\ -\lambda_2 \left[\sum_{k\in\boldsymbol{\mathcal{V}},k\neq i} r_{ik}\cdot \text{sign}(\delta_{ik}) - \sum_{k\in\boldsymbol{\mathcal{V}},k\neq j} r_{jk}\cdot \text{sign}(\delta_{jk})\right]\end{aligned} \tag{14}$$

The requirement concerning the switching communication topology is that the time varying graph $\hat{\boldsymbol{\mathcal{G}}}(t)$ stays weakly connected during, at least, a certain "minimal percentage" of time. This is formalized by the next Assumption.

Assumption 1: *There are positive constants ε and T, with $\varepsilon \leq T$, such that during the receding horizon time interval $\boldsymbol{\mathcal{I}}(t) = (t, t+T)$, $\hat{\boldsymbol{\mathcal{G}}}(t)$ is weakly connected[1] along a subinterval $\boldsymbol{\mathcal{S}}(t) \subseteq \boldsymbol{\mathcal{I}}(t)$, possibly formed by the union of disjoint subintervals, whose overall length is at least equal to ε.*

The meaning of Assumption 1 is clarified by the Figure 1, namely the overall duration of the disjoint grey subintervals during which the instantaneous digraph $\hat{\boldsymbol{\mathcal{G}}}(t)$ is weakly connected should be not less than the constant ε. We are now in a position to state the main result of the paper.

Theorem 1: *Consider the agents' dynamics (1), which satisfies (2), and let Assumption 1 be in force. Then, the discontinuous local interaction rule (5), (9)-(11) with tuning parameters selected according to*

$$\lambda_1 \geq 0 \quad , \quad \lambda_2 \geq \frac{2T\cdot\Pi}{\varepsilon} + \mu^2 \quad , \quad \mu \neq 0, \tag{15}$$

provides the approximate consensus condition (4) with

$$\Gamma = [2\cdot(T-\varepsilon) + \xi]\cdot\Pi, \tag{16}$$

where $\xi > 0$ is an arbitrary infinitesimally small positive parameter and the transient time t_r is upper bounded as follows

$$t_r \leq \left(\frac{T}{\varepsilon\mu^2}\right)\cdot \max_{i,j\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\mathcal{V}}} |x_i(0) - x_j(0)| \tag{17}$$

Proof:
Consider

$$V(t) = |\delta_{ij}(t)| \tag{18}$$

as a candidate Lyapunov function, where

$$(i,j) = \text{argmax}_{(i,j)\in\boldsymbol{\mathcal{V}}\times\boldsymbol{\mathcal{V}}} |\delta_{ij}(t)| \tag{19}$$

[1] A digraph is called weakly connected if every pair of nodes are connected by an undirected path [19].

in such a way that, without loss of generality, index i will correspond to an agent carrying the maximal value at time t among all the agents in the network, and, dually, index j will correspond to an agent carrying the minimal value, i.e.

$$x_i(t) = \sup_{h\in\mathcal{V}} x_h(t), \quad x_j(t) = \inf_{h\in\mathcal{V}} x_h(t) \tag{20}$$

Let us preliminarily address the case $\varepsilon < T$. It is worth to emphasize that the chosen Lyapunov function (18) is continuous at those time instants at which either i or j will change its value. Clearly, the vanishing of $V(t)$ implies the exact consensus condition among the agents of the network, while small values for $V(t)$ correspond to a practical consensus condition as in (4). Note that the considered Lyapunov function is locally Lipschitz and it is not differentiable when $\delta_{ij}(t) = 0$. Thus, we refer for stability analysis to the *Lyapunov Generalized Theorem* for non-smooth analysis reported in [20], which makes use of the *Clarke's Generalized Gradient* [21]. However, we can observe that $\delta_{ij}(t) = 0$ holds only when the exact consensus condition is in force, which will bring some useful simplification in the stability analysis.

In the remainder, we refer to the computation method illustrated in [20], where a Lyapunov analysis based on an analogous sum-of-absolute-value Lyapunov function was dealt with. All the necessary technicalities justifying the correctness of adopting the chain rule to compute the time derivative of $V(t)$, which exists almost everywhere in the form of a suitable set-valued map, are not reported here, and the reader is referred, e.g., to [9], [20], [22] where discontinuous systems and non-smooth Lyapunov tools analogous to those involved in the present analysis were discussed in detail.

The time-derivative of $V(t)$ along the solutions of the deviation error dynamics (14) takes the following set-valued form

$$\begin{aligned}\dot{V}(t) &= \mathrm{SIGN}(\delta_{ij}(t)) \cdot \dot{\delta}_{ij}(t) = \\ &= \mathrm{SIGN}(\delta_{ij}) \cdot (\vartheta_i - \vartheta_j) \\ &\quad - \lambda_1 \cdot \mathrm{SIGN}(\delta_{ij}) \sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \delta_{ik} \\ &\quad + \lambda_1 \cdot \mathrm{SIGN}(\delta_{ij}) \sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \delta_{jk} \\ &\quad - \lambda_2 \cdot \mathrm{SIGN}(\delta_{ij}) \sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \mathrm{sign}(\delta_{ik}) \\ &\quad + \lambda_2 \cdot \mathrm{SIGN}(\delta_{ij}) \sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \mathrm{sign}(\delta_{jk})\end{aligned} \tag{21}$$

where $\mathrm{SIGN}(\delta_{ij}(t))$, the generalized gradient of $V(t)$ (see [20]), is the multi-valued function

$$\mathrm{SIGN}(\delta_{ij}(t)) = \begin{cases} 1 & \text{if} \quad \delta_{ij}(t) > 0 \\ [-1,1] & \text{if} \quad \delta_{ij}(t) = 0 \\ -1 & \text{if} \quad \delta_{ij}(t) < 0 \end{cases} \tag{22}$$

Note that by definition, and considering (20), as long as $V(t) \neq 0$ we have $\mathrm{SIGN}(\delta_{Mm}(t)) = 1$. Furthermore due to the uniform boundedness of the disturbance (2), the next estimation is in force

$$|\vartheta_M - \vartheta_m| \leq 2\Pi \tag{23}$$

Thus, we can manipulate (21) so as to obtain

$$\begin{aligned}\dot{V}(t) \leq 2\cdot\Pi - \lambda_1 &\sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \delta_{ik} + \\ + \lambda_1 &\sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \delta_{jk} + \\ - \lambda_2 &\sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \mathrm{sign}(\delta_{ik}) + \\ + \lambda_2 &\sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \mathrm{sign}(\delta_{jk})\end{aligned} \tag{24}$$

Note that, in light of (20), irrespectively of the instantaneous current graph topology, all the state-dependent feedback terms in the right hand side of (24) are nonnegative, i.e.

$$\begin{aligned}-\lambda_1 \sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \delta_{ik} + \lambda_1 \sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \delta_{jk} + \\ - \lambda_2 \sum_{k\in\mathcal{V},k\neq i} r_{ik} \cdot \mathrm{sign}(\delta_{ik}) + \\ + \lambda_2 \sum_{k\in\mathcal{V},k\neq j} r_{jk} \cdot \mathrm{sign}(\delta_{jk}) \leq 0\end{aligned} \tag{25}$$

The receding horizon time interval $\mathcal{I}(t) = (t, t+T)$ is divided into the union of subinterval $\mathcal{S}(t)$, along which the graph is guaranteed to be weakly connected, and the complementary interval $\mathcal{I}(t) \setminus \mathcal{S}(t)$ during which nothing can be said about the connectivity properties of the switching graph. By virtue of (24) and (25) one can conclude that

$$\dot{V}(t) \leq 2\cdot\Pi, \quad t \in \mathcal{I}(t) \setminus \mathcal{S}(t). \tag{26}$$

It shall be noted that the pair (i,j) is not uniquely defined and there can be multiple agents carrying the maximal or minimal values x_i and x_j at time t. At those time instants when $\hat{\mathcal{G}}(t)$ is weakly connected, however, at least **one of the following conditions holds**:

1) among all agents carrying the maximal value, there is at least one of them which admits, among its neighbors, one agent with state value strictly less than x_i;
2) among all agents carrying the minimal value, there is at least one of them which admits, among its neighbors, one agent with state value strictly greater than x_i;

Suppose i (resp., j) is the agent for which the maximum (resp., minimum) is achieved at time t. If there are many such agents, we choose one, if any, which share an active edge with a neighbor having state value strictly less (resp., greater) than x_i (resp., x_j). If there are still many of such agents we choose any one of those, but commit to that until a new agent holds the maximum (resp., minimum) value.

As a consequence of the previous developments, at those time instants when $\hat{\mathcal{G}}(t)$ is weakly connected there exists at least an agent index $\bar{k}$, $\bar{k} \neq i$, $\bar{k} \neq j$, which satisfies at least one of the following conditions:

$$r_{i\bar{k}}(t) = 1 \quad , \quad \delta_{i\bar{k}} > 0 \tag{27}$$

$$r_{j\bar{k}}(t) = 1 \quad , \quad \delta_{j\bar{k}} < 0 \tag{28}$$

When either of (27) and (28) is in force, it follows that the right hand side of (24) can be upper-estimated as follows. Whenever $t \in \mathcal{S}(t)$ and $V(t) \neq 0$

$$\dot{V}(t) \leq 2 \cdot \Pi - \lambda_2 \quad t \in \mathcal{S}(t) \tag{29}$$

By construction the next relation holds:

$$V(t+T) - V(t) = \int_{\mathcal{S}(t)} \dot{V}(\tau)\, d\tau + \int_{\mathcal{I}(t)\setminus\mathcal{S}(t)} \dot{V}(\tau)\, d\tau \tag{30}$$

By noticing that the length of the subinterval $\mathcal{S}(t)$ is at least ε, according to the Assumption 1, it follows that the length of the interval $\mathcal{I}(t) \setminus \mathcal{S}(t)$ will not exceed the value of $T - \varepsilon$.

Thus, in light of (26) and (29), one can manipulate (30) as

$$\begin{aligned} V(t+T) - V(t) \leq \varepsilon\,(2\Pi - \lambda_2) + (T-\varepsilon)\, 2 \cdot \Pi = \\ = -\varepsilon\lambda_2 + 2T \cdot \Pi \end{aligned} \tag{31}$$

By plugging (15) into (31) one obtains the next condition

$$V(t+T) - V(t) \leq -\mu^2 \varepsilon. \tag{32}$$

which will be satisfied as long as $V(\tau) \neq 0 \ \forall \ \tau \in (t, t+T)$, thereby guaranteeing the existence of a finite t_r such that $V(t_r) = 0$. In order to evaluate an upper bound to the transient time t_r, denote $V_\kappa = V(\kappa T)$, and express (32) in the form of the difference equation

$$V_{\kappa+1} = V_\kappa - \mu^2 \varepsilon \tag{33}$$

which admits the solution

$$V_\kappa = V(0) - \kappa \cdot \mu^2 \varepsilon \tag{34}$$

From (34) it can be readily concluded that

$$t_r \leq \left(\frac{T}{\varepsilon\mu^2}\right) \cdot V(0) = \left(\frac{T}{\varepsilon\mu^2}\right) \cdot \max_{i,j \in \mathcal{V}\times\mathcal{V}} |x_i(0) - x_j(0)| \tag{35}$$

which is according to (17). We now prove that, at $t \geq t_r$, the Lyapunov function $V(t)$ undergoes bounded fluctuations preserving the consensus accuracy established by (4) and (16). Define

$$V_S = \sup_{t \geq t_r} V(t) \tag{36}$$

which sets the ultimate precision of the approximate consensus condition. If, at any time t' one has that $V(t') = 0$ then along the time interval $t \in (t', t'+T)$ the Lyapunov function $V(t)$ may deviate form zero, at most, by a quantity $2(T-\varepsilon)\Pi$, which is obtained by integrating (26) for a time $T - \varepsilon$ (the maximal consecutive time interval in which the graph is disconnected, according to the Assumption 1 starting from the zero initial condition. Thereby, the domain

$$V(t) \leq 2(T-\varepsilon)\Pi. \tag{37}$$

is positively invariant at any $t \geq t_r$.

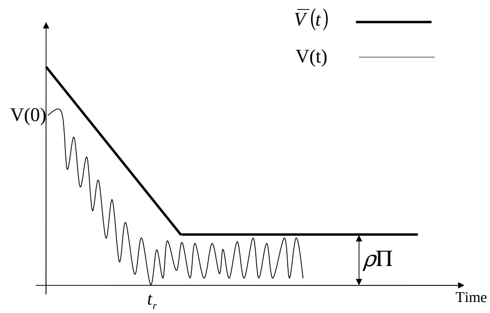

Fig. 2. Actual and majorant curves of V(t)

Now let us address the case in which $\varepsilon = T$, i.e. the time varying graph is weakly connected at all times. The previous analysis has shown that there exists a finite time t_r, satisfying (17), at which exact consensus is achieved, i.e. $V(t_r) = 0$. Unfortunately, $V(t) = 0$ cannot be an equilibrium state at $t \geq t_r$ due to the fact that all the local control laws $u_i(t)$ are identically zero when $V(t) = 0$ (as a consequence of all δ_{ij}'s being zero and in view of the adopted definition (8) of the sign function) while the disturbances $\vartheta_i(t)$ are not. On the other hand, an infinitesimal deviation of $V(t)$ from zero will restore the convergence features of the algorithm, steering immediately $V(t)$ back to zero. This phenomenon, local instability of the ideal consensus condition $V(t) = 0$ when the disturbances are acting, can be characterized by an infinitesimal increase of Γ as follows:

$$\Gamma \leq [2(T-\varepsilon)+\xi]\Pi \tag{38}$$

where ξ is an arbitrarily small positive real number. Theorem 1 is proven.

■

Remark 2: Note that the transient time, which satisfies (35), can be made arbitrarily small by taking the design parameter μ in (15) large enough. It can be defined a μ-dependent majorant curve, illustrated in Figure 2, such that

$$V(t) \leq \bar{V}(t) = \max\left\{V(0) - \mu^2 \varepsilon \frac{t}{T} + \Gamma, \Gamma\right\}, \tag{39}$$

It is also worth to remark that the tuning of the gain λ_2 does not require the perfect knowledge of the time varying network topology, and it is carried out on the basis of an upper bound to the noise magnitude and an upper bound to the ratio T/ε that sets the relative amount of time during which the network is weakly connected.

IV. NUMERICAL SIMULATION

To demonstrate the effectiveness of the proposed local interaction protocol, a network of 20 agents is considered, which interact through a randomly chosen directed communication network with switching topology. Each agent, modeled as in (1), has a randomly chosen initial state $x_{i0} \in [0,5]$. The disturbances are selected according to

$$\vartheta_i(t) = \eta_i(t) + \alpha_i + \beta_i \cdot \sin(20 \cdot t + \phi_i), \quad i = 1, \ldots, 20 \tag{40}$$

where $\eta_i(t)$ is a bounded uniformly distributed random signal, α_i is a random constant, and the pair β_i, ϕ_i are the characteristic parameter of the harmonic part of the disturbance. All the underlying disturbance parameters have been randomly chosen in such a way to guarantee the bound $|\vartheta_i(t)| \leq \Pi = 2.5 \ \forall \ i$.

The communication topology is set by a randomly chosen time-varying graph $\mathcal{G}(t)$ such that at most $|\mathcal{E}| = 30$ edges can be simultaneously active. The random edge selection policy is implemented in such a way that the requirement of Assumption 1 is met. The value $T = 0.01s$ is used in all tests while different choices for ε have been considered for the sake of comparison.

Four tests, using different values of ε and of the control gains λ_1, λ_2 have been considered, according to the next tabular representation.

$$\begin{array}{llll} \text{TEST1}: & \varepsilon = T, & \lambda_1 = 0, & \lambda_2 = 6 \\ \text{TEST2}: & \varepsilon = 0.5T, & \lambda_1 = 0, & \lambda_2 = 11 \\ \text{TEST3}: & \varepsilon = 0.05T, & \lambda_1 = 0, & \lambda_2 = 101 \\ \text{TEST4}: & \varepsilon = 0.5T, & \lambda_1 = 5, & \lambda_2 = 11 \end{array}$$

The chosen control gains are always according to the design inequalities (15). The continuous time network (1) has been simulated numerically by using the Euler fixed-step solver with sampling time $T_s = 10^{-4}$. Figure 3 and Figure 4 display, respectively, the time evolutions of the agent state variables, and of the corresponding Lyapunov function $V(t)$, relative to the first three tests. It can be verified that in all tests agents are synchronized after a finite transient time. Particularly, Figure

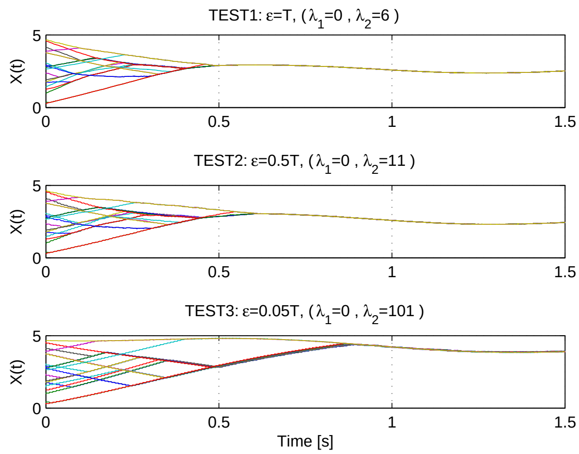

Fig. 3. Time evolution of the clock variables for TEST1-3 (right).

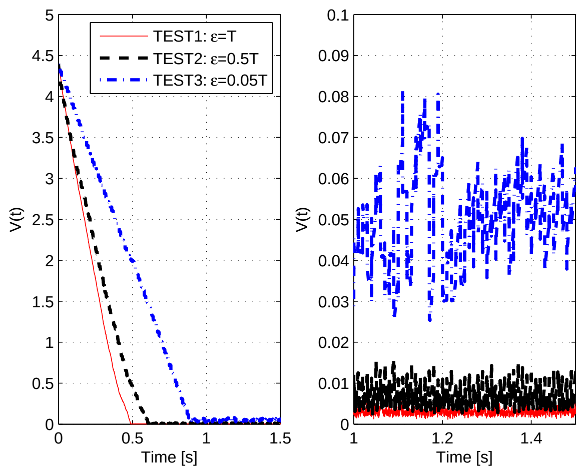

Fig. 4. Transient evolution (left) and steady state accuracy (right) of the Lyapunov function $V(t)$.

4 shows the negative impact of an increasing difference $T-\varepsilon$ on the steady state accuracy, in accordance with conditions (4) and (16).

With reference to TEST2 and TEST4, Figure 5 shows how the introduction of the linear control component in the consensus protocol (5)-(7) speeds up the achievement of consensus without causing chattering, as it would be the case by increasing the parameter λ_2 instead.

Figure 6 shows the Lyapunov function relative to an additional conclusive test (TEST5) where, under the same conditions of TEST1, the external perturbations have been removed ($\vartheta_i(t) = \Pi = 0$). A small residual synchronization error is still present, even if the achievement of a theoretically-exact consensus condition would be expected in this condition due to (16). The source of this error is, however, of purely numerical nature and the size of the residual set teds to zero while the sampling-time T_s is progressively reduced.

V. CONCLUSIONS AND FUTURE WORKS

In this paper a distributed algorithm, based on the mixed use of continuous and discontinuous local interaction rules, is suggested to solve the finite-time consensus problem in a network of continuous time integrators with additive disturbances. It has been proven that the network converges in finite-time to an approximate consensus condition. Numerical simulations have been provided to corroborate the analytical results. Among the most interesting directions for next research, more complex agent's dynamics are currently under investigation. Convergence under discrete time implementation of the proposed interaction rule, whose proof demands different and more involved Lyapunov analysis, is under study as well.

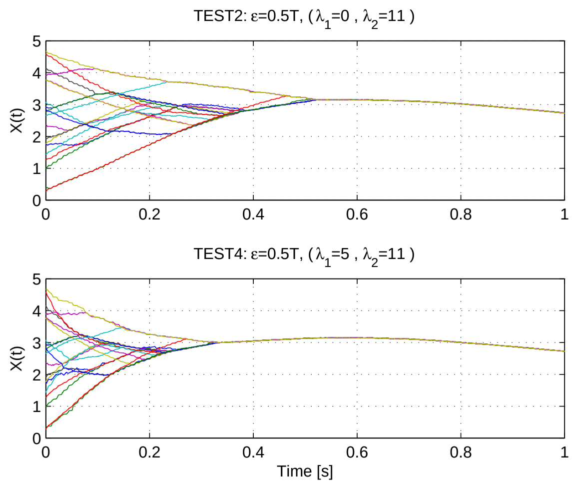

Fig. 5. Transient evolution of the agent states in TEST2 and TEST4.

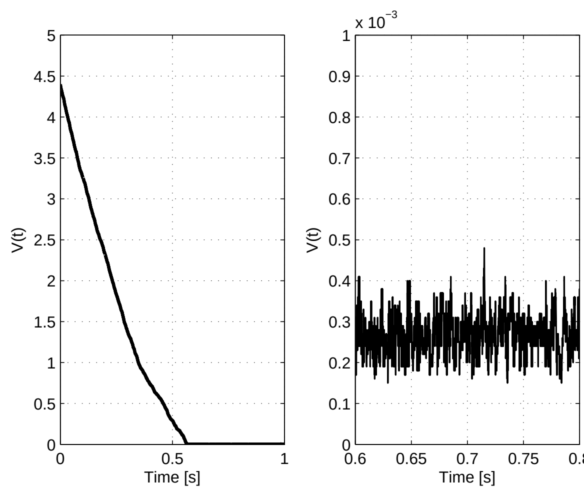

Fig. 6. Transient evolution (left) and steady state accuracy (right) of the Lyapunov function $V(t)$ in TEST 5.

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