Some New Results on Supervisory Control of Petri Nets with Decentralized Monitor Places

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Abstract

This paper presents two new results on the problem of determining a set of decentralized controllers for place/transition nets to enforce a global specification on the net behavior. Both the global specification and the decentralized specifications are given in terms of Generalized Mutual Exclusion Constraints (GMECs). First, an algorithm to select a decentralized specification that finds a compromise between fairness among variables and the maximal cardinality is proposed assuming that the support of each decentralized GMEC is a singleton. Then, a maximal solution in terms of permissiveness and fairness among places is proposed removing the previous assumption.

Published as:

F. Basile, A. Giua, C. Seatzu, ”Some new results on supervisory control of Petri nets with decentralized monitor places,” *17th IFAC World Congress* (Seoul, Korea), July 2008.

I. Introduction

This paper focuses on decentralized supervisory control of Petri Nets (PNs). Decentralized supervisory control has received a great attention in the Discrete Event Systems (DES) area in the last decade (5) but usually formal languages and automata based approaches have been preferred (1; 4; 6; 7) although the compact representation of PNs may potentially help in reducing the complexity of the problem.

In particular, we study the problem of determining a set of local supervisors when the global specification is given by a set of Generalized Mutual Exclusion Constraints (GMECs) $(\boldsymbol{W}, \boldsymbol{k})$, where $\boldsymbol{W} = [\boldsymbol{w}_1^T, \boldsymbol{w}_2^T, ...\boldsymbol{w}_{n_c}^T]^T$, $\boldsymbol{k} = [k_1, k_2, ...k_{n_c}]^T$ and the set of legal markings is $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k}) = \{\boldsymbol{m} \in \mathbb{N}^m \mid \boldsymbol{W} \cdot \boldsymbol{m} \leq \boldsymbol{k}\}$. A control architecture without central coordinator and communication between local supervisors is here considered. The set of places is partitioned into ν disjoint sets P_j, and the j-th local supervisor may enforce only places in P_j to assume a certain set of values.

In (2) under the assumption that: (A1) all weights are positive, i.e., $\boldsymbol{W} \geq \boldsymbol{0}$, $\boldsymbol{k} \geq \boldsymbol{0}$, (A2) all transitions are controllable and observable, (A3) the support of each decentralized GMEC is a singleton, thus $\nu = m$ and $P_j = \{p_j\}$, for $j = 1, \ldots, m$, it was shown that this problem can be solved by computing an *integer inner box* $\mathcal{I}(\boldsymbol{u}) = \{\boldsymbol{m} \in \mathbb{N}^m \mid \boldsymbol{m} \leq \boldsymbol{u}\}$ included in the set of legal markings defined by the global GMEC $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$. In particular, the problem of finding a *maximal integer inner box* $\mathcal{I} \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$, i.e. an inner box such that there does not exist a box $\tilde{\mathcal{I}} \neq \mathcal{I}$ and $\mathcal{I} \subsetneq \tilde{\mathcal{I}} \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$, was considered. A solution that aims to guarantee fairness among places was proposed, that can be computed using a simple iterative algorithm.

Based on the approach in (2) two new results are presented in this paper.

(1) We first show that, if assumption (A3) is removed allowing the support of decentralized GMECs to include two or more places, a maximal set of legal markings can be obtained directly from the solution computed under assumption (A3).

(2) We propose an algorithm to optimize the permissiveness of the closed loop behavior under decentralized control by selecting with an heuristic rule the decentralized specifications that find a compromise between fairness among variables and the maximal cardinality of the set of legal markings under decentralized control. This procedure works under assumptions (A1),(A2) and (A3). Furthermore, an example shows that the selection is improved if additional GMECs coming from net properties (e.g., place bounds) are added to the control specification.

II. Basic definitions

A. Petri nets

A Place/Transition (P/T) is represented by $N = (P, T, \boldsymbol{Pre}, \boldsymbol{Post})$ where: P is a set of m *places*; T is a set of n *transitions*; $P \cap T = \emptyset$, $P \cup T \neq \emptyset$; $\boldsymbol{Pre}$ ($\boldsymbol{Post}$) is the $m \times n$ sized, natural valued, pre-(post-)incidence matrix. We denote by $\boldsymbol{C}$ the incidence matrix, by the $m \times 1$ vector $\boldsymbol{m} : P \to \mathbb{N}$ the net marking, by $\boldsymbol{m}[t\rangle\boldsymbol{m}'$ the fact that an enabled transition t may fire at $\boldsymbol{m}$ yielding $\boldsymbol{m}'$. A *firing sequence* from $\boldsymbol{m}_0$ is a (possibly empty) sequence of transitions $\sigma = t_1, \ldots, t_k$ such that $\boldsymbol{m}_0[t_1\rangle\boldsymbol{m}_1[t_2\rangle\boldsymbol{m}_2 \ldots [t_k\rangle\boldsymbol{m}_k$. A P/T system $\langle N, \boldsymbol{m}_0\rangle$ is a P/T net N with an initial marking $\boldsymbol{m}_0$. A marking $\boldsymbol{m}$ is reachable in $\langle N, \boldsymbol{m}_0\rangle$ iff there exists a firing sequence σ such that $\boldsymbol{m}_0[\sigma\rangle\boldsymbol{m}$. Given a net system $\langle N, \boldsymbol{m}_0\rangle$ the set of reachable markings is denoted $R(N, \boldsymbol{m}_0)$. Consider a net system $\langle N, \boldsymbol{m}_0\rangle$, the bound of place p is $B(p) = \max\{\boldsymbol{m}(p) | \forall \boldsymbol{m} \in R(N, \boldsymbol{m}_0)\}$.

B. Generalized Mutual Exclusion Constraints

A *Generalized Mutual Exclusion Constraint* (GMEC) is a couple $(\boldsymbol{w}, k)$ where $\boldsymbol{w} : P \to \mathbb{Z}$ is an m dimensional row vector and $k \in \mathbb{Z}$. A GMEC defines a set of *legal markings*:

$$\mathcal{M}(\boldsymbol{w}, k) = \{\boldsymbol{m} \in \mathbb{N}^m \mid \boldsymbol{w} \cdot \boldsymbol{m} \leq k\}.$$

The non legal markings are called *forbidden markings*. A controlling agent, called *supervisor*, must ensure the forbidden markings will be not reachable. So the set of legal markings under control is $\mathcal{M}_c(\boldsymbol{w}, k) = \mathcal{M}(\boldsymbol{w}, k) \cap R(N, \boldsymbol{m}_0)$. We call *support* of $(\boldsymbol{w}, k)$ the set $Q_{\boldsymbol{w}} = \{p \in P | \boldsymbol{w}(p) \neq 0\}$.

A set of GMECs $(\boldsymbol{W}, \boldsymbol{k})$, with $\boldsymbol{W} = [\boldsymbol{w}_1^T, \ldots \boldsymbol{w}_{n_c}^T]^T$, $\boldsymbol{k} = [k_1, \ldots k_{n_c}]^T$, defines the set of legal markings $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k}) = \{\boldsymbol{m} \in \mathbb{N}^m \mid \boldsymbol{W} \cdot \boldsymbol{m} \leq \boldsymbol{k}\}$. We call *support* of $(\boldsymbol{W}, \boldsymbol{k})$ the set $Q_{\boldsymbol{W}} = \{p \in P \cap \left(\cup_{j=1}^{n_c} Q_{\boldsymbol{w}_j}\right)\}$. It has been shown in (3) that a set of n_c GMECs can be enforced adding to the controlled net a set of n_c places called *monitors*, provided that the initial marking is legal.

C. Geometrical definitions

A *box* is a set of real vectors defined as

$$\mathcal{B}(\boldsymbol{l}, \boldsymbol{u}) = \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\},$$

where $\boldsymbol{l}$ and $\boldsymbol{u}$ are real d-vectors. If $\boldsymbol{x} \in \mathbb{N}^d$ we call $\mathcal{B}(\boldsymbol{0}, \boldsymbol{u})$ *integer box* and we denote it simply as $\mathcal{I}(\boldsymbol{u})$.

An *hypercube* is a box such that $\boldsymbol{u} = \boldsymbol{l} + \lambda \boldsymbol{e}$, where λ is a scalar and $\boldsymbol{e}$ denotes the d-vector of ones; an *integer hypercube* is an integer box $\mathcal{I}(\boldsymbol{u})$ such that $\boldsymbol{u} = \lambda \boldsymbol{e}$ where λ is a positive integer scalar.

Given a set $\mathcal{S}$, we denote by $|\mathcal{S}|$ its cardinality.

Definition 1: An integer box $\mathcal{I}(\boldsymbol{u}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ is a *maximal cardinality inner box* if there does not exist an inner box $\mathcal{I}(\tilde{\boldsymbol{u}}) \neq \mathcal{I}(\boldsymbol{u})$ such that $\mathcal{I}(\tilde{\boldsymbol{u}}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ and $|\mathcal{I}(\boldsymbol{u})| < |\mathcal{I}(\tilde{\boldsymbol{u}})|$. ■
The following results hold.

- The cardinality of an integer inner box $\mathcal{I}(\boldsymbol{u})$ is equal to $\prod_{p \in P} (\boldsymbol{u}(p) + 1)$ since each edge includes all integer numbers from 0 to $\boldsymbol{u}(p)$.
- The cardinality of an integer hypercube $\mathcal{I}(\tau \boldsymbol{e})$ having a vertex in the origin with edge length τ is equal to $(\tau + 1)^m$.

Note that determining a maximal integer hypercube is equivalent to determine a maximal cardinality integer hypercube since all edges have the same length.

III. Problem statement

Let $\langle N, \boldsymbol{m}_{p0} \rangle$ be a P/T system to be controlled, where $N = (P, T, \boldsymbol{Pre}, \boldsymbol{Post})$. Assume that a global specification is given in terms of a GMEC $(\boldsymbol{W}, \boldsymbol{k})$. Without loss of generality we take $Q_{\boldsymbol{W}} = P$, i.e., all places are bounded by the constraint. If such is not the case, we can simply apply the proposed procedure to the projection on $Q_{\boldsymbol{W}}$. Assume that the set of places P is partitioned into ν subsets $P_1, \ldots, P_\nu$.

We want to determine a set of *decentralized* GMECs $(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$ whose support is P_j, with $j = 1, \ldots, \nu$, such that

$$\cap_{j=1}^{\nu} \mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k}). \tag{1}$$

A set of $\mathcal{M}^{(j)} = \mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$, $j = 1, \ldots, \nu$, satisfying (1) is called a *set of decentralized legal markings sets*.

The choice of the decentralized GMECs, and thus of sets $\mathcal{M}^{(j)}$'s, is obviously not unique, and depends in general on the dimension $n_c^{(j)}$ of the decentralized GMECs $(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$.

In the rest of the paper we will make the following two assumptions.

(A1) Weights are positive, i.e., $\boldsymbol{W} \geq \mathbf{0}$, $\boldsymbol{k} \geq \mathbf{0}$.

(A2) Transitions are controllable and observable.

IV. Background on Previous results

In this section we recall some results we presented in (2) that hold under the following additional assumption.

(A3) The support of each decentralized GMEC is a singleton, thus $\nu = m$ and $P_j = \{p_j\}$, for $j = 1, \ldots, m$.

By assumption (A3) it follows that $n_c^{(j)} = 1$ and the effect of each decentralized GMEC is that of imposing an upper bound on the corresponding place. Thus, the set $\cap_{j=1}^{\nu} \mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$ can be regarded as the integer box $\mathcal{I}(\boldsymbol{u})$ where $\boldsymbol{u}$ is an m-integer vector whose j-th component denotes the bound induced by the j-the decentralized GMEC on place p_j.

Definition 2: An integer box $\mathcal{I}(\boldsymbol{u}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ is a *maximal integer inner box* if there does not exist an inner box $\mathcal{I}(\tilde{\boldsymbol{u}}) \neq \mathcal{I}(\boldsymbol{u})$ such that $\mathcal{I}(\boldsymbol{u}) \subsetneq \mathcal{I}(\tilde{\boldsymbol{u}}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$. ■

Note that the maximal integer inner box is in general not unique.

Proposition 3: (2) An integer box $\mathcal{I}(\boldsymbol{u})$ is a *maximal integer inner box* in $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ where $\boldsymbol{W} \geq \mathbf{0}$ and $\boldsymbol{k} \geq \mathbf{0}$, if and only if $\forall\, p \in P$:

$$\min_{i \in \{1,\ldots,n_c\}} \frac{k_i - \boldsymbol{w}_i \cdot \boldsymbol{u}}{\boldsymbol{w}_i(p)} < 1.$$

■

Thus, an integer box $\mathcal{I}(\boldsymbol{u})$ is maximal if and only if in each direction there exists at least one constraint that is saturated.

The following proposition provides a criterion to determine the *maximal integer hypercube* in $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$.

Proposition 4: Let $\bar{\mathcal{M}} = \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ be a marking set containing the null marking ($\boldsymbol{m} = \mathbf{0}$). Let us denote as $\tau(\bar{\mathcal{M}}) = \max\,\{\tau \in \mathbb{N} \mid \mathcal{I}(\tau \boldsymbol{e}) \subseteq \bar{\mathcal{M}}\}$.

It holds $\tau(\bar{\mathcal{M}}) = \min\limits_{i=1,\ldots,n_c} \tau(i, \bar{\mathcal{M}})$ where

$$\tau(i, \bar{\mathcal{M}}) = \left\lfloor \frac{k_i}{\sum_{p \in P} \boldsymbol{w}_i(p)} \right\rfloor$$

and $\lfloor\,\rfloor$ denotes the floor operator. ■

Different criteria can be used to determine a maximal integer inner box $\mathcal{I}(\boldsymbol{u}^*)$ in $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$ (2). The following algorithm, that ensures maximality and fairness among places, has been presented in (2).

Algorithm 5: [Maximal inner box computation]

1. Let $\tau_0 = 0$, $\boldsymbol{W}^0 = \boldsymbol{W}$, $\boldsymbol{k}^0 = \boldsymbol{k}$, $\boldsymbol{u}_0 = \boldsymbol{0}$,
$U_0 = \{1, \ldots, m\}$.

2. For $s = 1$ to m do

2.1. let $\mathcal{M}_{s-1} = \mathcal{M}(\boldsymbol{W}^{s-1}, \boldsymbol{k}^{s-1})$

2.2. let $\tau_s = \tau(\mathcal{M}_{s-1})$ (see Proposition 4)

2.3. let $\bar{\jmath}_s$ be an index arbitrarily chosen in

$$J_s = \left\{ \bar{\jmath} \in \mathbb{N} \;\middle|\; s^{s-1}_{\bar{\imath},\bar{\jmath}} = \min_{\substack{j \in U_{s-1} \\ i \in \{1,\ldots,n_c\}}} s^{s-1}_{i,j} \right\}$$

where $s^{s-1}_{i,j} = \dfrac{k_i^{s-1} - \tau_s \boldsymbol{w}_i^{s-1} \cdot \boldsymbol{1}}{\boldsymbol{w}_i(p_j)}$

2.4. for $i = 1$ to n_c do

let $\boldsymbol{w}_i^s(p_j) = \begin{cases} 0 & j = \bar{\jmath}_s \\ \boldsymbol{w}_i^{s-1}(p_j) & \text{otherwise} \end{cases}$

let $k_i^s = k_i^{s-1} - \tau_s \cdot \boldsymbol{w}_i(p_{\bar{\jmath}_s})$

2.5. let $\boldsymbol{u}_s(p_j) = \begin{cases} \tau_s & j = \bar{\jmath}_s \\ \boldsymbol{u}_{s-1}(p_j) & \text{otherwise} \end{cases}$

2.6. let $U_s = U_{s-1} \setminus \{\bar{\jmath}_s\}$

3. If the resulting sequence of τ's is

$\tau_1 \le \tau_2 \cdots \le \tau_\mu = \cdots = \tau_m$, then

3.1. let $\bar{\boldsymbol{u}}^0 = \boldsymbol{u}_m$,

3.2. for $s = 1$ to $\mu - 1$ do

3.2.1. for $j = 1, \ldots, m$, $j \neq \bar{\jmath}_s$

let $\bar{\boldsymbol{u}}^s(p_j) = \bar{\boldsymbol{u}}^{s-1}(p_j)$

3.2.2. let

$$\bar{\boldsymbol{u}}^s(p_{\bar{\jmath}_s}) = \begin{cases} \bar{\boldsymbol{u}}^{s-1}(p_{\bar{\jmath}_s}) & \text{if } \tau_s < \tau_{s+1} \\ \bar{\boldsymbol{u}}^{s-1}(p_{\bar{\jmath}_s}) + \displaystyle\min_{i \in \{1,\ldots,n_c\}} \left\lfloor \frac{k_i - \boldsymbol{w}_i \cdot \bar{\boldsymbol{u}}^{s-1}}{\boldsymbol{w}_i(p_{\bar{\jmath}_s})} \right\rfloor & \text{if } \tau_s = \tau_{s+1} \end{cases}$$

3.3. let $\boldsymbol{u}^* = \bar{\boldsymbol{u}}^{\mu-1}$.

else let $\boldsymbol{u}^* = \boldsymbol{u}_m$. ■

In simple words the algorithm first requires m iterative steps. At each step s we define a GMEC $(\boldsymbol{W}^s, \boldsymbol{k}^s)$, choosing at the initial step $(\boldsymbol{W}^0, \boldsymbol{k}^0) = (\boldsymbol{W}, \boldsymbol{k})$. We denote $\mathcal{M}_s = \mathcal{M}(\boldsymbol{W}^s, \boldsymbol{k}^s)$. At step s we compute the maximal integer hypercube in $\mathcal{M}_{s-1}$ using Proposition 4, and denote τ_s the corresponding edge. At each step we eliminate one place appropriately chosen from the support of the current GMEC, and assign to it an upper bound which coincides with the edge of the current hypercube. Thus, if $p_{\bar{\jmath}_s}$ is the place we eliminate at step s, it results $\boldsymbol{u}^*(p_{\bar{\jmath}_s}) = \tau_s$. In (2) we proved that the solution resulting at step 2 is maximal when the sequence of τ's is strictly increasing, apart from the tail of the sequence that may keep constant. On the contrary no guarantee is given if two or more τ's that are not in the tail are equal. Therefore, at step 3 we look for all variables to which it corresponds the same upper bound that is different from τ_m, and we verify if their upper bounds may be further increased. If so, we increase them as much as possible in accordance with the given constraints, and go further with our exploration.

V. An heuristic rule to maximize the cardinality

In this section we consider assumptions (A1) to (A3) and consider the problem of finding an inner box that is maximal not only with respect to set inclusion but also to cardinality. The motivation is to find a trade-off between fairness among places and permissiveness of the system.

The solution we propose starts from Algorithm 5 and differs from it essentially at step 1 where Algorithm 6 – which is formalized in the sequel — is executed in place of steps 1 and 2 of Algorithm 5.

The algorithm can be summarized as follows.

- It is based on m iterative steps. At each step s we define a GMEC $(\boldsymbol{W}^s, \boldsymbol{k}^s)$, choosing at the initial step $(\boldsymbol{W}^0, \boldsymbol{k}^0) = (\boldsymbol{W}, \boldsymbol{k})$. We denote $\mathcal{M}_s = \mathcal{M}(\boldsymbol{W}^s, \boldsymbol{k}^s)$.
- At step s we compute the maximal integer hypercube in $\mathcal{M}_{s-1}$ using Proposition 4, and denote τ_s the corresponding edge.
- At each step we eliminate one place appropriately chosen from the support of the current GMEC, and assign to it an upper bound which coincides with the edge of the current hypercube. We choose to eliminate at each step the place that corresponds to the smallest *relative slack* of constraint i with respect to place p_j as
$$s_{i,j} = \frac{k_i - \tau \boldsymbol{w}_i \cdot \boldsymbol{1}}{\boldsymbol{w}_i(p_j)}.$$

- If $\tau_s > \tau_{s-1}$ we compute the edges of the hypercube at the next iteration both keeping the current value of τ_s unaltered and reducing it of one unit ($\bar{\tau}_s = \tau_s - 1$).
- The number of points we have in the hyperplane defined by the variables that are eliminated at iterations s and $s+1$ in both cases ($\bar{V}_s$ and V_s) is compared.
- If the reduction of τ_s produces an improvement in this respect, the upper bound assigned to the variable that is eliminated at iteration s is set equal to $\bar{\tau}_s = \tau_s - 1$, otherwise it is set equal to τ_s.

Note that a more sophisticated heuristic consists in investigating further possibilities, namely comparing all the results obtained when the upper bound at iteration s assumes all integer values in $[\tau_{s-1}, \tau_s]$. We have not presented this case here because we found out no advantage in all the considered cases.

Algorithm 6: [Maximal cardinality inner box computation]

1. Let $\tau_0 = 0$, $\boldsymbol{W}^0 = \boldsymbol{W}$, $\boldsymbol{k}^0 = \boldsymbol{k}$, $\boldsymbol{u}_0 = \boldsymbol{0}$,
$U_0 = \{1, \ldots, m\}$.

2. For $s = 1$ to m do

2.1. let $\mathcal{M}_{s-1} = \mathcal{M}(\boldsymbol{W}^{s-1}, \boldsymbol{k}^{s-1})$

2.2. let $\tau_s = \tau(\mathcal{M}_{s-1})$ (see Proposition 4)

2.3. let $\bar{\jmath}_s$ be an index arbitrarily chosen in

$$J_s = \left\{ \bar{\jmath} \in \mathbb{N} \;\middle|\; s_{i,\bar{\jmath}}^{s-1} = \min_{\substack{j \in U_{s-1} \\ i \in \{1,\ldots,n_c\}}} s_{i,j}^{s-1} \right\}$$

where $s_{i,j}^{s-1} = \dfrac{k_i^{s-1} - \tau_s \boldsymbol{w}_i^{s-1} \cdot \boldsymbol{1}}{\boldsymbol{w}_i(p_j)}$

2.4. for $i = 1$ to n_c do

let $\boldsymbol{w}_i^s(p_j) = \begin{cases} 0 & j = \bar{\jmath}_s \\ \boldsymbol{w}_i^{s-1}(p_j) & \text{otherwise} \end{cases}$

let $k_i^s = k_i^{s-1} - \tau_s \cdot \boldsymbol{w}_i(p_{\bar{\jmath}_s})$

2.5. if $\tau_s > \tau_{s-1}$

let $\bar{\tau}_s = \tau_s - 1$

for $i = 1$ to n_c do

let $\bar{k}_i^s = k_i^{s-1} - \bar{\tau}_s \cdot w_i(p_{\jmath_s})$

$\mathcal{M}_s = \mathcal{M}(\boldsymbol{W}^s, \boldsymbol{k}^s)$, $\bar{\mathcal{M}}_s = \mathcal{M}(\boldsymbol{W}^s, \bar{\boldsymbol{k}}^s)$

let $\bar{V}_s = (\bar{\tau}_s + 1)(\tau(\bar{\mathcal{M}}_s) + 1)^{m-s}$

let $V_s = (\tau_s + 1)(\tau(\mathcal{M}_s) + 1)^{m-s}$

if $\bar{V}_s > V_s$ then $\tau_s = \bar{\tau}_s,\ k^s = \bar{k}^s$

2.6. let $\boldsymbol{u}_s(p_j) = \begin{cases} \tau_s & j = \bar{\jmath}_s \\ \boldsymbol{u}_{s-1}(p_j) & \text{otherwise} \end{cases}$

2.7. let $U_s = U_{s-1} \setminus \{\bar{\jmath}_s\}$

3. let $\bar{\boldsymbol{u}}^0 = \boldsymbol{u}^m$.

4. Execute Step 3.2 of Algorithm 5.

5. let $\boldsymbol{u}^* = \bar{\boldsymbol{u}}^{\mu-1}$. ■

In the next example we show how Algorithm 6 works, but also that it is possible to improve it if additional constraints obtained from the net system to be controlled are taken into account. Algorithm 6 at Step 2.3 in order to guarantee a certain fairness among places uses the relative slack $s_{i,j}^{s-1}$ to select which place has to be eliminated at step s. Additional constraints reduce the number of places which at the step s have the same relative slack, and often help to select decentralized specifications which optimize the closed-loop permissiveness since decentralized specifications redundant with respect to net behavior are discarded.

Example 7: Consider the set of legal markings $\mathcal{M}(\boldsymbol{w}, k) = \{\boldsymbol{m} \mid 3\boldsymbol{m}(p_1)+\boldsymbol{m}(p_2)+\boldsymbol{m}(p_3) \leq 5\}$. Applying Algorithm 6 we obtain:

$s = 1$

$\tau_1 = \tau(\mathcal{M}_1) = \lfloor \frac{5}{5} \rfloor = 1,\ J_1 = \{1\}$
$\bar{\tau}_1 = \tau(\bar{\mathcal{M}}_1) = \tau_1 - 1 = 0$
$\tau(\mathcal{M}_2) = 1;\ \tau(\bar{\mathcal{M}}_2) = 2$
Since $(0+1)(2+1)^2 = 9 > (1+1)(1+1)^2 = 8$, we set $\tau_1 = 0$
$\boldsymbol{w}_1^1 = \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right],\ k_1^1 = 5 - 3\tau_1 = 5$
$\boldsymbol{u}_1 = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right],\ U_1 = \{1\}.$

$s = 2$

$\tau_2 = \tau(\mathcal{M}_2) = \lfloor \frac{5}{2} \rfloor = 2,\ J_2 = \{2, 3\}$ choose $\bar{\jmath}_2 = 2$
$\bar{\tau}_2 = \tau(\bar{\mathcal{M}}_2) = \tau_2 - 1 = 1$
$\tau(\mathcal{M}_3) = 3;\ \tau(\bar{\mathcal{M}}_3) = 4$
Since $(2+1)(3+1) = 12 > (1+1)(4+1) = 10$, we set $\tau_2 = 2$
$\boldsymbol{w}_1^2 = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right],\ k_1^2 = 5 - \tau_2 = 3$
$\boldsymbol{u}_1^2 = \left[\begin{array}{ccc} 0 & 2 & 0 \end{array}\right],\ U_2 = \{1, 2\}.$

$s = 3$

$\tau_3 = \tau(\mathcal{M}_3) = \lfloor \frac{3}{1} \rfloor = 3,\ J_3 = \{3\}$
$\boldsymbol{w}_1^3 = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right],\ k_1^3 = 3 - \tau_3 = 0$
$\boldsymbol{u}^* = \boldsymbol{u}_3 = \left[\begin{array}{ccc} 0 & 2 & 3 \end{array}\right],\ U_3 = \{1, 2, 3\}.$

Thus,

$\mathcal{M}(\boldsymbol{w}^{(1)}, k^{(1)}) = \{\boldsymbol{m} \mid \boldsymbol{m}(p_1) \leq 0\},$
$\mathcal{M}(\boldsymbol{w}^{(2)}, k^{(2)}) = \{\boldsymbol{m} \mid \boldsymbol{m}(p_2) \leq 2\},$
$\mathcal{M}(\boldsymbol{w}^{(3)}, k^{(3)}) = \{\boldsymbol{m} \mid \boldsymbol{m}(p_3) \leq 3\},$

and the cardinality of the set $\cap_{i=1}^{3} \mathcal{M}(\boldsymbol{w}^{(i)}, k^{(i)})$ is $\prod_{s=1}^{3} (\tau(\mathcal{M}^s) + 1) = 12$.

Note that by selecting $\boldsymbol{m}(p_3)$ instead of $\boldsymbol{m}(p_2)$ at step $s = 2$, we obtain

$\mathcal{M}(\boldsymbol{w}^{(1)}, k^{(1)}) = \{\boldsymbol{m} \,|\boldsymbol{m}(p_1) \leq 0\}$,
$\mathcal{M}(\boldsymbol{w}^{(2)}, k^{(2)}) = \{\boldsymbol{m} \,|\boldsymbol{m}(p_2) \leq 3\}$,
$\mathcal{M}(\boldsymbol{w}^{(3)}, k^{(3)}) = \{\boldsymbol{m} \,|\boldsymbol{m}(p_3) \leq 2\}$,
but nothing changes in terms of cardinality. On the contrary the solution resulting from Algorithm 5 has a cardinality equal to 8.

Now, let $B(p_1) = 3$, $B(p_2) = 2$, $B(p_3) = 5$. Assume we also consider such bounds as additional GMECs. In such a case, we initially set

$$\boldsymbol{W}_0 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{k}_0 = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 5 \end{bmatrix}.$$

From Algorithm 6 we obtain $\boldsymbol{u}^* = \boldsymbol{u}_3 = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$. Thus, by taking into account additional constraints coming from the net system, only one solution is selected and it has a closed-loop permissiveness greater than the one discarded since p_2 cannot have more than two tokens. ■

VI. Sets P_j are not singleton

Let us consider the problem statement given in Section III under assumptions (A1) and (A2).

We propose a solution that consists of ν decentralized supervisors (monitors) respectively specified by a set of n_c decentralized GMECs for each partition, i.e., we choose $n_c^{(j)} = n_c$ for all j's. This implies that $\boldsymbol{W}^{(j)}$ is a $n_c \times m$ positive integer matrix and $\boldsymbol{k}^{(j)}$ is a n_c integer vector.

Under assumptions (A1) and (A2) we prove that a maximal solution can be immediately obtained starting from a solution resulting from Algorithm 5, where a maximal solution is formally defined as follows.

Definition 8: Consider a global GMEC $(\boldsymbol{W}, \boldsymbol{k})$, a partition of the set of places P into ν disjoint subsets $P_1, \ldots, P_\nu$, and a set of decentralized legal markings sets $\mathcal{M}^{(j)}$, $j = 1, \ldots, \nu$.

We say that $\mathcal{M}^{(j)}$'s are a *maximal* set of decentralized legal markings sets if there does not exist a set of decentralized legal markings sets still defined by GMECs, $\bar{\mathcal{M}}^{(j)} = (\bar{\boldsymbol{W}}^{(j)}, \bar{\boldsymbol{k}}^{(j)}) \neq \mathcal{M}^{(j)}$, such that $\cap_{j=1}^{\nu} \mathcal{M}^{(j)} \subsetneq \cap_{j=1}^{\nu} \bar{\mathcal{M}}^{(j)} \subseteq \mathcal{M}\boldsymbol{W}, \boldsymbol{k})$. ■

Definition 9: Let $\mathcal{I}(\boldsymbol{u}^*)$ be a maximal inner box in $(\boldsymbol{W}, \boldsymbol{k})$. We define a set of ν decentralized GMECs $(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$, $j = 1 \ldots \nu$, where $\boldsymbol{W}^{(j)} \in \mathbb{N}^{n_c \times m}$ and $\boldsymbol{k}^{(j)} \in \mathbb{N}^{n_c}$, as follows:

$$\begin{aligned} &\boldsymbol{w}_i^{(j)}(p) = \begin{cases} \boldsymbol{w}_i(p) & \text{if } p \in P_j \\ 0 & \text{otherwise} \end{cases} \\ &k_i^{(j)} = \boldsymbol{w}_i^{(j)} \boldsymbol{u}^* \end{aligned} \tag{2}$$

for all $i = 1 \ldots n_c$ and all $p \in P$. ■

Note that $\sum_{j=1}^{\nu} \boldsymbol{w}_i^{(j)} = \boldsymbol{w}_i$ and

$$k_i^{(j)} = \sum_{j=1}^{\nu} \boldsymbol{w}_i^{(j)} \boldsymbol{u}^* = k_i - s_i$$

where $s_i \geq 0$ is the distance of the maximum vertex of inner box $\mathcal{I}(\boldsymbol{u}^*)$ from the i-th constraint.

The following result holds.

Proposition 10: Let $\mathcal{I}(\boldsymbol{u}^*)$ be a maximal inner box. The set of decentralized GMECs $(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$ given by Definition 9 is a maximal set of decentralized legal markings sets of $\mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$.

Proof: We first prove that $\cap_{j=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)}) \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$.

If $\boldsymbol{m} \in \cap_{j=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)})$, it follows that for all $i = 1, \dots, n_c$ it holds

$$\begin{array}{l} \sum_{j=1}^{\nu} \boldsymbol{w}_i^{(j)}\boldsymbol{m} = \boldsymbol{w}_i\boldsymbol{m} \leq \\ \sum_{j=1}^{\nu} k_i^{(j)} = \sum_{j=1}^{\nu} \boldsymbol{w}_i^{(j)}\boldsymbol{u}^* = k_i - s_i > 0. \end{array}$$

Then, we conclude $\boldsymbol{m} \in \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$.

Assume that there exists another set of decentralized legal marking sets $\mathcal{M}^{(j)}$, with $j = 1 \dots \nu$, such that $\cap_{j=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)}) \subsetneq \cap_{j=1}^{\nu}\mathcal{M}^{(j)} \subseteq \mathcal{M}(\boldsymbol{W}, \boldsymbol{k})$.

Then, $\exists \tilde{\boldsymbol{m}} \in \cap_{j=1}^{\nu}\mathcal{M}^{(j)}$ but $\tilde{\boldsymbol{m}} \notin \cap_{i=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(i)}, \boldsymbol{k}^{(i)})$.

Since $\tilde{\boldsymbol{m}} \notin \cap_{i=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(i)}, \boldsymbol{k}^{(i)})$, it follows that

$$\exists \bar{\imath}, \bar{\jmath} \text{ s.t. } \boldsymbol{w}_{\bar{\imath}}^{(\bar{\jmath})}\tilde{\boldsymbol{m}} > k_{\bar{\imath}}^{(\bar{\jmath})} = \boldsymbol{w}_{\bar{\imath}}^{(\bar{\jmath})}\boldsymbol{u}^*,$$

i.e. $\exists \bar{\jmath}$ s.t. $\tilde{\boldsymbol{m}} \notin \mathcal{M}(\boldsymbol{W}^{(\bar{\jmath})}, \boldsymbol{k}^{(\bar{\jmath})})$.

Since the net marking is a positive integer vector, $\exists \tilde{\boldsymbol{m}}' = \boldsymbol{m} + \vec{\varepsilon}_k$ such that $p_k \in P_{\bar{\jmath}}$, $\boldsymbol{m} \in \mathcal{M}(\boldsymbol{W}^{(\bar{\jmath})}, \boldsymbol{k}^{(\bar{\jmath})})$, $\tilde{\boldsymbol{m}}' \notin \mathcal{M}(\boldsymbol{W}^{(\bar{\jmath})}, \boldsymbol{k}^{(\bar{\jmath})})$.

Hence $\tilde{\boldsymbol{m}}'(p) \geq \boldsymbol{u}^*(p) \;\; \forall p \in P_{\bar{\jmath}} \setminus p_k$ and $\tilde{\boldsymbol{m}}'(p_k) \geq \boldsymbol{u}^*(p_k) + 1$.

Being $\mathcal{M}^{(j)}$ a legal decentralized marking set with respect to P_j without loss of generality we can assume that $\tilde{\boldsymbol{m}}'(p) = \boldsymbol{u}^*(p), \; \forall p \notin P_j$ since $\boldsymbol{u}^* \in \cap_{j=1}^{\nu}\mathcal{M}(\boldsymbol{W}^{(j)}, \boldsymbol{k}^{(j)}) \subset \cap_{j=1}^{\nu}\mathcal{M}^{(j)}$.

Thus $\tilde{\boldsymbol{m}}' \gneq \boldsymbol{u}^*$ but $\boldsymbol{w}_i\tilde{\boldsymbol{m}}' = \boldsymbol{w}_i\boldsymbol{u}^* + \boldsymbol{w}_i(p_k) \leq k_i \forall i$. This is a contradiction, being $\mathcal{I}(\boldsymbol{u}^*)$ a maximal cardinality inner box. ■

Example 11: Let the set of legal markings be

$$\mathcal{M}(\boldsymbol{w}, k) = \{\boldsymbol{m} \mid 3\boldsymbol{m}(p_1) + \boldsymbol{m}(p_2) + \boldsymbol{m}(p_3) \leq 5\}.$$

Assume that the set of places is partitioned into two disjoint sets: $P_1 = \{p_1, p_2\}$ and $P_2 = \{p_3\}$.

To determine a maximal set of decentralized marking sets, we first apply Algorithm 5 to obtain a maximal inner box in $\mathcal{M}(\boldsymbol{w}, k)$. To this aim we initially set $\boldsymbol{w}^0 = \left[\begin{array}{ccc} 3 & 1 & 1 \end{array}\right]$, $k^0 = 5$; then applying iteratively the algorithm we obtain:

— $s = 1$

$\tau_1 = \tau(\mathcal{M}_1) = \lfloor \frac{5}{5} \rfloor = 1, \;\; J_1 = \{1\}$
$\boldsymbol{w}_1^1 = \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right], \; k_1^1 = 5 - 3\tau_1 = 2$
$\boldsymbol{u}_1 = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right], \; U_1 = \{1\}.$

— $s = 2$

$\tau_2 = \tau(\mathcal{M}_2) = \lfloor \frac{2}{2} \rfloor = 1, \;\; J_2 = \{2, 3\}$ choose $\bar{\jmath}_2 = 2$

$$\boldsymbol{w}_1^2 = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array}\right], \; k_1^2 = 2 - \tau_2 = 1$$
$$\boldsymbol{u}_1^2 = \left[\begin{array}{ccc} 1 & 1 & 0 \end{array}\right], \; U_2 = \{1,2\}.$$

— $s = 3$

$$\tau_3 = \tau(\mathcal{M}_3) = \lfloor \tfrac{1}{1} \rfloor = 1, \; J_3 = \{3\}$$
$$\boldsymbol{w}_1^3 = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right], \; k_1^3 = 1 - \tau_3 = 0$$
$$\boldsymbol{u}^* = \boldsymbol{u}_3 = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right], \; U_3 = \{1,2,3\}.$$

Finally, by Definition 9 we obtain $\mathcal{M}(\boldsymbol{w}^{(1)}, k^{(1)}) = \{\boldsymbol{m} \mid 3\boldsymbol{m}(p_1)+\boldsymbol{m}(p_2) \leq 4\}$, $\mathcal{M}(\boldsymbol{w}^{(2)}, k^{(2)}) = \{\boldsymbol{m} \mid \boldsymbol{m}(p_3) \leq 1\}$. ■

VII. Conclusions and Future Works

In this paper we have investigated the problem of determining a set of decentralized GMECs which are able to impose a global specification on the net behavior given in terms of a set of n_c GMECs. Our future efforts in this topic will be devoted to the case of partitions of places that are not given a priori.

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