Switched system optimal control: An application to buck-boost converter

Daniele Corona, Jean Buisson, Bart De Schutter, Alessandro Giua

Abstract— **In this paper we extend a technique developed to design a feedback stabilizing control law for a class of autonomous switched system. More specifically we extend the *switching table procedure* to a particular class of switched systems the dynamics of which either do not have an equilibrium point or, if they do, it is not common. This study is motivated by the application of the DC-DC *buck-boost converter*. The design of the control law is based on dynamic programming arguments and it is a partition of the state space into switching look-up tables. A comparison with a Lyapunov based technique is also discussed.**

I. INTRODUCTION

The switched systems [20] form a particular class of hybrid systems where the occurrence of a discrete event, controlled or uncontrolled, triggers the change in the mode of the system. As a consequence of the highly sophisticated technology in electronics observed in the last decades, countless physical plants, machines and devices integrate discrete and continuous behavior and they can be modeled in this framework. This progress now permits a high level of not only accuracy in measurements (used for instance to detect passing of thresholds), but also of precision in performing discrete-event driven evolution. An important class of switched systems, sometimes called *autonomous* [20], is characterized by a sole control action provided by the switching signal. One of the milestone paper in the field is by Branicky [2], where, through an elementary example, he highlights some paradoxical behaviors of this class.

Among many application fields we consider power converters (Boost, Buck, multilevel converters), that are widespread used in industry, as variable speed DC motor drives, computer power supply, cell phones and cameras. They are electrical circuits controlled by switches (transistors, diodes), used to adapt the energy supplied by a power source to a load. Aiming at reducing switching losses and EMI (Electromagnetic Interference) of power converters, a lot of soft switching techniques are developed so that high efficiency, small size and low weight can be achieved. In nominal conditions, these circuits have been designed so that the switching action does not provoke discontinuity.

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Some control approaches use continuous models. Practically, these devices are controlled through a Pulse-Width-Modulation (PWM) where the switching behavior of the closed loop system is averaged with a nonlinear model [19]. Continuous control approaches are then used, among which passivity based control [19] and sliding mode control [21].

Alternatively these devices are good candidate for hybrid modeling, analysis and control. In this context, they can be modeled by switched systems (without jumps). For a complete, general study on analysis and design of switched system recent books have been published [16], [20]. In particular in [20] stability, robustness, controllability and optimal control are studied. In the context of stability analysis and design a standard technique is to investigate the conditions of existence of a common [17] or multiple Lyapunov function [2], or to refer to some geometric [13] approaches. The issue of stabilizing a switched system can be transformed into a nonconvex programming problem, for which LMI [9], [11] or iterative methods [14] may be used. Properties of uniform stability for a switched system were studied by Hespanha in [12].

A possible technique used to stabilize switched systems is described in [5] and it is based on an optimal control approach. As explained in [8], this method, called *Switching Table Procedure* (STP), is viable in the case when all dynamics admit a common equilibrium point. Here we provide an extension of the method to the case where the system is affine and the dynamics have no common equilibrium or no equilibrium.

II. THE BUCK-BOOST CONVERTER

In order to derive models for DC-DC converters, different energy based approaches, such as circuit theory, bond graphs, Euler Lagrange, Hamiltonian approach can be used. For switching systems, extensions have been proposed for the Hamiltonian approach [10] or for the bond graph approach [3]. In most of these systems, one physical switch is controlled (e.g. transistor), while the other one may be not (e.g. diode).

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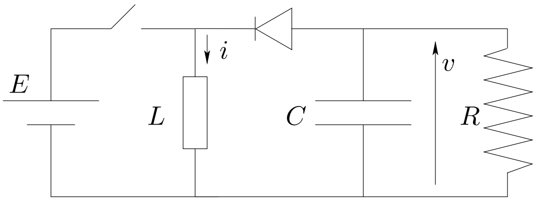

Fig. 1: *Circuit scheme of the buck-boost converter.*

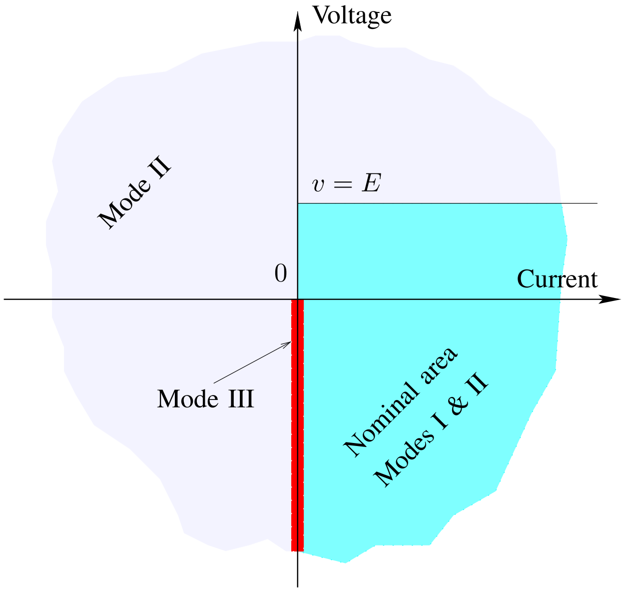

Fig. 2: *Partition of the state space for the different modes of the converter.*

In a normal operating mode of an ideal converter both switches occurr simultaneously.

A simple circuit representation of the buck-boost converter is depicted in Figure 1. We consider a simplified version of the buck-boost converter with ideal components. In particular the continuous source E has a negligible internal resistance and infinite power. No energy is lost in the inductor L nor in the capacitor C. The diode has an ideal characteristic, hence it has no voltage drop in conducting mode and switches exactly at zero voltage level.

It can be seen that the converter *theoretically* has four possible operating modes. We label them with the variable ρ and we denote, as in Figure 1, by v the voltage on the capacitor (positive when upwards) and by i the current across the inductor (positive when downwards). The four modes are:

I. switch closed and diode blocked ($\rho = 1$),
II. switch open and diode conducting ($\rho = 2$),
III. switch open and diode blocked ($\rho = 3$),
IV. switch closed and diode conducting ($\rho = 4$).

In nominal behavior only modes (I) and (II) are involved. The nominal working area of the space state is $\mathcal{N} \equiv \{(i, v) \in \mathbb{R}^2 : i \geq 0, v \leq E\}$ depicted in Figure 2 in the cyan area (right-bottom area of the figure). The four modes are represented by the nodes of the oriented graph in Figure 3. The arcs indicate the discrete transitions from one mode to another; the controlled switches are solid lines, while the diode switches, depending on the state of the system, are dashed. A description of the behavior is described in the following paragraph.

In state (I) the battery transfers energy into the inductor, in form of a magnetic field, while, on the load side, the capacitor is feeding the load. After some time the switch is opened and the system goes to mode (II) where the energy stored in the inductor can now flow towards the load and the capacitor. Then the controller may close the switch again to mode (I) and so on. If the duration in mode (II) is protracted all the magnetic energy is transferred to the load and the buck-boost converter switches to the *discontinuous* mode (III) [22]. Hence this state is reached when the condition $i = 0, v < 0$ is attained. In this mode the current remains null and the capacitor is feeding the load. From (III) it is possible to switch to (I) by closing the switch. Finally let us observe that mode (IV) is in fact critical, because it imposes two different voltage levels in the same point (v on the anode and E on the cathode of the diode in conducting mode). This is a harmful behavior both for the system and for security reasons. It can be avoided by opening the switch, leading the system to the passive mode (II), whenever the voltage level v increases to reach the value E imposed on the cathode by the generator. In other words, if for some bad initial conditions, or for any other reason like resonance or disturbances, the voltage v overtakes E, a *safe* controller must immediately open the switch leading to mode (II). Mode (IV) is, to some extent, a *fault* mode. Let us denote by x the state space, i.e., the couple i, v, hence let $x = [i, v]^{\mathrm{T}}$. The differential equation for each mode of the system associated to each location in the oriented graph in Figure 3, are the following. In location (I)

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}, \qquad (1)$$

in location (II)

$$\dot{x} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x, \qquad (2)$$

in location (III)

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} x. \qquad (3)$$

Note that we do not provide a model for location (IV) which ideally does not exist, because it leads to inconsistency with the laws of electrical networks. For this reason, assuming that initial conditions are

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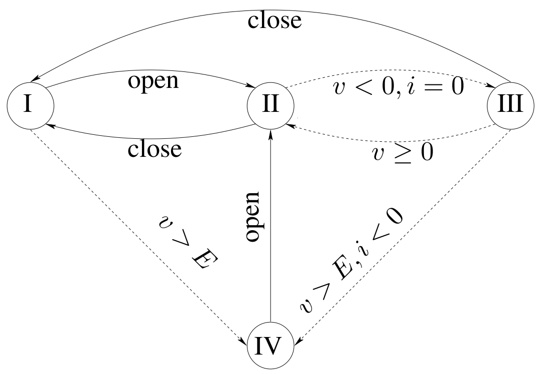

Fig. 3: *Oriented graph of the switching behavior of the converter. Solid line: controlled switches, dashed line: diode state-based switches.*

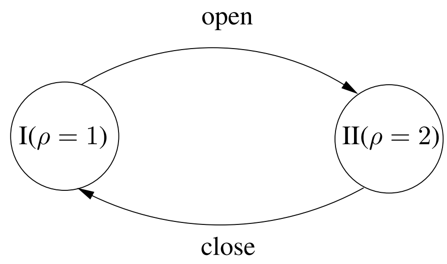

Fig. 4: *Oriented graph representing only the 2 nominal operating modes.*

well-posed, it is allowed to remove location (IV) from the model. The attention may thus be focused on a model that only contains three states and whose dynamics are given above. Furthermore we may also make the assumption that the controller of the switch is fast enough to prevent the complete discharge of the inductor during the evolution in location (II). This additional assumption allows to also disregard the presence of the third dynamics. This framework is practically relevant, especially when the working point of the converter is deep enough inside the nominal working area depicted in Figure 2.

These assumptions lead to approximate the model in Figure 2 to a much simpler one that is depicted in Figure 4. The dynamics associated to the locations are summarized in the following equation [4]:

$$\dot{x} = \begin{bmatrix} 0 & \frac{\rho-1}{L} \\ \frac{1-\rho}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{2-\rho}{L}E \\ 0 \end{bmatrix} = A_\rho x + F_\rho, \tag{4}$$

where $\rho \in \{1, 2\}$ is the switching signal that indicates the active mode of the system. The system is described in terms of the state variables and the control signal $\rho(t)$, which switches among the possible modes in order to stabilize to a specific operating point x_{p}.

III. The switching table procedure

The method, based on the switching table procedure (STP), used to obtain the control law is described in [8] and [5], Chapter 7. It can be applied to the class of switched autonomous systems, $\dot{x} = A_\rho x$, denoted by $\{A_\rho\}_{\rho\in\mathcal{S}}$, where $\mathcal{S}$ is a set of s modes indexed by ρ. It consists in determining a partition of the state space (by means of look-up tables) that indicates what mode A_ρ should be active for the current state value. This can be done by associating a weight matrix Q_ρ to each operating mode and solving, for every possible initial point x_0, the problem

$$\begin{aligned} J(x_0, \rho_0) = \min_{\rho(t)} & \int_0^{+\infty} x^{\mathrm{T}} Q_{\rho(t)} x dt \\ \text{s.t.} \quad & \dot{x}(t) = A_{\rho(t)} x(t) \\ & x(0) = x_0 \\ & \rho(0) = \rho_0. \end{aligned} \tag{5}$$

We now briefly sketch how the STP can be used for stabilizing purposes. This will be done in three steps. A complete description and proofs can be found in [6], [8], [18]. Initially we assume that only a finite number N of switches is available. In this framework the basic assumption is that at least one dynamics of $\{A_\rho\}_{\rho\in\mathcal{S}}$ is stable. Then we show how the procedure can be extended to the case of $N = \infty$. Finally, we relax the condition that at least one dynamics of $\{A_\rho\}_{\rho\in\mathcal{S}}$ is stable and show how the STP can be used as a design tool for stabilizing control laws.

A. Step 1

In the first step we show that the optimal control law for the optimization problem (5) takes the form of a state feedback. When k out of N switches are available the current *hybrid system state* (x, ρ) indicates, via a look-up table $\mathcal{C}_k^\rho$, whether a switch from the current dynamics A_{ρ_k} to $A_{\rho_{k-1}}$, should occur. The look-up table $\mathcal{C}_k^\rho$ is a partition of the state space into different regions $\mathcal{R}_\rho$ labeled with the target mode $\rho \in \mathcal{S}$ to switch to whenever the continuous part of the evolution $x \in \mathcal{R}_\rho$. For autonomous systems and quadratic cost these partitions are *homogeneous*, i.e., if a strategy is valid for a specific $\bar{x}$, then it is also valid for any point $\lambda\bar{x}$, $\lambda \in \mathbb{R}^+$, allowing to restrict the interest to a unitary semi-sphere Σ_n.

The tables are constructed *recursively*, on the increasing number k of remaining switches, so, using the information already computed when $k-1$ switches are available. The procedure is iterated until $k = N$.

The method is based on dynamic programming arguments, i.e., on the minimization of a *residual cost* function defined as follows: assume that k switches are remaining and the current hybrid state is (y, ρ), where $y \in \Sigma_n$. The residual cost evaluates the cost of the evolution as a function of the time $t \geq 0$ evolving in the current mode $\rho \in \mathcal{S}$ and the mode $\sigma \in \mathcal{S}$ into which the system will switch after time t has elapsed. From that point on, i.e., from $z = e^{A_\rho t}y$ and $k-1$ remaining switches, the optimal strategy is already known from the previous computation.

By minimizing the residual cost function over the couple $t \geq 0$ and $\sigma \in \mathcal{S}$ we obtain the optimal arguments $t^*(y,\rho), \sigma^*(y,\rho)$ that allow to build, point by point, the table $\mathcal{C}_k^\rho$. The procedure is initialized by computing the residual cost function with 0 switches as:

$$J_0^*(y,\rho) \triangleq \begin{cases} y' Z_\rho y & \text{if } A_\rho \text{ is stable} \\ +\infty & \text{else,} \end{cases} \tag{6}$$

where Z_ρ is the unique solution of the equation $A_\rho^{\mathrm{T}} Z_\rho + Z\rho A_\rho = -Q_\rho$, which allows to construct $\mathcal{C}_0^\rho$. This technique can be extended to the switched system where also *autonomous* switches (i.e., governed by the crossing of thresholds in the state space) may occur [7]. In this case the residual cost function to be minimized over time t and σ is piecewise continuous, with as many breakpoints as the number of autonomous thresholds crossed during the time evolution.

B. Step 2

Now if the system is allowed to switch indefinitely we have the following result [6]: there exists a sufficiently big $\bar{N}$ such that for all $N > \bar{N}+1$ it holds $\mathcal{C}_N^\rho \equiv \mathcal{C}_{\bar{N}+1}^\rho$.

The proof of this result is based on the fact that for every initial point (y,ρ) the value of the cost converges with the increasing number of switches. This allows one to compute with a finite procedure the optimal tables for a switching law when N goes to infinity. In fact, it holds that for all $\rho \in \mathcal{S}$,

$$\mathcal{C}_\infty^\rho \triangleq \lim_{N\to\infty} \mathcal{C}_N^\rho \equiv \mathcal{C}_{\bar{N}+1}^\rho.$$

Furthermore, if the switched system automaton graph is *totally connected*, i.e., for all $\rho, \sigma \in \mathcal{S}$, with $\rho \neq \sigma$, there exists an oriented arc of the automaton graph from node ρ to node σ, it holds for all $\rho, \sigma \in \mathcal{S}$

$$\mathcal{C}_\infty^\rho \equiv \mathcal{C}_\infty^\sigma \equiv \mathcal{C}_\infty.$$

Hence, there is a unique table for all modes.

To construct the table $\mathcal{C}_\infty$ the value of $\bar{N}$ is needed. We leave to further investigation a method to compute $\bar{N}$ in advance; so far the approach consists in constructing tables until a convergence criterion[1], is met.

[1]Threshold on the improvements in the value of cost function or on the variation of labels for the population of points.

C. Step 3

In the third step we show how the STP can be used to obtain an optimal stabilizing switching signal in the case when all dynamics of $\{A_\rho\}_{\rho\in\mathcal{S}}$ are unstable. In such a condition, if we apply the STP without any adjustments, the residual cost, at the initial stage of *zero* remaining switches, will be equal to ∞ due to (6). Hence for all k we have $J_k^*(y,\rho) = +\infty$ as well. This difficulty can be avoided by the introduction in $\{A_\rho\}_{\rho\in\mathcal{S}}$ of a stable *dummy* dynamics A_{s+1}, that serves to give a finite value to the function $J_0^*(y,\rho)$.

In other words we consider an *augmented* system, $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$ with $|\tilde{\mathcal{S}}| = |\mathcal{S}|+1$, obtained by joining to $\{A_\rho\}_{\rho\in\mathcal{S}}$ a stable dynamics, as in Figure 5, where (a) is the augmented system of (b) and A_3 is stable. Informally the new dynamics serve as a *launch pad* for the STP. The basic idea is the following: if the partition $\tilde{\mathcal{C}}_\infty$, solution of the same optimal control problem for the *augmented* system, does not contain the label relative to A_{s+1}, then the table $\tilde{\mathcal{C}}_\infty$ is also a solution for $\{A_\rho\}_{\rho\in\mathcal{S}}$. To make sure that the dynamics A_{s+1} is only used if absolutely necessary, i.e., the original switched system is *not* stabilizable, we associate to it a very high cost. This result is supported by the following theorem:

Theorem 3.1 ([8]): Consider a switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$, and an optimal control problem with $N = \infty$ and weight matrices $Q_\rho > 0$, $\rho \in \mathcal{S}$. Define an augmented $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$ and a corresponding optimal control problem, with $Q_{s+1} = qQ$, $q \in \mathbb{R}^+$, $Q > 0$. We have that:

1) If the switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$ is *globally exponentially stabilizable* [15], then there exists a $q \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$ does not contain the label associated to A_{s+1}.
2) If there exists a $q \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$, computed by solving an optimal control problem on $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$, does not contain the label associated to A_{s+1}, then the switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$ is asymptotically stabilizable. □

The above theorem provides an efficient way to deal with the problem of determining an asymptotic stabilizing switching law for a switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$ with linear unstable modes, that can be summarized in the following steps.

1) Associate to the switched system an optimal control problem with $N = \infty$;
2) Define an augmented system $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$ by adding a stable dynamics and an augmented optimal control problem with $Q_{s+1} = qQ$, where q is a very large positive real number and Q is any positive definite matrix;
3) Construct the table $\tilde{\mathcal{C}}_\infty$ solving an optimal control problem on $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$;
4) If this table does not contain the label associated

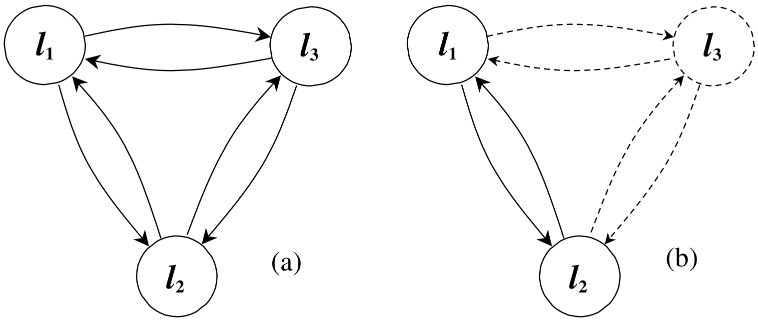

Fig. 5: *(a) Augmented switched system* $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$, $\tilde{\mathcal{S}} \equiv \{1,2,3\}$. *(b) Switched system* $\{A_\rho\}_{\rho\in\mathcal{S}}$.

to the stable mode A_{s+1}, then the table $\tilde{\mathcal{C}}_\infty$ coincides with table $\mathcal{C}_\infty$.

We do not provide an a priori rule to establish whether the switched system is stabilizable and in such a case, an analytical way to compute an appropriate value of q in advance. The solution of the problem of *knowing* whether the system is stabilizable remains in the general case undecidable [1].

IV. Extension of the STP procedure

In this section an extension of the STP as described previously is considered. In particular we study the possibility of using the STP as a design tool to regulate a switched affine system to a desired point of the state space $x_\mathrm{p} \in \mathbb{R}^n$.

We can define the following problem:

Problem 1: Given a switched affine system of the form

$$\dot{x} = \tilde{A}_\rho x + \tilde{F}_\rho, \tag{7}$$

$\rho \in \mathcal{S}$, the corresponding automaton graph of which is totally connected, design the switching signal $\rho(t)$ so that the state x is steered to a desired value x_p. □

In the particular case when x_p is a stable equilibrium point of one of the modes of the switched affine system (7), let us say $\bar{\rho}$, Problem 1 has a straightforward solution: execute any finite switching sequence with final element $\bar{\rho}$. Once in location $\bar{\rho}$ the system will autonomously reach the stable equilibrium point and no further control action is needed.

This scenario is however very particular, because it requires that the specific point x_p solves the strong condition $\tilde{A}_\rho x_\mathrm{p} + \tilde{F}_\rho = 0$ for at least one $\rho \in \mathcal{S}$. In several applications the point x_p is a *specifically required* working point hence it does not necessarily solve the above condition.

Hence we study now the case of designing, for a switched affine system (7), a feedback control law for the switching signal $\rho(t)$ that regulates the state to a generic desired value x_p, assuming that this point is *not* an equilibrium for any mode of the system.

In order to apply the STP to this framework we associate to the system above an LQ criterion to minimize. As explained above we consider a set of positive definite weight matrices $\tilde{Q}_\rho$ for each mode of the switched affine system, that penalizes the offset from the target x_p. Therefore we want to solve

$$\begin{aligned} J(x_0,\rho_0) = \min_{\rho(t)} & \int_0^{+\infty} (x-x_\mathrm{p})^\mathrm{T}\tilde{Q}_{\rho(t)}(x-x_\mathrm{p})dt \\ \text{s.t.} \quad & \dot{x}(t) = \tilde{A}_{\rho(t)}x(t) + \tilde{F}_{\rho(t)} \\ & x(0) = x_0 \\ & \rho(0) = \rho_0. \end{aligned} \tag{8}$$

It is convenient to perform a shift of the state space so that its origin is centered in x_p. Thus we define a new state space reference system $\tilde{y} \in \mathbb{R}^n$ such that $\tilde{y} = x - x_\mathrm{p}$. In this new set of coordinates the affine switched system becomes $\dot{\tilde{y}} = \tilde{A}_\rho\tilde{y} + F_\rho$, where $F = A_\rho x_\mathrm{p} + \tilde{F}_\rho$ and problem (8) becomes

$$\begin{aligned} J(\tilde{y}_0,\rho_0) = \min_{\rho(t)} & \int_0^{+\infty} \tilde{y}^\mathrm{T}\tilde{Q}_{\rho(t)}\tilde{y}dt \\ \text{s.t.} \quad & \dot{\tilde{y}}(t) = \tilde{A}_{\rho(t)}\tilde{y}(t) + F_{\rho(t)} \\ & \tilde{y}(0) = \tilde{y}_0 \\ & \rho(0) = \rho_0. \end{aligned} \tag{9}$$

The next step is to reformulate the switched affine system $\dot{\tilde{y}} = \tilde{A}_\rho\tilde{y} + F_\rho$ as a switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$. To this purpose we consider [18] an augmented space variable $y \in \mathbb{R}^{n+1}$ obtained by extending the original state space vector with an additional variable y_{n+1} and governed by the dynamics $\dot{y} = A_\rho y$, where

$$A_\rho = \begin{bmatrix} \tilde{A}_\rho & F_\rho \\ 0 & 0 \end{bmatrix}$$

and weight matrix $Q_\rho = \begin{bmatrix} \tilde{Q}_\rho & 0 \\ 0 & 0 \end{bmatrix}$. The dummy variable y_{n+1} remains constant for any initial state, thus, $y(t) = [\tilde{y}(t), y_{n+1}(0)]^\mathrm{T}$. If $y_{n+1}(0) = 1$ the problem

$$\begin{aligned} J(y,\rho_0) = \min_{\rho(t)} & \int_0^{+\infty} y^\mathrm{T}Q_{\rho(t)}ydt \\ \text{s.t.} \quad & \dot{y} = A_{\rho(t)}y \\ & y(0) = [\tilde{y}_0^\mathrm{T}, 1]^\mathrm{T} \\ & \rho(0) = \rho_0 \end{aligned} \tag{10}$$

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is equivalent to (9).

It is clear that the matrices A_ρ of the switched system $\{A_\rho\}_{\rho\in\mathcal{S}}$ are *all* unstable, not only because of the null eigenvalue introduced by the dummy variable, but also because x_p is not an equilibrium point for any of the original modes. The objective of the switching control law for the new switched system is to steer the vector field y towards $y_\mathrm{eq} = [0,\ldots,0,1]^\mathrm{T}$. In this case in fact the original system has reached the target x_p.

Consider now an augmented switched system $\{A_\rho\}_{\rho\in\tilde{\mathcal{S}}}$ of $\{A_\rho\}_{\rho\in\mathcal{S}}$ as described in Section III-C, with

$$A_{s+1} = \begin{bmatrix} \tilde{A}_{s+1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $Q_{s+1} = \begin{bmatrix} \tilde{Q}_{s+1} & 0 \\ 0 & 0 \end{bmatrix}$, and the assumption:

Assumption 4.1: Matrix $\tilde{A}_{s+1}$ is Hurwitz and matrix $\tilde{Q}_{s+1}$ is positive definite. □

We can now prove the following proposition

Proposition 4.2: Under the assumption above Theorem 3.1 holds despite the fact that matrix Q_{s+1} is not strictly positive definite and dynamics A_{s+1} is not strictly stable.

Proof: It is an immediate consequence of the fact that for every possible initial state y_0, the cost function, for the mode $s+1$,

$$J(y_0, s+1) = \int_0^{+\infty} y^\mathrm{T} Q_{s+1} y dt$$

is finite. In fact, by construction and assumption, the first n components of y are integrable, and the component $n+1$ is constant but it has a null weight. □

This result allows one to use the STP to stabilize an all-unstable-modes switched affine system to a desired specific point of the state space. Furthermore this is done by minimizing a quadratic criterion that penalizes the distance of the current state from the desired target point. Note however that, as stated in Theorem 3.1, the STP is guaranteed to find a state feedback switching signal in the particular case that the switched system is globally stabilizable.

As an application of this new result we consider the case study of the buck-boost converter.

V. Experimental set-up

Consider the buck-boost converter in Figure 1. The numerical values of the physical system are normalized, hence we chose $E = 1$, $L = 1$, $C = 1$ and $R = 1$. The initial step of the implementation is to adapt the physical system described in Section II to the method in Sections III and IV. First we select the set-point $x_\mathrm{p} = [2, -1]^\mathrm{T}$ and transform the affine equations (4) into the form $\dot{y} = A_\rho y$, by the introduction of an additional state variable. This leads us to work in $\mathbb{R}^3$,

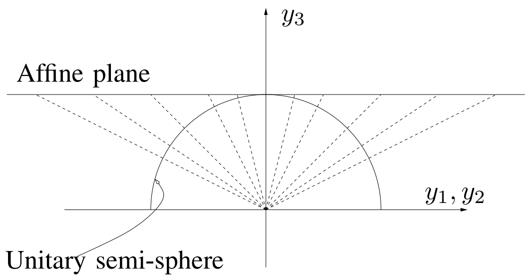

Fig. 6: *Side view of the projection on the affine plane $y_3 = 1$ of the table $\mathcal{C}_\infty$ obtained on the unitary semi-sphere.*

TABLE I: *Critical tuning parameters of STP applied to the buck-boost converter.*

Number of switches	$N = 25$
Number of samples	$N_\mathrm{s} = 2000$
Time horizon	$\tau_\mathrm{max} = 500$
Number of points (azimuth)	$N_\varphi = 15$
Weight of stable mode	$q = 1000$

and more precisely along the affine plane $y_3 = 1$. This transformation allows to preserve important properties of STP. Then we define the stable dynamics, i.e., a dummy dynamics A_3 that converges *autonomously* to the set point x_p. A choice [4] of this dynamics can be obtained by solving on ρ $A_\rho x_\mathrm{p} + F_\rho = 0$, yielding to $A_3 = \begin{bmatrix} 0 & 0.5 \\ -0.5 & -1 \end{bmatrix}$ and $F_3 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$. In this case the obtained dynamics has also a physical interpretation: in the neighborhood of its equilibrium point x_p it approximates the sliding surface of the system with infinite switching rate. The weight matrices for both modes are chosen as the identity. The dummy dynamics (Section III-C) is penalized with a factor of $q = 10^3$.

In order to speed up the STP algorithm a revised version of the infinite-horizon cost was implemented. In particular it was preferred, both for computation time savings and accuracy improvement, to compute it analytically. We set $N = 25$ switches the convergence level of the switching tables. The discretization of the $\mathbb{R}^3$ unitary semi-sphere is obtained in polar coordinates by sampling the zenith angle φ with $N_\varphi = 15$ samples and the azimuth angle with $N_\vartheta = 60$ samples[2]. This leads to a total number of 574 points distributed (as described in [6]) on the semi-sphere. In Table I we provide the parameters of the STP we have used.

The obtained partition $\mathcal{C}_\infty$ is computed in $\mathbb{R}^3$, but the meaningful part is the intersection with the affine plane $y_3 = 1$. This can be obtained by *projecting* the $\mathbb{R}^3$ solution along the affine plane, with an imaginary light point in the origin that beams radially,

[2]For the semi-sphere, the range of zenith angle is $\pi/2$, 4 times less then the range of the azimuth angle, which is 2π.

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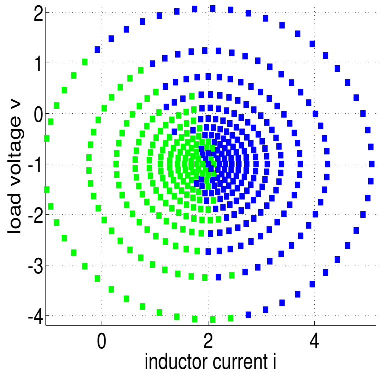

Fig. 7: *Table $\mathcal{C}_\infty$ obtained for buck-boost converter and parameters in Table I. The green side, located on the left, imposes a closed switch (A_1), and the blue side an open switch (A_2).*

as illustrated by Figure 6. We chose this particular distribution of points because we want to have a higher degree of precision around the origin, that corresponds to the working point. Note that only the labels (colors in this case) associated to dynamics A_1 and A_2 appear.

In Figure 7 we depict the table $\mathcal{C}_\infty$ restricted to the subspace $y_3 = 1$, that should be used during the simulation. When the state $y = [y_1, y_2]^{\mathrm{T}}$ is in the green area, then the mode $\rho = 1$, (closed switch in Figure 1) must be active. On the contrary, when $y = [y_1, y_2]^{\mathrm{T}}$ is in the blue area, then the mode $\rho = 2$ (open switch) must be active.

VI. Simulation Results

The synthesis of the control law was obtained by running the programs that implement the STP, with the tuning parameters of the algorithm listed in Table I. The computations were carried out with Matlab 7, on a 2 GHz Pentium Centrino, requiring a total off-line computation time of about 8.95×10^4 seconds. The resulting control law, depicted in Figure 7, is affected by numerical error along the switching surface. This is often observed in those examples where the solution for the switching sequence collapses into a sliding surface. Note that it is possible to smoothen the solution by means of 2-dimensional filtering algorithms. In fact we decided not to follow this way because it results into a suboptimal solution. Additionally it is intuitive that the nonoptimality of the numerical solution decreases with higher granularity of the state space discretization, but the proof is not straightforward and is of interest for future research.

The table $\mathcal{C}_\infty$ is used then for the simulation. Due to the state space discretization the on-line controller decides the best strategy by choosing the information contained in the *closest neighbor* point to the current state value. Other policies, for example based on averaging the indication contained in a surrounding of points are also possible, with the additional advantage of filtering out part of the numerical disturbance.

TABLE II: *Comparison of the solution obtained with the STP and with Lyapunov-based method for different initial points (Figure 8).*

Label (Fig. 8)	Point	Cost (STP)	Cost (Lyap)
A	$[4, 0]^{\mathrm{T}}$	16.2	19.8
B	$[4, 4]^{\mathrm{T}}$	34.1	38.3
C	$[0, 4]^{\mathrm{T}}$	6.0	6.1
D	$[-4, 4]^{\mathrm{T}}$	25.4	25.8
E	$[-4, 0]^{\mathrm{T}}$	24.2	24.6
F	$[-4, -4]^{\mathrm{T}}$	28.3	28.7
G	$[0, -4]^{\mathrm{T}}$	6.6	7.6
H	$[4, -4]^{\mathrm{T}}$	20.9	31.4

The table $\mathcal{C}_\infty$ was tested on 8 different initial points, listed in Table II. The corresponding trajectories are plotted in Figure 8.a. Note that the optimal strategy is to remain in the initial location until the switching surface is hit. From there on a chattering behavior is activated and the state is steered towards the equilibrium point along the sliding surface. Theoretically this should occur at an infinite frequency. In practice the switching occurs at the sampling time rate of the simulation tool, which in these examples was set to 1 ms (a faster sampling time did not increase the accuracy of the trajectory and the cost value). Note that it is also possible to impose a minimum permanence time in each location, provided that the delay does not overtake the discharge period of the inductor, leading the converter to behave in the discontinuous mode (III) described in Section II.

Another possible control law can be designed with a Lyapunov based method [4]. In this approach, based on physical considerations, a unique Lyapunov function, which is not computed but directly derived from the model, is proposed. It allows to stabilize a physical switched affine system around a non common equilibrium point using different strategies such as maximum descent or minimum switching. The obtained control law is reported in Figure 8.b, where in addition the trajectories from the 8 different initial points are plotted.

In Table II we report the performances for the trajectories obtained by using the two methods. It can be seen that both laws are stabilizing and it is relevant to observe that in addition the STP provides minimization of a performance.

VII. Conclusions

A standard case study, the buck-boost converter, was considered as an example for extending the *switching table procedure*, presented in [5]. In particular we have shown how to regulate to a generic point a switched system composed of dynamics with different equilibria or no equilibria. The procedure is based on dynamic

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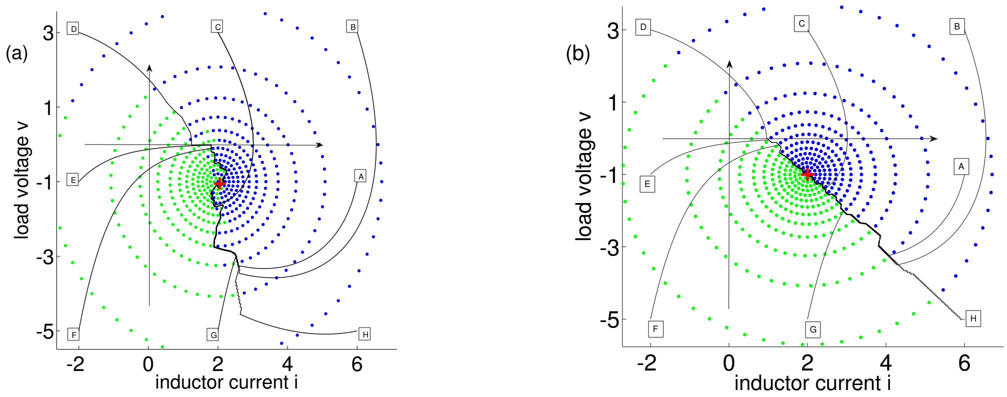

Fig. 8: *State trajectories from different initial points resulting (a) from the STP solution, (b) from solution based on common Lyapunov function.*

programming and principle of convergence for infinite time horizon methodologies. We have shown how the STP can be successfully applied to regulate the system state to a desired target point. Improvements of the procedure in terms of computational complexity and dimensionality curse define further research lines, and so does the crucial role of the tuning of parameters. Despite the presence of numerical error, the obtained solution proved to be very efficient for the considered application.

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