ON SAMPLING CONTINUOUS TIMED PETRI NETS: REACHABILITY "EQUIVALENCE" UNDER INFINITE SERVERS SEMANTICS

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Abstract: This paper addresses a sampling problem for timed continuous Petri nets under infinite servers semantics. Different representations of the continuous Petri net system are given, the first two in terms of piecewise linear system and the third one, for the controlled continuous Petri nets systems, in terms of a particular linear constrained system with null dynamic matrix. The last one is used to obtain the discrete-time representation. An upper bound on sample period is given in order to preserve important information of timed continuous nets, in particular the positiveness of the markings. The reachability space of the sampled system in relation to autonomous continuous Petri nets is also studied.

Keywords: Petri-nets, Continuous-time systems, Discrete-time systems.

1. INTRODUCTION

Discrete Petri nets (PNs) (Silva, 1993) are a mathematical formalism with an appealing graphical representation for the description of discrete-event systems, successfully used for modeling, analysis and synthesis of such systems. To study performance evaluation, timing should be introduced and timed PNs are obtained.

Discrete PNs may suffer the state explosion problem, when the number of tokens is large. As in the case of other formalisms (e.g. integer programming or queuing networks), continuous relaxation can provide a good approximation for discrete models under certain circumstances (Silva and Recalde, 2002).

Continuous Petri nets (contPNs) are a formalism in which the marking of each place is a non-negative real number (David and Alla, 2004) (Silva and Recalde, 2002). As in discrete case, timing can be associated to transitions resulting in timed contPNs. Controllers and observers can be designed for this class of systems but taking into account that probably they need to be implemented on some computer, the sampling of the continuous system is required.

For finite servers semantics, sampling is not a hard problem because the flow of the transitions is constant inside each invariant behavior (IB) state (David and Alla, 2004), and the times at which IB state changes occur can be computed. This

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allows to tackle the problem as an event-driven control (Júlvez *et al.*, 2004). However, it seems that infinite servers semantics usually provides a much better approximation of the discrete system than finite servers semantics (Mahulea *et al.*, 2006). Under infinite servers semantics, there is not an easy way to compute the equivalent to these IB states, so sampling is an important issue.

In classical *Systems and Signal Theory*, it is well-known that the *Sampling theorem* (frequently known as the Nyquist-Shannon sampling theorem) provides an upper bound for the sampling period of limited bandwidth signals in order "not to loose information". Here, it is shown that sampling at "too low rate", spurious solutions can appear, in particular negative markings. In this paper, for timed contPNs, an upper bound on the sampling period is given in order to avoid spurious solutions. In other words, for the sampled timed contPNs, some "equivalence results" regarding the reachability space of sampled timed contPNs and (autonomous) contPNs are presented.

2. CONTINUOUS PETRI NETS

Definition *2.1.* A contPN system is a pair $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$, where: (1) $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is the net structure with set of places P, set of transitions T, pre and post incidence matrices $\boldsymbol{Pre}, \boldsymbol{Post} : P \times T \rightarrow \mathbb{N}$; and (2) $\boldsymbol{m_0} : P \rightarrow \mathbb{R}_{\geq 0}$ is the initial marking (or distributed state).

The number of places of a net is $n = |P|$ and the number of transitions is $m = |T|$. We also denote $\boldsymbol{m}(\tau)$ the marking at time τ and in discrete time we denote $\boldsymbol{m}(k)$ the marking at sampling instant k ($\tau = k \cdot \Theta$, where Θ is the sampling period). The token load contained in place p_i at marking $\boldsymbol{m}$ is denoted m_i. Finally, *preset* and *postset* of a node $X \in P \cup T$ are denoted $^\bullet X$ and $X^\bullet$, respectively.

A transition $t_j \in T$ is *enabled* at $\boldsymbol{m}$ iff $\forall p_i \in {}^\bullet t_j, m_i > 0$, and its *enabling degree* is

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i}{Pre(p_i, t_j)} \right\}.$$

An enabled transition t can fire in any real amount $0 \leq \alpha \leq enab(t, \boldsymbol{m})$ leading to a new marking $\boldsymbol{m'} = \boldsymbol{m} + \alpha \boldsymbol{C}(\cdot, t)$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *incidence matrix*; this firing is also denoted $\boldsymbol{m}[t(\alpha)\rangle \boldsymbol{m'}$.

In general, if $\boldsymbol{m}$ is reachable from $\boldsymbol{m_0}$ through a sequence $\sigma = t_{r_1}(\alpha_1) t_{r_2}(\alpha_2) \ldots t_{r_k}(\alpha_k)$, and we denote by $\boldsymbol{\sigma} : T \rightarrow \mathbb{R}_{\geq 0}$ the *firing count vector* whose component associated to a transition t_j is

$$\sigma_j = \sum_{h \in H(\sigma, t_j)} \alpha_h,$$

where $\quad H(\sigma, t_j) = \{h = 1, \ldots, k \mid t_{r_h} = t_j\}$,

then we can write: $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, which is called the *fundamental equation*.

The basic difference between discrete and continuous PN is that the components of the markings and firing count vectors are not restricted to take value in the set of natural numbers but in the non-negative reals. The set of markings that are reachable with a finite firing sequence for a given system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is denoted as $RS^{un}(\mathcal{N}, \boldsymbol{m_0})$.

Definition *2.2.* Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a contPN system and $RS^{un}(\mathcal{N}, \boldsymbol{m_0})$ the set of *reachable markings*, i.e., the set of markings $\boldsymbol{m} \in \mathbb{R}^m_{\geq 0}$ such that a finite fireable sequence $\boldsymbol{\sigma} = t_{a_1}(\alpha_1) \cdots t_{a_k}(\alpha_k)$ exists, and $\boldsymbol{m_0} \xrightarrow{t_{a_1}(\alpha_1)} \boldsymbol{m_1} \xrightarrow{t_{a_2}(\alpha_2)} \boldsymbol{m_2} \cdots \xrightarrow{t_{a_k}(\alpha_k)} \boldsymbol{m_k} = \boldsymbol{m}$, where $t_{a_i} \in T$ and $\alpha_i \in \mathbb{R}^+$.

A relaxation of this space can be considered allowing an infinite firing sequence and lim-reachable space is obtained:

Definition *2.3.* Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a continuous system. A marking $\boldsymbol{m}$ is *lim-reachable* iff a sequence of reachable markings $\{\boldsymbol{m_i}\}_{i \geq 1}$ exists such that $\boldsymbol{m_0} \xrightarrow{\sigma_1} \boldsymbol{m_1} \xrightarrow{\sigma_2} \boldsymbol{m_2} \cdots \xrightarrow{\sigma_i} \boldsymbol{m_i} \cdots$ and $\lim_{i \to \infty} \boldsymbol{m_i} = \boldsymbol{m}$. The set of lim-reachable markings is denoted as $lim - RS^{un}(\mathcal{N}, \boldsymbol{m_0})$.

Definition *2.4.* A (deterministically) timed contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ is a contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ together with a vector $\boldsymbol{\lambda} : T \rightarrow \mathbb{R}_{>0}$, where λ_j is the firing rate of transition t_j.

Now, the fundamental equation depends on time: $\boldsymbol{m}(\tau) = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$, where $\boldsymbol{\sigma}(\tau)$ denotes the firing count vector in the interval $[0, \tau]$. Deriving it with respect to time the following is obtained: $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. The derivative of firing vector represents the *flow* of the timed model $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$. Depending on how the flow of the transition is defined many firing semantics are possible; the most used in literature are *finite servers semantics* (or *constant speed*) and *infinite server semantics* (or *variable speed*) (Recalde and Silva, 2001) (David and Alla, 2004).

This paper deals with *infinite server semantics* in which the flow of transition t_j is given by:

$$f_j = \lambda_j \min_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i}{Pre(p_i, t_j)} \right\} \tag{1}$$

Example *2.5.* Let us consider the net system in Fig. 1. The flow of transitions are:

$$\left\{ \begin{array}{l} f_1 = \lambda_1 \cdot \min \left\{ \dfrac{m_1}{2}, m_3 \right\} \\ f_2 = \lambda_2 \cdot \min \{m_1, m_2\} \end{array} \right. .$$

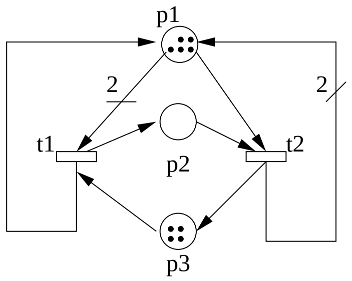

Fig. 1. Continuous PN system.

Thus, the state space representation of this unforced system ($\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \boldsymbol{f}(\tau)$) is:

$$\begin{cases} \dot{m}_1 = -\lambda_1 \cdot \min\left\{\dfrac{m_1}{2}, m_3\right\} + \lambda_2 \cdot \min\{m_1, m_2\} \\ \dot{m}_2 = \lambda_1 \cdot \min\left\{\dfrac{m_1}{2}, m_3\right\} - \lambda_2 \cdot \min\{m_1, m_2\} \\ \dot{m}_3 = -\lambda_1 \cdot \min\left\{\dfrac{m_1}{2}, m_3\right\} + \lambda_2 \cdot \min\{m_1, m_2\} \end{cases} \quad (2)$$

■

Because the flow of a transition depends on its enabling degree which is based on the minimum function, a timed contPN under infinite servers semantics is a piecewise linear system. In fact, if we define

$$s = \prod_{t \in \boldsymbol{T}} |^{\bullet}t|,$$

the state space of a timed contPN can be *partitioned* [1] as follows: $R_1 \cup \cdots \cup R_s$, where each set R_k (for $k = 1, \ldots, s$) denotes a *region* (eventually empty) where the flow is limited by the same subset of places (one for each transition). For a given region R_k, we can define the *constraint matrix* $\boldsymbol{\Pi_k} : T \times P \to \mathbb{R}$ such that:

$$\boldsymbol{\Pi_k}(t_j, p_i) = \begin{cases} \dfrac{1}{Pre(p_i, t_j)}, \text{ if } (\forall \boldsymbol{m} \in R_k) \\ \dfrac{m_i}{Pre(p_i, t_j)} = \min\limits_{p_h \in {}^{\bullet}t_j} \left\{\dfrac{m_h}{Pre(p_h, t_j)}\right\}; \\ 0, \text{ otherwise.} \end{cases} \quad (3)$$

***Example** 2.6.* For the system sketched in Fig. 1, the flow of t_1 can be restricted by the marking of p_1 or p_3 and the flow of t_2 can be restricted by the marking of p_1 or p_2. The number of regions in this case is $s = 4$ and they are defined as follows:

- R_1: $\frac{m_1}{2} \le m_3$ and $m_1 \le m_2$ with $\boldsymbol{\Pi}_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- R_2: $\frac{m_1}{2} \le m_3$ and $m_1 \ge m_2$ with $\boldsymbol{\Pi}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
- R_3: $\frac{m_1}{2} \ge m_3$ and $m_1 \le m_2$ with $\boldsymbol{\Pi}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
- R_4: $\frac{m_1}{2} \ge m_3$ and $m_1 \ge m_2$ with $\boldsymbol{\Pi}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

■

[1] These partitions are disjoint except possibly on the borders.

If marking $\boldsymbol{m}$ belongs to R_k, we denote $\boldsymbol{\Pi}(\boldsymbol{m}) = \boldsymbol{\Pi}_k$ the corresponding constraint matrix. Furthermore, the firing rate of transitions can also be represented by a diagonal matrix $\boldsymbol{\Lambda} : T \times T \to \mathbb{R}_{>0}$, where

$$\Lambda(t_j, t_h) = \begin{cases} \lambda_j \text{ if } j = h \\ 0, \text{ otherwise} \end{cases}$$

Using this notation, the non-linear flow of the transitions at a given marking $\boldsymbol{m}$ (see eq. (1) for f_j) can be written as:

$$\boldsymbol{f} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} \quad (4)$$

We now consider net systems subject to external control actions, and assume that the only admissible control law consists in *slowing down* the firing speed of transitions (Silva and Recalde, 2004).

***Definition** 2.7.* The flow of the forced (or controlled) timed contPN is denoted as $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$, with $0 \le \boldsymbol{u}(\tau) \le \boldsymbol{f}(\tau)$, $\boldsymbol{u}(\tau)$ represents the control input.

Therefore, the control input will be dynamically upper bounded by the flow of the corresponding unforced system. Under these conditions, the overall behavior of the system is ruled by the following system (Mahulea *et al.*, 2005):

$$\begin{cases} \dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot [\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}(\tau)) \cdot \boldsymbol{m}(\tau) - \boldsymbol{u}(\tau)] \\ 0 \le \boldsymbol{u}(\tau) \le \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}(\tau)) \cdot \boldsymbol{m}(\tau) \end{cases} \quad (5)$$

This is a particular hybrid system: a piecewise linear system with autonomous switches and dynamic (or state-based) constraints in the input.

***Example** 2.8.* Let us consider the net system in Fig. 1 with $\boldsymbol{\lambda} = [5, 1]^T$. It is ruled by the following set of systems of the form (5):

$$\boldsymbol{m} \in R_1 : \begin{cases} \dot{\boldsymbol{m}}(\tau) = \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ \frac{3}{2} & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \end{bmatrix} \boldsymbol{m}(\tau) - \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{u}(\tau) \\ \boldsymbol{0} \le \boldsymbol{u}(\tau) \le \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{m}(\tau) \end{cases}$$

$$\boldsymbol{m} \in R_2 : \begin{cases} \dot{\boldsymbol{m}}(\tau) = \begin{bmatrix} -\frac{5}{2} & 1 & 0 \\ \frac{5}{2} & -1 & 0 \\ -\frac{5}{2} & 1 & 0 \end{bmatrix} \boldsymbol{m}(\tau) - \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{u}(\tau) \\ \boldsymbol{0} \le \boldsymbol{u}(\tau) \le \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{m}(\tau) \end{cases}$$

$$\boldsymbol{m} \in R_3 : \begin{cases} \dot{\boldsymbol{m}}(\tau) = \begin{bmatrix} 1 & 0 & -5 \\ -1 & 0 & 5 \\ 1 & 0 & -5 \end{bmatrix} \boldsymbol{m}(\tau) - \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{u}(\tau) \\ \boldsymbol{0} \le \boldsymbol{u}(\tau) \le \begin{bmatrix} 0 & 0 & 5 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{m}(\tau) \end{cases}$$

$$\boldsymbol{m} \in R_4\colon \begin{cases} \dot{\boldsymbol{m}}(\tau) = \begin{bmatrix} 0 & 1 & -5 \\ 0 & -1 & 5 \\ 0 & 1 & -5 \end{bmatrix} \boldsymbol{m}(\tau) - \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{u}(\tau) \\ \boldsymbol{0} \leq \boldsymbol{u}(\tau) \leq \begin{bmatrix} 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{m}(\tau) \end{cases}$$

■

As a final remark, it should be noted that in this paper we assume that all transitions are controllable [2], i.e., may be slowed down. It may also be possible to extend the approach to deal with uncontrollability of certain transitions. If transition t_j cannot be controlled, then it is obvious that the control input must be $u_j = 0$ at every time instant.

3. A CONSTRAINED LINEAR REPRESENTATION OF CONTINUOUS PN

The system in the eq. (5) is a *piecewise linear* system with a dynamical constraint on the control input $\boldsymbol{u}$ that depends on the current value of the system state $\boldsymbol{m}$. In this section we provide an alternative expression that takes the form of a simple *linear* system with dynamical constraints on the control input.

***Proposition** 3.1.* Any piecewise linear constrained model of the form (5) can be rewritten, by suitably defining a matrix $\boldsymbol{G}$, as a linear constrained model of the form:

$$\begin{cases} \dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \boldsymbol{w}(\tau) \\ \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}(\tau) \\ \boldsymbol{m}(\tau) \end{bmatrix} \leq \boldsymbol{0} \\ \boldsymbol{w}(\tau) \geq \boldsymbol{0} \end{cases} \tag{6}$$

that we call *continuous time controlled contPN* model, or *ct-contPN* model for short. The initial value of the state system is $\boldsymbol{m}(0) = \boldsymbol{m_0} \geq \boldsymbol{0}$.

Proof: The equivalence of the dynamic equations immediately follows by replacing $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$ in (6) being $\boldsymbol{f}(\tau)$ defined as in (4).

Concerning the constraints on the input, we first observe that, by virtute of (4), constraints in (5) can be rewritten as $\boldsymbol{0} \leq \boldsymbol{w}(\tau) \leq \boldsymbol{f}(\tau)$, i.e., $\forall j = 1, \cdots, n$, and at any marking $\boldsymbol{m}$,

$$0 \leq w_j \leq \lambda_j \min_{p_i \in {}^\bullet t_j} \left(\frac{m_i}{Pre(p_i, t_j)} \right)$$

that is equivalent to the following set of equations

$$0 \leq w_j \leq \lambda_j \frac{m_i}{Pre(p_i, t_j)} \qquad (\forall p_i \in {}^\bullet t_j).$$

All these equations can be combined as

$$\boldsymbol{0} \leq \boldsymbol{Q} \cdot \boldsymbol{w} \leq \boldsymbol{R} \cdot \boldsymbol{m}$$

where matrices $\boldsymbol{Q}$ $(q \times n)$ and $\boldsymbol{R}$ $(q \times m)$ have as many rows as there are "pre" arcs in the net, i.e., $q = \sum_{t \in T} |{}^\bullet t|$.

In particular, given a pre arc (p_i, t_j) the corresponding row of $\boldsymbol{Q}$ is the vector

$$\left[\underbrace{0 \ \cdots \ 0 \ 1}_{j} \ 0 \ \cdots \ 0 \right],$$

while corresponding row of $\boldsymbol{R}$ is the vector

$$\left[\underbrace{0 \ \cdots \ 0 \ \frac{\lambda_j}{Pre(p_i, t_j)}}_{i} \ 0 \ \cdots \ 0 \right].$$

If we let $\boldsymbol{G} = \begin{bmatrix} \boldsymbol{Q} & -\boldsymbol{R} \end{bmatrix}$ we obtain the constraints in the last two equations of (6). □

The system in eq. (6) is a linear system with a *dynamic-matrix* equal to $\boldsymbol{0}$ and an *input matrix* equal to the *token flow matrix* of the contPN. Note however, that there is still a dynamical constraint on the system inputs that depends on the value of the system state $\boldsymbol{m}$.

4. ON SAMPLED (OR DISCRETE-TIME) CONTINUOUS PETRI NETS MODELS

Let us obtain a discrete-time representation of continuous-time continuous Petri net under infinite servers semantics. Sampling should preserve the important information of the original model (for example the positiveness of the markings). This is studied in the next section through the equivalence of the reachability graph of the discrete-time model and the untimed model (not the reachability graph of discrete-time with continuous time). In this section the discretization is defined together with a bound for the sampling period.

The system given by the eq. (6) represents a continuous-time system and can be discretized. A first order discretization method is used here and we are proving that under some conditions, it ensures the reachability equivalence.

***Definition** 4.1.* Consider a ct-contPN as in eq. (6) and let Θ be a sampling period $(\tau = k \cdot \Theta)$. The *discrete-time controlled contPN* or *dt-contPN* $\langle \mathcal{N}, \lambda, \boldsymbol{m_0}, \theta \rangle$ can be written as follows:

$$\begin{cases} \boldsymbol{m}(k+1) = \boldsymbol{m}(k) + \Theta \cdot \boldsymbol{C} \cdot \boldsymbol{w}(k) \\ \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}(k) \\ \boldsymbol{m}(k) \end{bmatrix} \leq \boldsymbol{0} \\ \boldsymbol{w}(k) \geq \boldsymbol{0} \end{cases} \tag{7}$$

The initial value of the state of this system is $\boldsymbol{m}(0) = \boldsymbol{m_0} \geq \boldsymbol{0}$.

[2] We use "controllable" in the supervisory control sense. In (Mahulea *et al.*, 2005) the concept is referred as *control-feasible*.

The reachability space of dt-contPN can be defined as follows.

***Definition** 4.2.* We denote $RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta)$ the set of markings $\boldsymbol{m} \in \mathbb{R}_{\geq 0}$ such that there exists a finite input sequence $\boldsymbol{w} = \boldsymbol{w_1} \cdots \boldsymbol{w_k}$ and $\boldsymbol{m}(0) \xrightarrow{\boldsymbol{w_1}} \boldsymbol{m}(1) \xrightarrow{\boldsymbol{w_2}} \boldsymbol{m}(2) \cdots \xrightarrow{\boldsymbol{w_k}} \boldsymbol{m}(k) = \boldsymbol{m}$, where $\boldsymbol{0} \leq \boldsymbol{w}(k) \leq \boldsymbol{f}(k)\ \forall k$, and $\boldsymbol{f}(k)$ is the flow of the unforced system at time $k \cdot \Theta$.

***Example** 4.3.* Let us consider the net system in Fig. 1 with $\Theta = 1$, $\boldsymbol{\lambda} = [5, 1]^T$. Then the discrete-time representation is given by:

$$\begin{cases} \boldsymbol{m}(k+1) = \boldsymbol{m}(k) + \boldsymbol{C}\boldsymbol{w}(k) \\ w_1(k) - \dfrac{\lambda_1}{2} \cdot m_1(k) \leq 0 \\ w_1(k) - \lambda_1 \cdot m_3(k) \leq 0 \\ w_2(k) - \lambda_2 \cdot m_1(k) \leq 0 \\ w_2(k) - \lambda_2 \cdot m_2(k) \leq 0 \\ \boldsymbol{w}(k), \boldsymbol{m}(k+1) \geq \boldsymbol{0} \end{cases} \tag{8}$$

and

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & -\dfrac{5}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \tag{9}$$

It is important to stress that, although the evolution of a sampled contPN occurs in discrete steps, *discrete time evolutions* and *untimed evolutions* are not the same. As an example, while an untimed net can be seen evolving sequentially, executing a single transition firing at each step (because they are executed at the same time instant), a dt-contPN may evolve in concurrent steps where more than one transition fires. We denote such a concurrent step as follows:

$$\boldsymbol{m}[\{t_{i_1}(\alpha_1), t_{i_2}(\alpha_2), \ldots, t_{i_k}(\alpha_k)\}\rangle \boldsymbol{m}'.$$

In unforced ct-contPN under infinite servers semantics, the positiveness of the marking is ensured if the initial marking $\boldsymbol{m_0}$ is positive, because the flow of a transition goes to zero whenever one of the input places is empty (Silva and Recalde, 2004).

In a dt-contPN, this is not always true. Let us consider the net in Fig. 1, with $\boldsymbol{m_0} = [1.1, 3.9, 0.1]^T$, $\boldsymbol{\lambda} = [5, 1]^T$, $\Theta = 0.5$. Assume transition t_2 is stopped ($w_2(0) = 0$), then $m_3(1) = m_3(0) - \Theta \cdot w_1(0) = 0.1 - 0.5 \cdot w_1(0)$. But $w_1(0)$ is upper bounded by $\lambda_1 \cdot m_3(0) = 5 \cdot 0.1 = 0.5$. If the maximum value is chosen, then $m_3(1)$ will be negative!!!

This can be avoided if the sampling period is small enough. Let Θ be a sampling period such that for all $p \in P$ it holds that:

$$\sum_{t_j \in p^\bullet} \lambda_j \Theta < 1 \tag{10}$$

***Proposition** 4.4.* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ be a dt-contPN system with $\boldsymbol{m_0} \geq \boldsymbol{0}$ and Θ verifying (10).

(1) Any marking reachable from $\boldsymbol{m_0}$ is non negative, i.e., $RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta) \subseteq \mathbb{R}^m_{\geq 0}$.
(2) A place cannot be emptied with a finite sequence of firings, i.e., if $m_0(p) > 0$, then for all $\boldsymbol{m} \in RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta)$ it also holds $m(p) > 0$.

Proof: Let us consider a place p_i with ${p_i}^\bullet = \{t_1, t_2, \cdots, t_j\}$ and $m_i(k) > 0$. Then: $m_i(k+1) = m_i(k) + \Theta\boldsymbol{C}(i,:)\boldsymbol{w}(k) \geq m_i(k) - \Theta(\lambda_1 + \lambda_2 + \cdots + \lambda_j)m_i(k) = m_i(k)\left(1 - \sum_{t_j \in p^\bullet} \lambda_j \Theta\right) > 0$ □

In the rest of the paper we will assume that all nets are sampled with a sampling period Θ that satisfies (10).

***Corollary** 4.5.* If a marking $\boldsymbol{m}$ is reachable in a dt-contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta \rangle$ with Θ verifying (10) then is reachable in the underlying untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ (i.e. $RS^{dt}(\mathcal{N}, \boldsymbol{m_0}, \Theta) \subseteq RS^{un}(\mathcal{N}, \boldsymbol{m_0})$).

In general the converse of Corollary 4.5 is not true: in fact, the second item of Proposition 4.4 shows that in a dt-contPN with Θ satisfying (10) it is never possible to empty a place (only at the limit, thus timed contPN can be deadlocked only at the limit), while this may be possible in an untimed net system. As an example, in the untimed net system in Fig. 1 from the marking shown it is possible to fire $t_1(2)t_1(0.5)$, thus emptying place p_1. This marking is clearly not reachable on the same net system if we associate to it a firing rate vector and choose a sampling period Θ satisfying (10).

In the next section, two relaxations can be done: (1) considering in the untimed case only those sequences that never empty a marked place or (2) allowing the lim-reachable markings of the discrete-timed model. These relaxations are the same as in continuous-time case (Mahulea *et al.*, 2005). So, in fact we will prove that under these relaxations and with the sampling period as in (10), the reachability space of the discrete-time model will be the same with reachability space of the continuous-time model.

5. REACHABILITY "EQUIVALENCE" BETWEEN SAMPLED AND CONTINUOUS MODELS

The condition (10) can be seen like a "kind of Sampling Theorem" for sampling linear-invariant systems: Θ should be small enough to maintain some properties as that in Proposition 4.4. But it does not mean that all information is preserved by sampling. The following result characterizes the reachability set of dt-contPN.

***Lemma** 5.1.* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta\rangle$ be a dt-contPN system and assume that in the underlying untimed net system it is possible from $\boldsymbol{m}$ to fire the sequence $\boldsymbol{m}[t_j(\alpha)\rangle\boldsymbol{m'}$ and that for a certain $a > 1$, for all $p \in {}^\bullet t_j$ it holds $m'(p) \geq m(p)/a$.

Then in $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta\rangle$ marking $\boldsymbol{m'}$ is reachable from marking $\boldsymbol{m}$ with a finite sequence of length

$$r = \left\lceil \frac{a}{\Theta\lambda_j} \right\rceil.$$

Proof: Let us first prove by induction that the firing of a sequence $[t_j(\alpha\Theta\lambda_j/a)\rangle$ can at least be repeated $r-1$ times in the discrete time net.

(Basic step) It is immediate to observe that $t_j(\alpha\Theta\lambda_j/a)$ can be fired from $\boldsymbol{m}$, since $\Theta\lambda_j/a < 1$. The new marking is $\boldsymbol{m_1} = (\alpha\Theta\lambda_j/a)\cdot\boldsymbol{m'} + (1-\alpha\Theta\lambda_j/a)\cdot\boldsymbol{m}$.

(Inductive step) Assume that at a given intermediate step $\boldsymbol{m_h} = \beta\boldsymbol{m'} + (1-\beta)\cdot\boldsymbol{m}$, with $0<\beta<1$. It can be observed that for all $p \in {}^\bullet t_j$, it holds $m_h(p) = \beta m'(p) + (1-\beta)m(p) \geq \beta\frac{m(p)}{a} + (1-\beta)\frac{m(p)}{a} = \frac{m(p)}{a}$, hence $t_j(\alpha\Theta\lambda_j/a)$ can be fired from $\boldsymbol{m_h}$, since $\Theta\lambda_j/a < 1$.

After $r-1$ firings $t_j(\alpha\Theta\lambda_j/a)$ can still be fired and it is sufficient to fire t_j for a quantity less or equal to that to reach $\boldsymbol{m'}$ in one step. □

According to the previous lemma, regardless of the initial token content in a place p, if an untimed sequence reduces the marking of p by at most a factor $1/a$, then an equivalent finite sequence exists in the dt-net system.

***Theorem** 5.2.* A marking $\boldsymbol{m}$ is reachable in a dt-contPN $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta\rangle$ system (with Θ satisfying (10)) iff it is reachable in the underlying untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0}\rangle$ with a sequence that never empties an already marked place.

Proof: Mathematically, a sequence

$$\boldsymbol{m}[t_{i_1}(\alpha_1)\rangle\boldsymbol{m_1}[t_{i_2}(\alpha_2)\rangle\boldsymbol{m_2}\cdots[t_{i_k}(\alpha_k)\}\rangle\boldsymbol{m_k} = \boldsymbol{m'}$$

never empties a marked place if the following condition is verified

$$(\forall j = 1,\ldots,k), (\forall p \in {}^\bullet t_{i_j}) m_j(p) > 0 \qquad (11)$$

(If) Applying the previous Lemma for each $\boldsymbol{m_1}$, $\boldsymbol{m_2}$, $\cdots$, $\boldsymbol{m_k}$ implies that $\boldsymbol{m'}$ is reachable with a finite sequence.

(Only if) Assume there is a finite sequence that reaches $\boldsymbol{m}$ in the dt-contPN, then there exists an equivalent firing sequence for the untimed net system, according to Corollary 4.5. It is also immediate to observe that condition (11) holds because in the dt-contPN a place cannot be emptied with a finite sequence, according to Prop. 4.4 part 2. □

One may wonder what happens if a marking $\boldsymbol{m}$ is reachable in the untimed PN but there exists no sequence satisfying condition (11). In this case it can be easily proved that the marking is *lim*-reachable in the timed net, i.e., it is reachable with an unbounded sequence of steps. The result is formally proved in Theorem 5.3 by showing how such an infinite sequence may be determined.

***Theorem** 5.3.* If a marking $\boldsymbol{m}$ is reachable in the untimed contPN system $\langle \mathcal{N}, \boldsymbol{m_0}\rangle$, then it is lim-rechable in a dt-contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0}, \Theta\rangle$ with Θ satisfying (10).

Proof: Assume that in the untimed net system $\boldsymbol{m_0}[t_{r_1}(\alpha_1)\rangle\boldsymbol{m_1}[t_{r_2}(\alpha_2)\rangle\boldsymbol{m_2}\cdots[t_{r_k}(\alpha_k)\rangle\boldsymbol{m_k} = \boldsymbol{m}$, and let us define $\sigma = t_{r_1}(\alpha_1)t_{r_2}(\alpha_2)\cdots t_{r_k}(\alpha_k)$.

We will prove that this sequence is equivalent to an infinite sequence $\sigma^1\sigma^2\cdots$ in which all the input places of the fired transitions are reduced by each firing by at most a factor 1/2. Thus, applying Lemma 5.1, it can be fired in the discrete time net. This infinite sequence will fire each transition in σ, but in a smaller amount, and repeat the process. It will be seen that the amount of firing of each transition converges to the value in σ.

For each round, the sequence is defined as $\sigma^i = t_{r_1}(\beta_{i,1}\alpha_1)t_{r_2}(\beta_{i,2}\alpha_2)\cdots t_{r_k}(\beta_{i,k}\alpha_k)$ where

$$\beta_{i,1} = 1/2^i \qquad (i = 1,2,\ldots),$$
$$\beta_{1,j} = 1/2^j, \qquad (j = 1,\ldots,k),$$
$$\beta_{i,j} = \frac{1}{2}\left(\sum_{l=1}^{i}\beta_{i,j-1} - \sum_{l=1}^{i-1}\beta_{i,j}\right) \qquad (i = 2,\ldots;\ j = 2,\ldots,k).$$

Intuitively, in the first round the proportion of firing is decreasing each time so that places are never emptied by more than one half. In the following rounds, it is taken into account how much the previous transitions in the sequence have been fired, and how much the actual transition has been fired until now, again to be sure that the reduction never exceeds one half.

Formally, consider an intermediate step in which $\sigma^1\ldots\sigma^{i-1}$ and only part of σ^i, namely,

$t_{r_1}(\beta_{i,1}\alpha_1)t_{r_2}(\beta_{i,2}\alpha_2)\cdots t_{r_{j-1}}(\beta_{i,j-1}\alpha_{j-1})$, have been fired. If we denote $\boldsymbol{c}_j = \alpha_j\boldsymbol{C}(\cdot, t_{r_j})$ the actual marking can be described as

$\boldsymbol{m}_{i,j-1} = \boldsymbol{m_0} + \left(\sum_{h=1}^{i}\beta_{h,1}\right)\boldsymbol{c}_1 + \cdots + \left(\sum_{h=1}^{i}\beta_{h,j-1}\right)\boldsymbol{c}_{j-1} + \left(\sum_{h=1}^{i-1}\beta_{h,j}\right)\boldsymbol{c}_j + \cdots + \left(\sum_{h=1}^{i-1}\beta_{k,j}\right)\boldsymbol{c}_k = \left(1 - \sum_{h=1}^{i}\beta_{h,1}\right)\boldsymbol{m} + \left(\sum_{h=1}^{i}\beta_{h,1}\right)\boldsymbol{m_1} + \left(\sum_{h=1}^{i}\beta_{h,2}\right)\boldsymbol{c}_2 + \cdots + \left(\sum_{h=1}^{i}\beta_{h,j-1}\right)\boldsymbol{c}_{j-1} + \left(\sum_{h=1}^{i-1}\beta_{h,j}\right)\boldsymbol{c}_j + \cdots + \left(\sum_{h=1}^{i-1}\beta_{k,j}\right)\boldsymbol{c}_k = \cdots =$

$\left(1-\sum_{h=1}^{i}\beta_{h,1}\right)\boldsymbol{m}+\left(\sum_{h=1}^{i}\beta_{h,1}-\sum_{h=1}^{i}\beta_{h,2}\right)\boldsymbol{m_1}+$
$\cdots+\left(\sum_{h=1}^{i}\beta_{h,j-1}-\sum_{h=1}^{i-1}\beta_{h,j}\right)\boldsymbol{m_{j-1}}+$
$\left(\sum_{h=1}^{i-1}\beta_{h,j}-\sum_{h=1}^{i-1}\beta_{h,j-1}\right)\boldsymbol{m_j}\cdots+$
$\left(\sum_{h=1}^{i}\beta_{h,n-1}-\sum_{h=1}^{i-1}\beta_{h,k}\right)\boldsymbol{m_{k-1}}+$
$\left(\sum_{h=1}^{i}\beta_{h,n}\right)\boldsymbol{m_k}$

Hence, $\boldsymbol{m_{i,j-1}}\geq(\sum_{h=1}^{i}\beta_{h,j-1}-\sum_{h=1}^{i-1}\beta_{h,j})\boldsymbol{m_{j-1}}$ and so t_{r_j} can be fired half of this amount and no place looses more that one half of its token content.

With respect to the convergence to σ, it can be proved that $\beta_{i,j}=\frac{(i+j-2)!}{(j-1)!(i-1)!}\cdot\frac{1}{2^{i+j-1}}$, which is the probability mass distribution of the negative binomial of parameters j, $1/2$. Applying induction, the proof is based on the fact that the cumulative distribution function F_j can be immediately expressed as a regularized incomplete beta function, i.e., $F_j(h)=I_{1/2}(j,h+1)$, and that a regularized incomplete beta function enjoys the following property:

$$I_{1/2}(a,b)-I_{1/2}(a+1,b)=\frac{(a+b-1)!}{(a)!(b-1)!}\cdot\frac{1}{2^{a+b}}.$$

Observe that $\beta_{1,j}=\frac{1}{2^j}=\frac{(1+j-2)!}{(j-1)!(1-1)!}\cdot\frac{1}{2^{1+j-1}}$, and that $\beta_{i,1}=\frac{1}{2^i}=\frac{(i+1-2)!}{(1-1)!(i-1)!}\cdot\frac{1}{2^{i+1-1}}$.

Applying induction "following the rows", assume it holds for $\beta_{l,k}$, with $1\leq l\leq i-1$ and $1\leq k\leq n$, and for $\beta_{i,k}$, with $1\leq k\leq j-1$. Let us prove it for $\beta_{i,j}$.

$$\beta_{i,j}=\frac{\sum_{l=1}^{i}\beta_{i,j-1}-\sum_{l=1}^{i-1}\beta_{i,j}}{2}=$$
$$\frac{\beta_{i,j-1}}{2}+\frac{\sum_{l=1}^{i-1}\beta_{i,j-1}-\sum_{l=1}^{i-1}\beta_{i,j}}{2}=$$
$$\frac{\beta_{i,j-1}}{2}+\frac{1}{2}\left(\sum_{l=1}^{i-1}\frac{\binom{l+j-3}{j-2}}{2^{l+j-2}}-\sum_{l=1}^{i-1}\frac{\binom{l+j-2}{j-1}}{2^{l+j-1}}\right)$$
$$=\frac{\beta_{i,j-1}}{2}+\frac{1}{2}\left(\sum_{l=0}^{i-2}\frac{\binom{l+j-2}{j-2}}{2^{l+j-1}}-\sum_{l=0}^{i-2}\frac{\binom{l+j-1}{j-1}}{2^{l+j}}\right)=$$
$$\frac{1}{2}\frac{\binom{i+j-3}{j-2}}{2^{i+j-2}}+\frac{1}{2}(I_{1/2}(j-1,i-1)-I_{1/2}(j,i-1))=$$
$$\frac{1}{2^{i+j-1}}\frac{(i+j-3)!}{(j-2)!(i-1)!}+\frac{1}{2^{i+j-1}}\frac{(i+j-3)!}{(j-1)!(i-2)!}=$$
$$\frac{1}{2^{i+j-1}}\frac{(i+j-2)!}{(j-1)!(i-1)!}$$

This means that the amount in which transition t_j is fired is α_j times a cumulative distribution function, and so in the limit it converges to α_j. $\square$

6. CONCLUSIONS

In this paper we provide a study of contPNs under infinite servers semantics. First, different ways of describing the behavior of controlled contPNs are presented, starting with a min-based non-linear system (eq.(1) plus $\dot{\boldsymbol{m}}=\boldsymbol{C}\cdot\boldsymbol{f}$), continuing with a piecewise linear form (eq. (5)) and ending with a linear constrained form (eq. (6)).

The linear constrained system is then discretized and we provide a *Sampling theorem* giving an upper bound on sampling period. The purpose of the Sampling theorem presented here is to preserve reachability conditions (in particular non-negativity of markings), not to reconstruct the original signal from the sampled one.

The reachability space of the sampled system is studied in the last part of the paper and some relations between this space and the space of the underlying untimed contPN are provided. In practice, the sampling rate may be higher (like in Nyquist-Shannon sampling theorem) if signal reconstruction is required. But this is a topic to be considered in a future work. Anyhow the classical sampling theorem for linear systems should be respected for all embedded ones.

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