Quantized optimal control of discrete-time systems

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Abstract

In this paper we consider a quantized discrete-time linear quadratic regulator (DLQR) problem, namely a DLQR problem where the input u may only take values in a given finite set $\mathcal{U}$. Based on our previous results on the optimal control of hybrid systems we show that the optimal control law for the quantized DLQR problem takes the form of a feedback control law, that can be obtained from a partition of the state space $\mathcal{C}$, computed off-line. The numerical simulations carried out enabled us to observe a particular structure of $\mathcal{C}$, related to the solution of the non-quantized DLQR problem.

The lines of our future research in this topic are described in details in the last section, devoted to conclusions and future work.

1 Introduction

In many applications, physical constraints lead to an optimal continuous control law such that the control input u may only take values from a finite set $\mathcal{U}$. This problem is usually referred to as quantized optimal control and, unlike other approaches where the objective function is that of finding a suitable input discretization, we will address it assuming that the set $\mathcal{U}$ is given.

Quantized control of continuous-time systems with continuous output measurements is a problem that can be framed as a standard optimal control problem with bounds on the input signal [7]. The problem becomes much more complicated if discrete-time systems are considered or if the output measurements are supposed to be quantized as well. In these cases, as an example it is usually impossible to stabilize a system on the origin and different notions, such as that of practical stabilizability or invariant and attractive sets, must be introduced [6].

In this paper, that is a preliminary contribution to this problem, we extend our previous results in the context of optimal control of switched systems [2] to the case of quantized optimal control. In fact, a quantized system can be seen as a switched affine system where a different affine dynamics is associated to each element of the set $\mathcal{U}$. Our goal here is that of stabilizing the system at the origin, and we assume that the coefficient matrix A is Hurwitz and the null input belongs to $\mathcal{U}$.

We are able to compute a partition $\mathcal{C}$ of the state space that provides a state feedback control law: a different value of $\mathcal{U}$ is associated to each point of the state space thus, depending on the current value of the continuous state, we know which is the value of the control input that has to be applied in order to minimize the chosen quadratic performance index.

The most burdensome part of the proposed procedure is the high computational complexity of the procedure to compute $\mathcal{C}$. Nevertheless, this computation is done off-line thus the on-line control can be determined by simply consulting a look-up table.

The main advantages of this approach can be briefly summarized as follows. (a) It provides a feedback control law; (b) it guarantees the optimality of the solution; (c) it can be trivially extended to the case of non scalar input.

Note that very few approaches to solve the quantized control problem have been proposed in the literature, that are based on a partition of the state space. As far as we know, the first one has been very recently proposed by in Bemporad in [1], based on multiparametric integer programming.

2 Problem statement

In this paper we consider the quantized DLQR problem:

$$\begin{cases} \min \; J(x_0) \triangleq \displaystyle\sum_{k=0}^{\infty} x_k^T Q x_k + R u_k^2 \\ \text{s.t.} \\ x_{k+1} = Ax_k + Bu_k \quad k = 0, 1, \ldots, \infty, \\ u_k \in \mathcal{U} \triangleq \{\bar{u}_1, \ldots, \bar{u}_s\} \end{cases} \tag{1}$$

where the initial state $x_0 \in \mathbb{R}^n$ is given; $A \in \mathbb{R}^{n \times n}$ is strictly Hurwitz; $u_k \in \mathbb{R}$; $B \in \mathbb{R}^n$; Q and R are definite weighting matrices; $\mathcal{U}$ is the finite set of admissible values of the control such that $\bar{u}_1 < \bar{u}_2 < \ldots < \bar{u}_s$ and $0 \in \mathcal{U}$.

As well known from optimal control theory, when u_k may take any values in $\mathbb{R}$ the optimal control law takes the form of a feedback control law:

$$u_k = -Kx_k, \quad k = 0, 1, \ldots, +\infty,$$

where K is obtained by solving an algebraic Riccati equation [3].

On the contrary, when the constraint $u_k \in \mathcal{U}$ is added, the optimal control law cannot be implemented as a feedback control law with constant gains, and highly depends on the initial state x_0. This makes its implementation unfeasible in many real applications.

The main contribution of this paper consists in showing that, provided that some computations are performed

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off-line in order to compute an appropriate partition of the state space $\mathcal{C}$, the optimal control law can still be implemented as a state feedback control law. More precisely, $\mathcal{C}$ represents a portion of the state space $\mathbb{R}^n$ in s regions $\mathcal{R}_i$, $i = 1, \ldots, s$, each one associated to a different value of $\bar{u}_i \in \mathcal{U}$. If the current state lies in a point of the region $\mathcal{R}_i$ of $\mathcal{C}$, then the control is taken equal to $\bar{u}_i$.

To prove this result we repeat the same arguments we derived in the context of optimal control of continuous-time switched systems [2], to the case of discrete-time switched systems.

3 Computation of $\mathcal{C}$ via optimal control

In this section we show that a state space partition $\mathcal{C}$ can be computed in order to determine a state-feedback control law for the quantized DLQR problem. In particular, we first recall some results on the optimal control of switched systems with a finite number of switches; then we show how these results can be extended to the case of an infinite number of switches; finally, we show how they can be used to solve the considered quantized DLQR problem.

3.1 Optimal control of switched systems with a finite number of switches

Let us consider the following class of discrete-time hybrid systems, commonly denoted as *switched linear autonomous systems*,

$$x_{k+1} = A_{i_k} x_k, \quad i_k \in \mathcal{S}, \tag{2}$$

where $x_k \in \mathbb{R}^n$, $i_k \in \mathcal{S}$ is the current mode and represents a control variable, $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ is a finite set of integers, each one associated with a matrix $A_i \in \mathbb{R}^{n \times n}$.

Let us consider the optimal control problem

$$\begin{cases} V_N^*(x_0, i_0) & \triangleq \min\limits_{\mathcal{Z},\mathcal{K}} \left\{ F(\mathcal{Z},\mathcal{K}) \triangleq \sum\limits_{k=0}^{\infty} x_k' Q_{i_k} x_k \right\} \\ \text{s.t.} & x_{k+1} = A_{i_k} x_k \\ & i_k = z_r \in \mathcal{S}, \quad \text{for } \; k_r \le k < k_{r+1}, \\ & \qquad\qquad\qquad\qquad r = 0, 1, \ldots, N \\ & 0 \le k_0 \le k_1 \le \ldots \le k_N < k_{N+1} = +\infty \end{cases} \tag{3}$$

where Q_i are positive semi-definite matrices, (x_0, z_0) is the initial state of the system, and $N < +\infty$ is the maximum number of allowed switches, that is given a priori.

In this optimization problem there are two types of decision variables:

- $\mathcal{K} \triangleq \{k_1, \ldots, k_N\}$ is a finite sequence of switching time indices;
- $\mathcal{Z} \triangleq \{z_1, \ldots, z_N\}$ is a finite sequence of modes.

In order to make (2) stabilizable on the origin, we assume the following:

Assumption 3.1. There exists at least one mode $i \in \mathcal{S}$ such that A_i is strictly Hurwitz. ■

The optimization problem (3) can be solved by simply repeating the results in [2] to the discrete-time case. In particular, for a given mode $i \in \mathcal{S}$ and for a given switch $r \in \{1, \ldots, N\}$ it is possible to construct a table $\mathcal{C}_r^i$ that partitions the state space $\mathbb{R}^n$ into s regions $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$. Whenever $z_{r-1} = i$ we use table $\mathcal{C}_r^i$ to determine if a switch should occur: as soon as the state reaches a point in the region $\mathcal{R}_j$ (with $j \neq i$) we will switch to mode $z_r = j$, while no switch will occur when the system's state belongs to $\mathcal{R}_i$.

To prove this result in [2] we shown constructively how the tables $\mathcal{C}_r^i$ can be computed off-line using a dynamic programming argument. We first shown how tables $\mathcal{C}_1^i$ ($i \in \mathcal{S}$) for the last switch can be determined. Then we show by induction how the tables $\mathcal{C}_r^i$ can be computed once the tables $\mathcal{C}_{r-1}^i$ are known.

3.2 Optimal control of switched systems with an infinite number of switches

In this section we recall the main results in [5] that enables us to conclude that, under appropriate assumptions, the above procedure can be extended to the case of $N = \infty$. In particular, we consider an optimal control problem of the form (3) where

(i) for at least one $i \in \mathcal{S}$, A_i is strictly Hurwitz;

(ii) for all $i \in \mathcal{S}$, $Q_i > 0$.

Proposition 3.2. *[5]* For any continuous initial state x_0, $x_0 \neq 0$, and $\forall\, \varepsilon > 0$, $\exists\, \bar{N}$ such that for all $N > \bar{N}$,

$$\frac{V_N^*(x_0, i) - V_{\bar{N}}{}^*(x_0, j)}{V_N^*(x_0, i)} < \varepsilon,$$

for all $i, j \in \mathcal{S}$.

According to the above result, one may use a given fixed relative tolerance ε to approximate two cost values, i.e.,

$$\frac{V_N^*(x, i) - V_{N'}{}^*(x, j)}{V_N^*(x, i)} < \varepsilon \implies V_N^*(x, i) \cong V_{N'}{}^*(x, j).$$

This result enables us to prove the following important theorem.

Theorem 3.3. *[5]* Given a fixed relative tolerance ε, if $\bar{N}$ is chosen as in Proposition 3.2 then for all $i, j \in \mathcal{S}$ it holds that $\mathcal{C}_{\bar{N}+1}^i = \mathcal{C}_{\bar{N}+1}^j$.

This result also allows one to conclude that

$$\forall\, i \in \mathcal{S}, \quad \mathcal{C}_\infty = \lim_{N \to \infty} \mathcal{C}_N^i,$$

i.e., *all tables converge to the same one*.

To construct the table $\mathcal{C}_\infty$ the value of $\bar{N}$ is needed. We do not provide so far any analytical way to compute $\bar{N}$, therefore our approach consists in constructing tables until a convergence criterion is met.

Table $\mathcal{C}_\infty$ can be used to compute the optimal feedback control law that solves an optimal control problem of the form (3) with $N = \infty$. More precisely, when an infinite number of switches is available, we only need to keep track of the table $\mathcal{C}_\infty$. If the current continuous state is x and the current mode is A_i, on the basis of the knowledge of the color of $\mathcal{C}_\infty$ in x, we decide if it is better to still evolve with the current dynamics A_i or switch to a different dynamics, that is univocally determined by the color of the table in x.

3.3 A feedback control law for the quantized DLQR problem

The above procedure can be effectively used to derive a control law for the quantized DLQR problem (1). In fact, the DLQR problem (1) can be rewritten as an optimal control problem in the form (3) where

$$A_i = \begin{bmatrix} A & B\bar{u}_i \\ 0_n^T & 1 \end{bmatrix},$$

$$Q_i = \begin{bmatrix} Q & 0 \\ 0 & R\bar{u}_i^2 \end{bmatrix}, \quad i \in \mathcal{S} \triangleq \{1, \ldots, s\},$$

0_n is the n-dimensional column vector of zeros,

$$N = +\infty,$$

and the state vector is

$$\begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} \in \mathbb{R}^{n+1}, \quad \text{with} \quad \tilde{x}_0 = 1.$$

Note that, because of the presence of 1 in the entry $(n+1, n+1)$ of A_i, matrices A_i are not strictly Hurwith for all $i \in \mathcal{S}$. Thus Assumption (i) of Section 3.2 is not satisfied. Nevertheless, by hypothesis $0 \in \mathcal{U}$, i.e., there exists a $j \in \mathcal{S}$ such that $\bar{u}_j = 0$, thus

$$A_j = \begin{bmatrix} A & 0_n \\ 0_n^T & 1 \end{bmatrix}, \quad Q_j = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

where A is strictly Hurwitz and $Q > 0$. This implies that the optimal cost associated with dynamics A_j is finite also at an infinite time horizon, that enables us to repeat all the arguments proposed in the previous Sections 3 and 3.2.

Now, once the table $\mathcal{C}_\infty$ is constructed, we define $\mathcal{C}$ as the projection of $\mathcal{C}_\infty$ in the n-dimensional x-space. At this point, following the procedure described above, we can compute the optimal control law of the quantized DLQR problem (1) as a state feedback control law.

4 Numerical examples

Let us finally present the results of some numerical examples that allow us to make some interesting remarks on the structure of $\mathcal{C}$.

Let us first introduce the following notation. We recall that in the problem statement we assumed that $\bar{u}_1 < \bar{u}_2 < \ldots < \bar{u}_s$. Now, we define

$$\mathcal{P} \triangleq \{x_0 \in \mathbb{R}^n \mid \quad \begin{aligned} &\bar{u}_1 \leq -Kx_k \leq \bar{u}_s, \\ &x_{k+1} = (A - BK)x_k, \\ &\qquad k = 0, 1, \ldots, +\infty\} \end{aligned} \tag{4}$$

where K is the solution of the unconstrained DLQR problem and is determined by solving an algebraic Riccati equation. Thus, the set $\mathcal{P} \subset \mathbb{R}^n$ is the set of points of the state space such that, if the system evolution starts from one point in $\mathcal{P}$ and the system is controlled with the feedback control law $u_k = -Kx_k$, then the control input never violates the constraint $u_k \in [\bar{u}_1, \bar{u}_s]$.

Now, let us consider a quantized DLQR problem of the form (1) where

$$A = \begin{bmatrix} 0.949 & -0.064 \\ 0.586 & 0.981 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

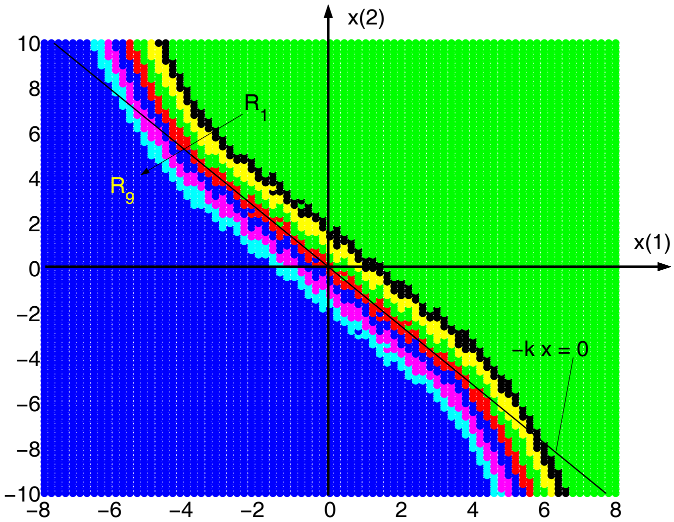

Figure 1. The switching table $\mathcal{C}$ relative to the example in Section 4.

$Q = I_2$ and $R = 1$.

Let $\mathcal{U} = \{u_{\min}, u_{\min} + \Delta u, u_{\min} + 2\Delta u, \ldots, u_{\max}\}$ where $u_{\min} = -1$, $u_{\max} = 1$ and $\Delta u = 0.25$, thus $|\mathcal{U}| = 9$.

Table $\mathcal{C}$ has the structure shown in Figure 1. Note that in this case $\bar{N} = 15$ was large enough to reach the convergence of the tables to the same one, i.e., to compute the state space partition $\mathcal{C}$.

The structure of $\mathcal{C}$ and $\mathcal{P}$ (see Figure 2) enables us to make the following interesting remarks.

Firstly, we observe that the set $\mathcal{P}$ satisfies the two inequalities

$$u_{\min} = \bar{u}_1 \leq -Kx \leq u_{\max} = \bar{u}_9.$$

Moreover, as better highlighted in Figure 3, we observe that within the set $\mathcal{P}$, regions $\mathcal{R}_j$, $j = 1, \ldots, 9$, are parallel to lines of equations $-Kx =$ constant.

More precisely, region $\mathcal{R}_1$ relative to the control input $\bar{u}_1$, is approximately delimited by the inequality

$$-Kx \leq \bar{u}_1 + \frac{\Delta u}{2};$$

regions $\mathcal{R}_j$, $j = 2, \ldots, 8$, are approximately delimited by the inequalities

$$\bar{u}_j - \frac{\Delta u}{2} < -Kx \leq \bar{u}_j + \frac{\Delta u}{2};$$

finally, region $\mathcal{R}_9$ is approximately delimited by the inequality

$$-Kx \geq \bar{u}_9 - \frac{\Delta u}{2}.$$

Analogous results have been obtained in all the other considered numerical examples.

Now, assume that the initial state is $x_0 = [2 \quad 2]^T$. The resulting state evolution is reported in Figure 4 where we have also pointed out, for each sampling time k, the corresponding value of u_k. The value of the performance index $J(x_0)$ is equal to 21.12. It is interesting to observe

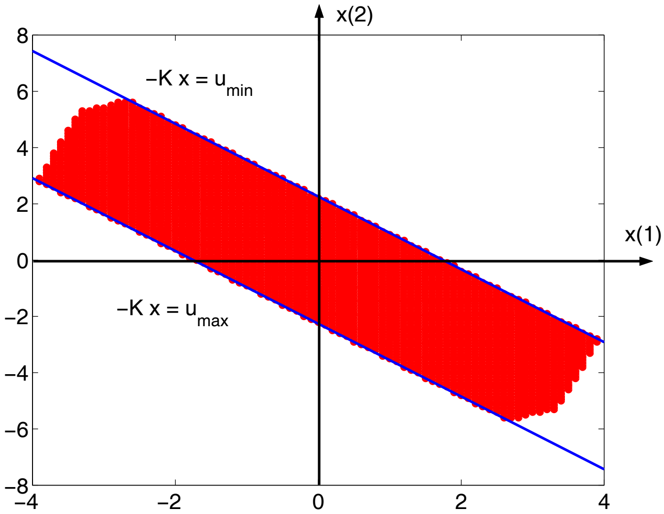

Figure 2. The set $\mathcal{P}$ relative to the example in Section 4.

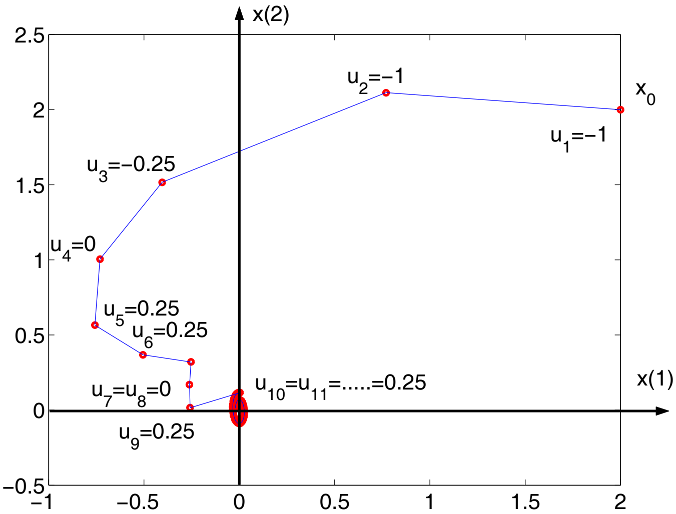

Figure 4. The state resulting state evolution when $x_0 = [2 \;\; 2]^T$.

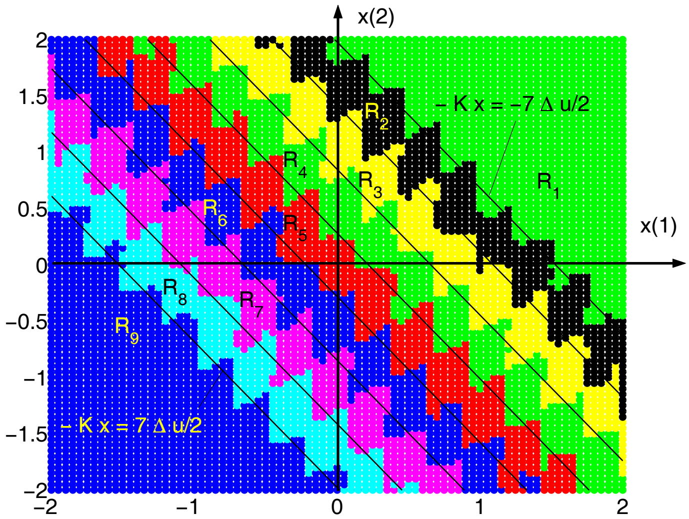

Figure 3. The structure of $\mathcal{C}$ within the set $\mathcal{P}$.

that if the constraint $u \in \mathcal{U}$ is relaxed and we solve the QP problem with the constraint $u_1 \leq u \leq u_9$ the resulting value of the performance index is 20.61.

5 Conclusions and future work

In this paper we have studied the problem of DLQR quantized control. We proposed a solution that is based on our previous results in the context of optimal control of switched affine systems. The main advantage of the approach is that is provides a feedback solution, based on a partition of the state space $\mathcal{C}$ that enables us to establish the optimal value of the control on the basis of the knowledge of the current state x.

The main restrictive assumption we have done here is that the coefficient matrix A of the controlled system is Hurwitz and the null input belongs to $\mathcal{U}$. What we plan to investigate in the future is how to extend the proposed approach to the case of systems with unstable matrix A.

Moreover, we plan to investigate how to stabilize the system either to a point different from the origin, or how to keep it within an attractive set (that we may be either assumed known a priori or unknown).

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