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SOME TIME ANALYSIS METHODS FOR CONTINUOUS AND HYBRID PETRI NETS

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Abstract: Continuous and hybrid Petri nets can be seen as relaxation of discrete nets, in which the firing of some or of all transitions is approximated with a fluid model. Several analysis techniques have been presented for studying these models, using either linear programming and incidence matrix analysis, or graph theory approaches. In this paper two of such approaches, one based on linear algebra and one based on graph theory, are used to compute the steady-state firing speed and steady-state marking of continuous weighted marked graphs. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Petri nets (PNs) were firstly introduced (Murata, 1989) to describe and analyze discrete event systems. Recently, however, several attempts have been made to extend the discrete PN formalism to also encompass hybrid systems, i.e., systems presenting both time-driven and event-driven dynamics. The first steps in this direction were taken by Alla and David (1987) who introduced a continuous Petri net (CPN) model and later a hybrid Petri net (HPN) model (Le Bail et al., 1991). Since then, several HPN models have been presented by different researchers. A list of relevant references can be found on the web (Biblio on HPN) and in a special issue on HPN recently published (Di Febbraro et al., 2001).

The particular HPN model we consider in this paper can be defined as “HPN with continuous places and transitions” (Di Febbraro et al., 2000) and is based on the original model of Alla and David (1998). In this class of models, the hybrid net contains two types of places: discrete places, containing tokens as in a discrete PN, and continuous places, containing fluid, i.e., non negative real quantities of marks. Thus, the marking of discrete (continuous) places represents the discrete (continuous) part of the state. The time-driven dynamics are represented by continuous transitions. Assigning a firing speed to a continuous transition can be seen as the counterpart of assigning a time delay to the transitions of a standard discrete PN model. When all nodes are discrete, such a model reduces to a classical timed Petri net; when all nodes are continuous it reduces to a continuous net (Alla and David, 1998). HPN models inherit all the advantages of PN models such as the ability to capture concurrency, synchronization and conflicts.

The study of structural properties of untimed models such as liveness and boundedness through the concept of invariants is thus possible; see Recalde and Silva (2000) for a discussion on this. Several procedures have been proposed to analyze qualitative and quantitative properties of such a timed model: timed hybrid automata (Allam and Alla, 1998), graph theory (Mostefaoui et al., 2000), dioid algebra (see Komenda et al., 2001), linear methods (Balduzzi et al., 2001), and of course simulation.

This paper aims to compare two complementary approaches for the analysis of timed properties: (a) the use of linear programming; (b) the graph theory algorithms that apply to particular subclasses of PNs such as continuous weighted marked graphs. The linear programming approach for the analysis and control of HPNs that we describe in this paper is taken from Balduzzi et al. (2000). An original results of the present paper, presented in section 3, consists in applying this approach to the computation of the steady-state behavior (in terms of firing speeds) of continuous weighted marked graphs. In section 4, we present another approach, based on graph theory, to determine the steady-state (in terms of both firing speeds and marking) of neutral continuous weighted marked graph --- modeling flow systems (chemical process for example) or high throughput production systems (packaging lines for instance).

2. CONTINUOUS AND HYBRID PETRI NETS

2.1. Definitions

We define a hybrid Petri net (Alla and David, 1998), as a structure $HP = \langle P, T, I, O, h, t, M(0)\rangle$ where: $P = \{p_1, \cdots, p_m\}$ is a finite set of m places; $T = \{t_1, \cdots, t_n\}$ is a finite set of n transitions; $I : P \times T \to \aleph$ and $O : P \times T \to \aleph$ are the input and output incidence mappings; $h : P \cup T \to \{C, D\}$ defines the set of continuous nodes ($h(x) = C$) and discrete nodes ($h(x) = D$); $t : T \to \Re^+$: associates a delay d_i to a discrete transition (drawn as a black box), and a maximal speed V_i to a continuous transitions (drawn as an empty box); $M(0) = [m_1(0) \quad \cdots \quad m_m(0)]^T$ is the initial marking. Continuous places (drawn with double circles), contain non negative real values as markings, while discrete places contain non negative integer values as markings.

We denote $P_C = \{p_1, \cdots, p_{m'}\}$ the set of the m' continuous places and $T_C = \{t_1, \cdots, t_{n'}\}$ the set of the n' continuous transitions, while we let $P_D = P - P_C$ and $T_D = T - T_C$. The marking at time q will be denoted $M(q) = [m_1(q) \quad \cdots \quad m_m(q)]^T$, while M_C (resp., M_D) is the restriction of M to the continuous (resp., discrete) places. We denote ${}^\circ t_j$ (resp., t_j°) the set of input (resp., output) places of transition t_j and ${}^\circ p_i$ (resp., p_i°) the set of input (resp., output) transitions of place p_i. The *incidence matrix* of the net is $W = O - I$.

A discrete transition $t \in T_D$ is *enabled* at a marking M if for all $p_i \in {}^\circ t$: $m_i \geq I(p_i, t)$. An enabled transition may fire yielding the marking $M' = M + W(\cdot, t)$. Note that the firing of a discrete transition may change the marking of both discrete and continuous places.

To every continuous transition $t_j \in T_C$ is associated an *instantaneous firing speed* (IFS), $v_j(q)$. It represents the quantity of markings by time unit that fires the continuous transition. Whatever the evolution, the instantaneous firing speed of t_j is lower or equal to its maximal firing speed, i.e. $v_j(q) \leq V_j$. For a constant maximal speed assumption, the instantaneous firing speed has a piecewise constant behavior between events. As will be clear in the following, two types of events may change an IFS vector: external events, i.e., discrete transition firings, and internal events, i.e., the fact that a place becomes empty thus changing the enabling state of its output transitions.

An empty place can be *fed*, i.e., supplied, by an input transition, which is enabled. Thus, as a flow can pass through an unmarked continuous place, this place can deliver a flow to its output transitions. Consequently, a continuous transition $t_j \in T_C$ *is enabled* at time q if and only if all its input discrete places ($p_d \in P_D$) have a marking at least equal to the weight $m_d(q) \geq I(p_d, t_j)$ and all its input continuous places p_i satisfy the following condition:

Either $m_i(q) > 0$ *or* p_i *is fed.*

If all input continuous places of t_j have a not null marking, t_j is called *strongly enabled* else t_j is called *weakly enabled*. Finally, transition t_j is *not enabled* if one of its empty input places is not fed.

The enabling state of a continuous transition $t_j \in T_C$ defines its *admissible instantaneous firing speed:*

- If t_j is not enabled then $v_j = 0$.
- If t_j is strongly enabled, then it may fire with any firing speed $v_j \in [0, V_j]$.
- If t_j is weakly enabled, then it may fire with any firing speed $v_j \in [0, \tilde{V}_j]$; in fact, it cannot remove more fluid from any empty input continuous place p than the quantity fed into p.

There are two ways of computing an IFS vector.

- If we consider an autonomous mode of operation, the IFS vector is "chosen" by the net: in this case it is common to consider a *maximal firing speed policy* --- it is the continuous counterpart of the "earliest firing policy" for discrete nets --- in which the firing of each transition is as high as possible. Note, however, that whenever there exists a conflict (an empty place whose inputting flow must be assigned to several output transitions) a conflict resolution policy, such as priority rules, must be specified.
- On the other hand, we may consider a mode of operation in which an IFS vector is chosen by the plant operator, i.e., it is a continuous control input to the plant. This more general approach is discussed in section 3.

In particular, if we disregard the discrete evolution (i.e., we consider purely continuous nets or we assume the discrete marking does not change), regardless of how the IFS vector has been computed, if we denote $v_j(q)$ the IFS at time q of a continuous transition t_j, the *continuous evolution* of a CPN is such that the marking $m_i(q)$ of a continuous place $p_i \in P_C$ varies according to:

$$\dot{m}_i(q) = \frac{\mathrm{d}\, m_i(q)}{\mathrm{d}q} = \sum_{j=1}^{n'} W(p_i, t_j) \cdot v_j(q) \qquad (2.1)$$

Whenever q is not necessary, it will be omitted.

2.2. Special structure of marked graphs

A *marked graph* (MG), also called event graph, is a PN where each place has exactly one input and one output transition and the weight associated to each arc is equal to 1. A marked graph in which a non negative real weight different from one can be associated to an arc is called a *weighted marked graph* (WMG). We use the acronym CWMG (HWMG) to denote a continuous (hybrid) WMG.

Let us now introduce a definition following (Balduzzi et al., 2000).

Definition 2.1. *A HPN is* continuous-conflict-free *(CCF) at a marking M if each empty place has at most one enabled output transition.* ÿ

At the light of this definition, we observe that marked graphs are structurally conflict free structures, in the sense that the fluid entering a place can only be removed by the firing of its unique output transition.

Finally, when there exists an oriented directed path, which connects any node (place or transition), to any other node of the graph, the PN is said to be strongly connected.

Based on these assumptions on the structure of the continuous Petri net model, section 4 will present the main results in terms of steady-state speed vector and marking.

3. ANALYSIS VIA LINEAR PROGRAMMING

In this section we discuss the use of linear programming as a tool for the analysis and control of HPN, following Balduzzi et al. (2000).

First of all, we slightly generalize the framework presented in section 2, by assuming that to each continuous transition t_j is assigned not only a maximal firing speed V_j but a minimal firing speed V'_j as well. As an example, considering a fluid analogy in which transitions can be seen as valves allowing the passage of fluids, a minimal firing speed greater than zero should be assigned to a transition that models a valve that cannot be completely turned off.

Secondly, in this section we do not assume that the continuous evolution of the HPN is autonomous or given by a pre-assigned evolution law. On the contrary, we take an IFS vector $\vec{v}$ to be a continuous control input that is applied to the plant, and the choice of a suitable (or optimal) IFS is the control problem we want to solve.

3.1. Admissible IFS vectors

Definition 3.1. *Given an HPN, let M be its present marking,* $T_e \subseteq T_C$ *be the set of continuous transitions enabled at M, and* $P_e = \{p_i \in P_C \mid m_i = 0\}$ *be the set of empty continuous places. Any admissible IFS vector* $\vec{v} = [v_1 \ \cdots \ v_{n'}]^T$ *is a feasible solution of the following linear set denoted* $S(M)$ *:*

$$\begin{cases} V_j - v_j \geq 0 & \forall t_j \in T_e & (a) \\ v_j - V'_j \geq 0 & \forall t_j \in T_e & (a') \\ v_j = 0 & \forall t_j \notin T_e & (b) \\ \sum_{j=1}^{n'} W(p,t_j) \cdot v_j \geq 0 & \forall p \in P_e & (c) \end{cases} \quad (3.1)$$

ÿ

The meaning of these equations is rather simple. Equations *(a)* and *(a')* represent the constraint imposed on $\vec{v}$ by the maximal and minimal firing speeds and hold for all enabled transitions. Equations *(b)* specify that the firing speed of non enabled transitions must be zero (note that the constraint imposed by the maximal and minimal firing speed only apply to enabled transitions, i.e., a disabled transition may have a null IFS even if its minimal firing speed is greater than zero). Finally, equations *(c)* specify that the net flow entering an empty continuous place must be greater of equal to zero (negative markings are not allowed).

Note that the constraint set (3.1) may have no admissible solutions. However, the following elementary result holds.

Definition 3.2. *A HPN is said to be* mfs-free *at a marking M if the constraint set (3.1) contains no equation of the type (a').* ÿ

Proposition 3.3. *If a HPN is mfs-free, the set of admissible solutions of (3.1) is non-empty.*

Proof: In this case $\vec{v} = \vec{0}$ is obviously an admissible solution. ÿ

It is important to observe that the constraint set $S(M)$ given by (3.1) may change as the marking of both discrete and continuous places changes. In fact, the discrete marking specifies the set of enabled and disabled continuous transitions, while the continuous marking specifies the set of empty continuous places.

In particular, if we disregard the discrete evolution (i.e., we consider purely continuous nets or we assume the discrete marking does not change) the constraint set $S(M)$ changes whenever a place becomes empty. Thus, we can give the following definition.

Definition 3.4. *Given a HPN with set of enabled continuous transitions* $T_e \subseteq T_C$*, we say that an admissible IFS vector* $\vec{v} = [v_1 \ \cdots \ v_{n'}]^T$ *is* a steady-state IFS *if the marking* M_C *is non-decreasing, i.e., for all continuous places* $p \in P_C$ *:*

$$\sum_{j=1}^{n'} W(p,t_j) \cdot v_j \geq 0$$

In particular, we say that a steady-state IFS is stationary *if the marking* M_C *is constant, i.e., for all continuous places* $p \in P_C$ *:*

$$\sum_{j=1}^{n'} W(p,t_j) \cdot v_j = 0$$

Clearly, stationary IFS vectors correspond to dynamics where the both the continuous marking and the constraint set $S(M)$ remain constant, while more generally steady-state IFS vectors correspond to dynamics where the continuous marking may increase but, because no continuous place may become empty, the constraint set $S(M)$ remains constant. We can state the following result.

Proposition 3.5. *Given a HPN with set of enabled continuous transitions* $T_e \subseteq T_C$*, an IFS vector* $\vec{v}$ *is a steady-state IFS vector if and only if it satisfies (3.1).c with* $P_e = P_C$*.*
Furthermore a steady-state vector $\vec{v}$ *is stationary if and only if equations (3.1).c hold with equality for all* $p \in P_C$*.*

3.2. Computation of an optimal IFS vector

We now consider the following *control problem*: "given a set $S(M)$ choose one among all admissible firing vectors". To this end, we introduce an objective function $f(\vec{v})$ and solve the following *optimization problem*:

$$\begin{array}{ll} \max & f(\vec{v}) \\ s.t. & \vec{v} \in S(M) \end{array}$$

The solution to this optimization problem is the desired solution of the control problem.

Let us briefly discuss different cases of practical interest in which this framework may be useful.

1. Maximize the overall flow, i.e., the total sum of the IFSs. This can be done choosing a function $f(\vec{v}) = \sum_{j=1}^{n'} v_j$.
2. Maximize the throughput, i.e., the sum of the IFSs of those transitions that can be considered as "outputs" of the plant. This can be done choosing a function $f(\vec{v}) = \sum_{j \in \Theta} v_j$, where Θ is the set of indices of all "output transitions".
3. Minimize the fluid stored in a subset of place P', i.e., minimize the derivative of the marking of all places $p \in P'$. This can be done choosing a function $f(\vec{v}) = -\sum_{p \in P'} \sum_{j=1}^{n'} W(p,t) \cdot v_j$, i.e., we maximize the sum over places in P' of the flow outputting (because of the minus sign) the places.

It is important to remark that the use of linear algebra makes it possible to use sensitivity analysis for evaluating how changes in the plant parameters may affect the solutions of these optimization problems (see Balduzzi et al., 2000).

3.3. Computation of a steady state IFS

The previously described formalism can also be applied to the computation of a steady-state IFS and steady-state marking for continuous Petri nets. In the general case, a CPN may admit more that one steady-state IFS vector; furthermore both steady-state IFS vector and steady-state marking may depend on the evolution of the net starting from the initial marking. For sake of simplicity, we consider, as in the next section, the case of CWMG, although the approach can be applied to more general classes of nets. In the case of CCF nets, the IFS of each transition may be maximized independently of all other ones (Balduzzi et al., 2000), thus the *maximal firing speed policy,* in which the firing of each transition is as high as possible, corresponds to choosing an objective function $f(\vec{v}) = \sum_{j=1}^{n'} v_j$.

Assuming a maximal firing speed policy is used, a steady-state IFS is the solution of:

$$\begin{cases} \max & \sum_{j=1}^{n'} v_j & & \\ s.t. & V_j - v_j \geq 0 & \forall t_j \in T_e & (a) \\ & v_j - V'_j \geq 0 & \forall t_j \in T_e & (a') \\ & v_j = 0 & \forall t_j \notin T_e & (b) \\ & \sum_{j=1}^{n'} W(p,t_j)\cdot v_j \geq 0 & \forall p \in P_C & (c) \end{cases} \qquad (3.2)$$

where the set of enabled transitions depends on the initial marking.

Proposition 3.6. *For CCF nets with no minimal firing speeds, system (3.2) has a solution and this solution is unique.*

Proof: If there are no minimal firing speeds, the constraint set of equations (3.2) is mfs-free and it admits at least a solution (see Proposition 3.3). Furthermore, for CCF nets (Balduzzi et al., 2000; theorem 13) showed that the solution of (3.2) is $\vec{v}^* = \begin{bmatrix} v_1^* & \cdots & v_{n'}^* \end{bmatrix}^T$ where each v_j^* is the optimal solution of

$$\begin{array}{ll} \max & v_j \\ s.t. & \vec{v} \in \hat{S} \end{array}$$

where $\hat{S}$ is the set of solutions of the constraint set of (3.2), and thus this solution is clearly unique. ÿ

Note, however, that this solution may be the null vector.

To compute the steady-state marking it is necessary to study the continuous evolution of the system, applying (3.1) from the initial marking, and adding new constraints as soon as a place becomes empty and a simple algorithm to do this is given in (Balduzzi et al., 2001). Although this problem will not be discussed in the paper, some comments about this can be found in the next subsection where two examples are discussed.

3.4 Examples

Consider the continuous WMG in Figure 3.1.a where $V'_1 = V'_2 = 0$. For this examples (3.2) rewrites as:

$$\begin{cases} \max & v_1 + v_2 & \\ s.t. & v_1 \leq 1; & v_2 \leq 3; \\ & -2v_1 + v_2 \geq 0; & v_1 - v_2 \geq 0; \end{cases}$$

whose unique solution is $\vec{v}^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. In this case, in fact, the circuit is absorbing and in steady state all transitions are dead (regardless of the initial marking). Note that the steady-marking in this example is $M(\infty) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and does not depend on the initial marking $M(0) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

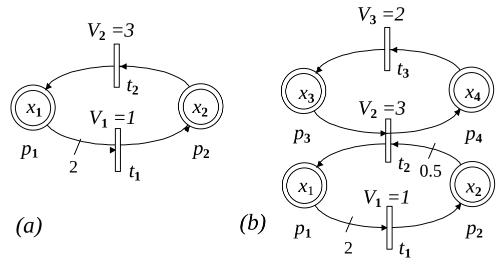

Figure 3.1. Two CWMG.

Consider the continuous WMG in Figure 3.1.b where $V'_1 = V'_2 = V'_3 = 0$. For this example, (3.2) rewrites as:

$$\begin{cases} \max & v_1 + v_2 + v_3 & \\ s.t. & v_1 \leq 1; & v_2 \leq 3; \\ & v_3 \leq 2; & -2v_1 + v_2 \geq 0; \\ & v_1 - 0.5\,v_2 \geq 0; & -v_2 + v_3 \geq 0; \\ & v_2 - v_3 \geq 0; & \end{cases}$$

whose unique solution is $\vec{v}^* = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T$. Note that the steady-marking in this case depends on the initial marking.

4. ANALYSIS OF CONTINUOUS WEIGHTED MARKED GRAPH VIA GRAPH THEORY

4.1. Notations

Given a CWMG N with *m* places and *n* transitions, we denote: $r_{ji} = \langle t_j, p_1, \cdots, p_r, t_i \rangle$: a path from transition t_j to transition t_i. For place p_i we denote (see Figure 4.1) $w_i = O(p_i, t_{w(i)})$ the weight of its unique input arc and $u_i = I(p_i, t_{u(i)})$ the weight of its unique output arc.

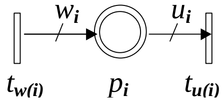

Figure 4.1. Notations on CWMG

According to the classification given in (Teruel et al., 1992), a circuit $g = \langle t_1, p_1, \cdots, p_n \rangle$, is either *neutral,* $G(g) = 1$, or *absorbing,* $G(g) < 1$, or finally, *generating,* $G(g) > 1$. Its gain is:

$$G(g) = \prod_{p_i \in g} \frac{w_i}{u_i}.$$

Under the assumption that the CWMG is strongly connected, it can be decomposed into elementary circuits. Following (Mostefaoui et al., 2000), it is possible to classify the graph according to the previous cases on circuits.

Definition 4.1. *A strongly connected CWMG, N is:*

- neutral *if and only if all its circuits are neutral,*
- absorbing *if and only if there exists an absorbing circuit,*
- generating *if and only if there exists at least one generating circuit and none absorbing circuit.* ÿ

Clearly, it has been established that a strongly connected CWMG is *live* if and only if all its elementary circuits are generating or neutral and there exists at least one place p_i in each circuit with a not null marking ($m_i(0) \geq 0$). In other terms, an absorbing graph reaches a deadlock situation. Thus, at the steady state, the IFS vector is the null vector and all places are empty. Consequently, in the rest of this paper, we will focus on neutral cases, whose are live. Some results on generating circuits can be found in Mostefaoui et al. (2000).

Now, if we assume that the continuous evolution of the CWMG is autonomous, i.e., we consider a maximal firing speed policy, in which the firing of each continuous transition is as high as possible, for these graphs, the fundamental equations (2.1), can be simplified as follows.

$$\dot{m}_i(q) = w_i.v_{w(i)}(q) - u_i.v_{u(i)}(q) \ .$$

The IFS of a weakly enabled transition t_j is:

$$v_j(q) = \min\left(V_j, \min_{p_i \in P_j(q)} \left(\frac{w_i}{u_i}.v_{w(i)}(q) \right)\right)$$

where $P_j(q) = \left\{ p_i \in (^\circ t_j) \middle| m_i(q) = 0 \right\}$ denotes the set of the input continuous places of t_j that are empty at time q .

From these definitions, the following property has been established for the neutral case.

Property 4.2. *In a neutral CWMG (supposed live), the IFS function is non-increasing in time, i.e.:*

$\forall t_j \in T_C, v_j(q_2) \leq v_j(q_1), \quad if \, q_2 > q_1$.

4.2. Steady-state IFS vector of a neutral CWMG

As previously defined in section 3, steady-state IFS vectors correspond to dynamics where the continuous marking may increase while the IFS vector remains constant. We present in this section, a method based on graph theory to determine the exact value of the firing speed vector at the steady state for CWMG (see (Mostefaoui et al, 2001) for proofs).

In the case of neutral CWMG, this vector is obtained independently of the value of its initial marking and without establishing the evolution graph.

Theorem 4.3. *At the stationary state,* the final IFS vector $\vec{v}^*$ *of a strongly connected continuous neutral weighted marked graph (supposed live) is given by:*

$$v_j^* = V_j \quad and \quad v_k^* = V_j \frac{z_k}{z_j}, \quad \forall k \neq j$$

where the bottleneck transition t_j verifies

$$\frac{V_j}{z_j} = \min_{t_k \in T_C} \left(\frac{V_k}{z_k} \right),$$

and z_k is the k^{th} component of the T-semiflow $\vec{z}$, solution of $W \cdot \vec{z} = \vec{0}$, where W is the incidence matrix of the net. ÿ

It is important to observe that the final speed vector, $\vec{v}^*$ is independent of the initial marking, i.e., at the steady state, the IFS of a neutral CWMG (supposed live) is unique whatever the initial marking and can be reached from any initial marking such that the CWMG is live.

4.3. Steady-state marking of a neutral CWMG

In the previous subsection we saw that the final IFS vector can directly be computed from the structure of the graph, thus determining which transition is strongly, weakly or not enabled. In neutral CWMG, however, the marking is also constant at the steady state and we show now how it can computed.

Let *N* be a neutral CWMG with *m* places and *n* transitions. In the following, these notations are used.

- T_B^* *:* the set of bottlenecks transitions of *N* at the final state, i.e. $T_B^* = \{ t_i \in T_C / v_i^* = V_i \}$
- $T_{\bar{B}}^{**}$ *:* the set of transitions of *N* which are not bottleneck, and which have more than one input place, i.e. $T_{\bar{B}}^{**} = \{ t_i \in T_C / |^\circ(t_i)| > 1 \; and \; t_i \notin T_B^* \}$
- Γ_{ji} *:* the set of elementary paths r_{ji} from transition t_j to transition t_i .
- Γ_i *:* the set of elementary paths r_{ji} from a transition of T_B^* to the given transition t_i , which do not contain another transition of T_B^* .
- For $t_i \in T_B^*$ and $p_k \in {}^\circ(t_i)$, $\Gamma_i'(p_k)$ is the set of paths $r_{ji} \in \Gamma_i$ which contains place p_k .

Definition 4.4. *For path r_{ji} , we define its weighted marking $N(r_{ji}, q)$ as the maximal number of marks that can arrive in the last place if t_j and t_i are not fired after time q . It is given by:*

$$N(r_{ji}, q) = u_r . \sum_{p_k \in r_{ji}} \frac{m(p_k, q)}{u_k} \prod_{p_p \in r_{si} / \left\{ \begin{array}{l} t_s = (p_k)^\circ \\ r_{si} \subset r_{ji} \end{array} \right.} \frac{w_p}{u_p}$$

where $p_r = r_{ji} \cap {}^\circ(t_j)$.

Example. For $\rho_{j3} = \langle t_j, p_1, t_1, p_2, t_2, p_3, t_3 \rangle$, the weighted marking is:

$$N(r_{ji}, q) = u_3 . \left(\frac{m_1}{u_1} . \frac{w_2}{u_2} . \frac{w_3}{u_3} + \frac{m_2}{u_2} . \frac{w_3}{u_3} + \frac{m_3}{u_3} \right)$$

Theorem 4.5. *In a neutral CWMG (supposed live), the final marking m_r^* of place p_r at the stationary state is given by:*

- *If* $t_i \in T_B^*$, $\forall p_r \in {}^\circ(t_i)$ *:*

$$m_r^* = \min_{r_{ji} \in \Gamma_i} (N(r_{ji}, q_0)) = \min_{r_{ji} \in \Gamma_i} (u_r . \sum_{p_k \in r_{ji}} \frac{m_k^0}{u_k} \prod_{p_p \in r_{si} / \left\{ \begin{array}{l} t_s = (p_k)^\circ \\ r_{si} \subset r_{ji} \end{array} \right.} \frac{w_p}{u_p})$$

- *If* $t_i \in T_{\bar{B}}^{**}$, $\forall p_r \in {}^\circ(t_i)$ *:*

$$m_r^* = \min_{r_{ji} \in \Gamma_i'(p_r)} (N(r_{ji}, q_0)) - u_r . \min_{\substack{r_{ji} \in \Gamma_i \\ p_s = r_{ji} \cap {}^\circ(t_i)}} (\frac{1}{u_s} . N(r_{ji}, q_0))$$

- *If* $t_i \notin (T_B^* \cup T_{\bar{B}}^{**})$, $\forall\, p_r \in {}^\circ(t_i)$: $m_r^* = 0$

Proof: see (Mostefaoui et al., 2001)

4.4 Example

Let us consider the previous example of a fluid system (see Figure 3.1b). This continuous PN is a neutral CWMG composed by two neutral elementary circuits: $g_1 = \langle t_1, p_2, t_2, p_1 \rangle$ and $g_2 = \langle t_3, p_3, t_2, p_4 \rangle$. The vector $\vec{z} = [1 \quad 2 \quad 2]^T$ is a T-semiflow. Applying theorem 4.3, transitions t_1 and t_3 are identified as bottleneck transitions, and the final firing speed vector is: $\vec{v}^* = [1 \quad 2 \quad 2]^T$.

We deduce: $T_B^* = \{t_1, t_3\}$, $T_{\bar{B}}^{**} = \{t_2\}$,

$\Gamma_1 = \{r_{11}, r_{31}\}$, $r_{11} = g_1$, $r_{31} = \langle t_3, p_3, t_2, p_1, t_1 \rangle$,

$\Gamma_2 = \{r_{12}, r_{32}\}$, $r_{12} = \langle t_1, p_2, t_2 \rangle$, $r_{32} = \langle t_3, p_3, t_2 \rangle$,

$\Gamma_3 = \{r_{13}, r_{33}\}$, $r_{13} = \langle t_1, p_2, t_2, p_4, t_3 \rangle$, $r_{33} = g_3$

By theorem 4.5:

- For place p_1:

$$m^*_1 = \min_{\rho_{j1} \in \Gamma_1} \left(N(\rho_{j1}, \theta_0) \right) = \min(N(\rho_{11}, \theta_0), N(\rho_{31}, \theta_0))$$

$$N(r_{31}, q_0) = u_1 . \left(\frac{m_1(q_0)}{u_1} + \frac{m_3(q_0)}{u_3} . \frac{w_1}{u_1} \right) = x_1 + x_3,$$

$$N(\rho_{11}, \theta_0) = u_1 . \left(\frac{m_1(\theta_0)}{u_1} + \frac{m_2(\theta_0)}{u_2} . \frac{w_1}{u_1} \right) = 2.x_2 + x_1,$$

$m^*_1 = \min(2.x_2 + x_1, x_1 + x_3)$.

- For place p_2: $\Gamma_2'(p_2) = \{r_{12}\}$,

$$m_2^* = N(r_{12}, q_0) - u_2 . \min\left(\frac{N(r_{12}, q_0)}{u_2}, \frac{N(r_{32}, q_0)}{u_3} \right),$$

$$N(r_{12}, q_0) = u_2 . \left(\frac{m_2(q_0)}{u_2} \right) = x_2,$$

$$N(r_{32}, q_0) = u_3 . \left(\frac{m_3(q_0)}{u_3} \right) = x_3$$

$m_2^* = x_2 - 0.5 \min(2.x_2, x_3)$.

- For place p_3: $\Gamma_2'(p_3) = \{r_{32}\}$,

$$m_3^* = N(\rho_{32}, \theta_0) - u_3 . \min\left(\frac{N(\rho_{12}, \theta_0)}{u_2}, \frac{N(\rho_{32}, \theta_0)}{u_3} \right),$$

$m_3^* = x_3 - \min(2.x_2, x_3)$

- For place p_4:

$$m^*_4 = \min_{\rho_{j3} \in \Gamma_3} \left(N(\rho_{j3}, \theta_0) \right) = \min(N(\rho_{13}, \theta_0), N(\rho_{33}, \theta_0))$$

$$N(\rho_{13}, \theta_0) = u_4 . \left(\frac{m_4(\theta_0)}{u_4} + \frac{m_2(\theta_0)}{u_2} . \frac{w_4}{u_4} \right) = x_4 + 2.x_2,$$

$$N(\rho_{33}, \theta_0) = u_4 . \left(\frac{m_4(\theta_0)}{u_4} + \frac{m_3(\theta_0)}{u_3} . \frac{w_4}{u_4} \right) = x_4 + x_3,$$

$m^*_4 = \min(x_3 + x_4, 2.x_2 + x_4)$.

Finally, if $2x_2 \geq x_3$ then $M(\infty) = [x_1 + x_3 \quad x_2 - 0.5\, x_3 \quad 0 \quad x_4 + x_3]^T$, else $M(\infty) = [x_1 + 2x_2 \quad 0 \quad x_3 - 2x_2 \quad x_4 + 2x_2]^T$.

5. CONCLUSIONS

The determination of the steady state, in terms of final marking and firing speed vectors is one of the main properties on Petri nets which characterises the timed dynamic behavior of hybrid systems. This paper gives two efficient methods based on linear programming and graph theory to compute the steady state. The linear programming approach is more general (can be applied not only to neutral weighted marked graphs but to a larger class of nets) but does not lead to an immediate computation of the steady-state marking. The graph theory approach leads to a closed form solution and allows one to also compute the steady-state marking but can only be presently applied to restricted structure of neutral weighted marked graphs.

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