

off-line in order to compute an appropriate partition of the state space \mathcal{C} , the optimal control law can still be implemented as a state feedback control law. More precisely, \mathcal{C} represents a portion of the state space \mathbb{R}^n in s regions \mathcal{R}_i , $i = 1, \dots, s$, each one associated to a different value of $\bar{u}_i \in \mathcal{U}$. If the current state lies in a point of the region \mathcal{R}_i of \mathcal{C} , then the control is taken equal to \bar{u}_i .

To prove this result we repeat the same arguments we derived in the context of optimal control of continuous-time switched systems [2], to the case of discrete-time switched systems.

3 Computation of \mathcal{C} via optimal control

In this section we show that a state space partition \mathcal{C} can be computed in order to determine a state-feedback control law for the quantized DLQR problem. In particular, we first recall some results on the optimal control of switched systems with a finite number of switches; then we show how these results can be extended to the case of an infinite number of switches; finally, we show how they can be used to solve the considered quantized DLQR problem.

3.1 Optimal control of switched systems with a finite number of switches

Let us consider the following class of discrete-time hybrid systems, commonly denoted as *switched linear autonomous systems*,

$$x_{k+1} = A_{i_k} x_k, \quad i_k \in \mathcal{S}, \quad (2)$$

where $x_k \in \mathbb{R}^n$, $i_k \in \mathcal{S}$ is the current mode and represents a control variable, $\mathcal{S} \triangleq \{1, 2, \dots, s\}$ is a finite set of integers, each one associated with a matrix $A_i \in \mathbb{R}^{n \times n}$.

Let us consider the optimal control problem

$$\left\{ \begin{array}{l} V_N^*(x_0, i_0) \triangleq \min_{\mathcal{Z}, \mathcal{K}} \left\{ F(\mathcal{Z}, \mathcal{K}) \triangleq \sum_{k=0}^{\infty} x_k' Q_{i_k} x_k \right\} \\ \text{s.t.} \quad \begin{array}{l} x_{k+1} = A_{i_k} x_k \\ i_k = z_r \in \mathcal{S}, \text{ for } k_r \leq k < k_{r+1}, \\ \quad \quad \quad r = 0, 1, \dots, N \\ 0 \leq k_0 \leq k_1 \leq \dots \leq k_N < k_{N+1} = +\infty \end{array} \end{array} \right. \quad (3)$$

where Q_i are positive semi-definite matrices, (x_0, z_0) is the initial state of the system, and $N < +\infty$ is the maximum number of allowed switches, that is given a priori.

In this optimization problem there are two types of decision variables:

- $\mathcal{K} \triangleq \{k_1, \dots, k_N\}$ is a finite sequence of switching time indices;
- $\mathcal{Z} \triangleq \{z_1, \dots, z_N\}$ is a finite sequence of modes.

In order to make (2) stabilizable on the origin, we assume the following:

Assumption 3.1. There exists at least one mode $i \in \mathcal{S}$ such that A_i is strictly Hurwitz. ■

The optimization problem (3) can be solved by simply repeating the results in [2] to the discrete-time case. In particular, for a given mode $i \in \mathcal{S}$ and for a given

switch $r \in \{1, \dots, N\}$ it is possible to construct a table \mathcal{C}_r^i that partitions the state space \mathbb{R}^n into s regions $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_s$. Whenever $z_{r-1} = i$ we use table \mathcal{C}_r^i to determine if a switch should occur: as soon as the state reaches a point in the region \mathcal{R}_j (with $j \neq i$) we will switch to mode $z_r = j$, while no switch will occur when the system's state belongs to \mathcal{R}_i .

To prove this result in [2] we shown constructively how the tables \mathcal{C}_r^i can be computed off-line using a dynamic programming argument. We first shown how tables \mathcal{C}_1^i ($i \in \mathcal{S}$) for the last switch can be determined. Then we show by induction how the tables \mathcal{C}_r^i can be computed once the tables \mathcal{C}_{r-1}^i are known.

3.2 Optimal control of switched systems with an infinite number of switches

In this section we recall the main results in [5] that enables us to conclude that, under appropriate assumptions, the above procedure can be extended to the case of $N = \infty$. In particular, we consider an optimal control problem of the form (3) where

(i) for at least one $i \in \mathcal{S}$, A_i is strictly Hurwitz;

(ii) for all $i \in \mathcal{S}$, $Q_i > 0$.

Proposition 3.2. [5] For any continuous initial state x_0 , $x_0 \neq 0$, and $\forall \varepsilon > 0$, $\exists \bar{N}$ such that for all $N > \bar{N}$,

$$\frac{V_N^*(x_0, i) - V_N^*(x_0, j)}{V_N^*(x_0, i)} < \varepsilon,$$

for all $i, j \in \mathcal{S}$.

According to the above result, one may use a given fixed relative tolerance ε to approximate two cost values, i.e.,

$$\frac{V_N^*(x, i) - V_N^*(x, j)}{V_N^*(x, i)} < \varepsilon \implies V_N^*(x, i) \cong V_N^*(x, j).$$

This result enables us to prove the following important theorem.

Theorem 3.3. [5] Given a fixed relative tolerance ε , if \bar{N} is chosen as in Proposition 3.2 then for all $i, j \in \mathcal{S}$ it holds that $\mathcal{C}_{\bar{N}+1}^i = \mathcal{C}_{\bar{N}+1}^j$.

This result also allows one to conclude that

$$\forall i \in \mathcal{S}, \quad \mathcal{C}_\infty = \lim_{N \rightarrow \infty} \mathcal{C}_N^i,$$

i.e., *all tables converge to the same one*.

To construct the table \mathcal{C}_∞ the value of \bar{N} is needed. We do not provide so far any analytical way to compute \bar{N} , therefore our approach consists in constructing tables until a convergence criterion is met.

Table \mathcal{C}_∞ can be used to compute the optimal feedback control law that solves an optimal control problem of the form (3) with $N = \infty$. More precisely, when an infinite number of switches is available, we only need to keep track of the table \mathcal{C}_∞ . If the current continuous state is x and the current mode is A_i , on the basis of the knowledge of the color of \mathcal{C}_∞ in x , we decide if it is better to still evolve with the current dynamics A_i or switch to a different dynamics, that is univocally determined by the color of the table in x .

3.3 A feedback control law for the quantized DLQR problem

The above procedure can be effectively used to derive a control law for the quantized DLQR problem (1). In fact, the DLQR problem (1) can be rewritten as an optimal control problem in the form (3) where

$$A_i = \begin{bmatrix} A & B\bar{u}_i \\ 0_n^T & 1 \end{bmatrix},$$

$$Q_i = \begin{bmatrix} Q & 0 \\ 0 & R\bar{u}_i^2 \end{bmatrix}, \quad i \in \mathcal{S} \triangleq \{1, \dots, s\},$$

0_n is the n -dimensional column vector of zeros,

$$N = +\infty,$$

and the state vector is

$$\begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} \in \mathbb{R}^{n+1}, \quad \text{with } \tilde{x}_0 = 1.$$

Note that, because of the presence of 1 in the entry $(n+1, n+1)$ of A_i , matrices A_i are not strictly Hurwitz for all $i \in \mathcal{S}$. Thus Assumption (i) of Section 3.2 is not satisfied. Nevertheless, by hypothesis $0 \in \mathcal{U}$, i.e., there exists a $j \in \mathcal{S}$ such that $\bar{u}_j = 0$, thus

$$A_j = \begin{bmatrix} A & 0_n \\ 0_n^T & 1 \end{bmatrix}, \quad Q_j = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

where A is strictly Hurwitz and $Q > 0$. This implies that the optimal cost associated with dynamics A_j is finite also at an infinite time horizon, that enables us to repeat all the arguments proposed in the previous Sections 3 and 3.2.

Now, once the table \mathcal{C}_∞ is constructed, we define \mathcal{C} as the projection of \mathcal{C}_∞ in the n -dimensional x -space. At this point, following the procedure described above, we can compute the optimal control law of the quantized DLQR problem (1) as a state feedback control law.

4 Numerical examples

Let us finally present the results of some numerical examples that allow us to make some interesting remarks on the structure of \mathcal{C} .

Let us first introduce the following notation. We recall that in the problem statement we assumed that $\bar{u}_1 < \bar{u}_2 < \dots < \bar{u}_s$. Now, we define

$$\mathcal{P} \triangleq \{x_0 \in \mathbb{R}^n \mid \bar{u}_1 \leq -Kx_k \leq \bar{u}_s, \\ x_{k+1} = (A - BK)x_k, \\ k = 0, 1, \dots, +\infty\} \quad (4)$$

where K is the solution of the unconstrained DLQR problem and is determined by solving an algebraic Riccati equation. Thus, the set $\mathcal{P} \subset \mathbb{R}^n$ is the set of points of the state space such that, if the system evolution starts from one point in \mathcal{P} and the system is controlled with the feedback control law $u_k = -Kx_k$, then the control input never violates the constraint $u_k \in [\bar{u}_1, \bar{u}_s]$.

Now, let us consider a quantized DLQR problem of the form (1) where

$$A = \begin{bmatrix} 0.949 & -0.064 \\ 0.586 & 0.981 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

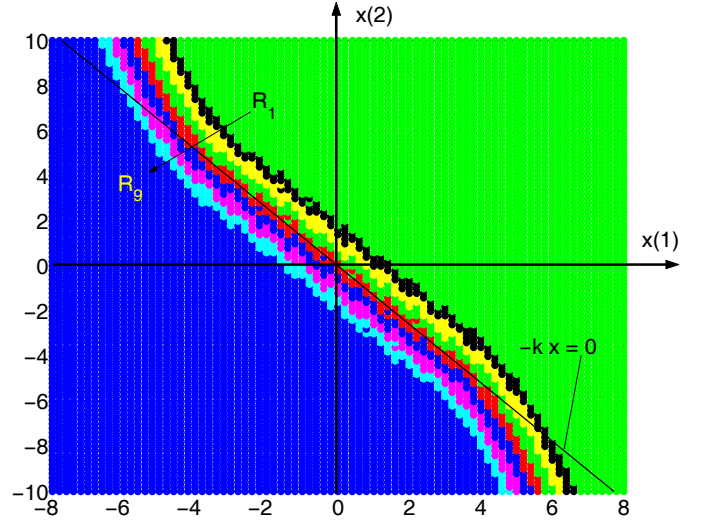


Figure 1. The switching table \mathcal{C} relative to the example in Section 4.

$Q = I_2$ and $R = 1$.

Let $\mathcal{U} = \{u_{\min}, u_{\min} + \Delta u, u_{\min} + 2\Delta u, \dots, u_{\max}\}$ where $u_{\min} = -1$, $u_{\max} = 1$ and $\Delta u = 0.25$, thus $|\mathcal{U}| = 9$.

Table \mathcal{C} has the structure shown in Figure 1. Note that in this case $\bar{N} = 15$ was large enough to reach the convergence of the tables to the same one, i.e., to compute the state space partition \mathcal{C} .

The structure of \mathcal{C} and \mathcal{P} (see Figure 2) enables us to make the following interesting remarks.

Firstly, we observe that the set \mathcal{P} satisfies the two inequalities

$$u_{\min} = \bar{u}_1 \leq -Kx \leq u_{\max} = \bar{u}_9.$$

Moreover, as better highlighted in Figure 3, we observe that within the set \mathcal{P} , regions \mathcal{R}_j , $j = 1, \dots, 9$, are parallel to lines of equations $-Kx = \text{constant}$.

More precisely, region \mathcal{R}_1 relative to the control input \bar{u}_1 , is approximately delimited by the inequality

$$-Kx \leq \bar{u}_1 + \frac{\Delta u}{2};$$

regions \mathcal{R}_j , $j = 2, \dots, 8$, are approximately delimited by the inequalities

$$\bar{u}_j - \frac{\Delta u}{2} < -Kx \leq \bar{u}_j + \frac{\Delta u}{2};$$

finally, region \mathcal{R}_9 is approximately delimited by the inequality

$$-Kx \geq \bar{u}_9 - \frac{\Delta u}{2}.$$

Analogous results have been obtained in all the other considered numerical examples.

Now, assume that the initial state is $x_0 = [2 \ 2]^T$. The resulting state evolution is reported in Figure 4 where we have also pointed out, for each sampling time k , the corresponding value of u_k . The value of the performance index $J(x_0)$ is equal to 21.12. It is interesting to observe

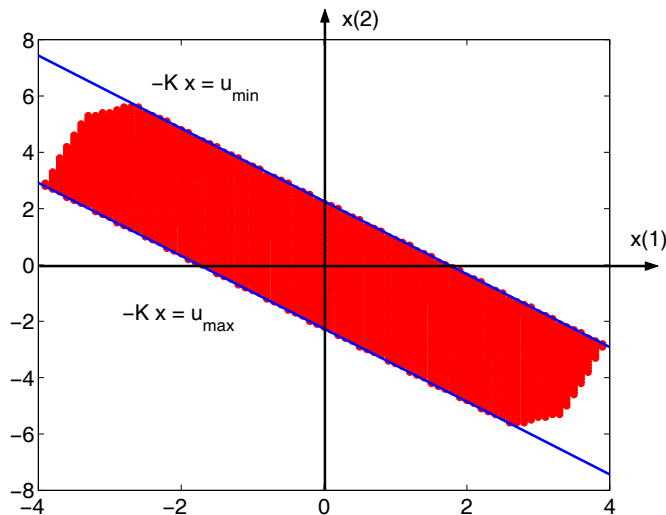


Figure 2. The set \mathcal{P} relative to the example in Section 4.

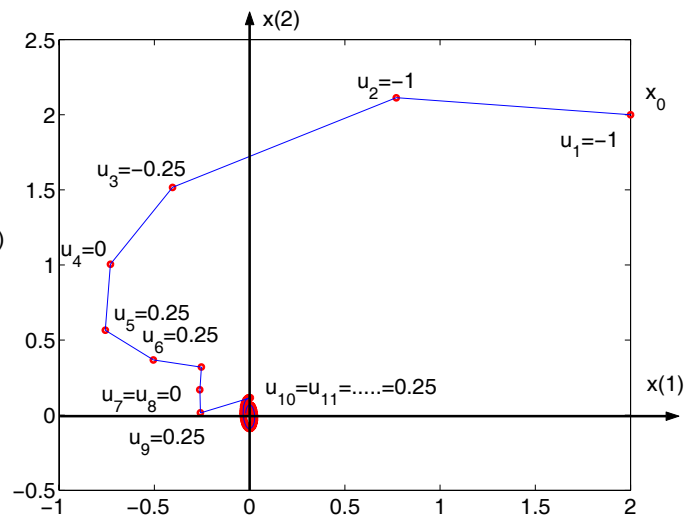


Figure 4. The state resulting state evolution when $x_0 = [2 \ 2]^T$.

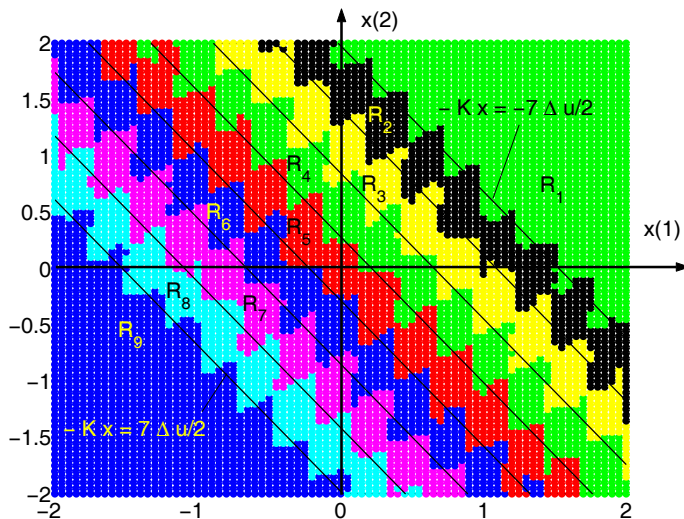


Figure 3. The structure of \mathcal{C} within the set \mathcal{P} .

that if the constraint $u \in \mathcal{U}$ is relaxed and we solve the QP problem with the constraint $u_1 \leq u \leq u_9$ the resulting value of the performance index is 20.61.

5 Conclusions and future work

In this paper we have studied the problem of DLQR quantized control. We proposed a solution that is based on our previous results in the context of optimal control of switched affine systems. The main advantage of the approach is that it provides a feedback solution, based on a partition of the state space \mathcal{C} that enables us to establish the optimal value of the control on the basis of the knowledge of the current state x .

The main restrictive assumption we have done here is that the coefficient matrix A of the controlled system is Hurwitz and the null input belongs to \mathcal{U} . What we plan to investigate in the future is how to extend the proposed approach to the case of systems with unstable matrix A .

Moreover, we plan to investigate how to stabilize the

system either to a point different from the origin, or how to keep it within an attractive set (that we may be either assumed known a priori or unknown).

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