ROBUST H_∞ CONTROL OF UNCERTAIN DISCRETE-TIME SWITCHING SYMMETRIC COMPOSITE SYSTEMS

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Abstract: Reduced–order design of decentralized H_∞ static output switching controllers is presented for uncertain discrete–time switching symmetric composite systems with state-dependent switching rule. First, a reduced-order control design model is constructed with the dimension equal to a subsystem's dimension of the original system. Then, quadratically stabilizing switching controller with a given H_∞ disturbance attenuation level is designed using linear matrix inequalities (LMI) for the design model. It is proved that when this controller together with the corresponding switching rule are implemented into each subsystem of the original system then such decentralized controller quadratically stabilizes the overall closed–loop system with H_∞ norm bound γ. The switching is decentralized into independent switching rules operating only on local subsystems states.

Keywords: Large–scale systems, switching, decentralized control, discrete–time systems, uncertainty, H-infinity, reduced–order models.

1. INTRODUCTION

Multi–controller switched schemes provide an effective and powerful mechanism to cope with highly complex systems and/or systems with large uncertainties. There are real world systems which are not stabilizable by means of any individual continuous state feedback controller. For such systems, multi–controller switching among smooth controllers provides a good conceptual framework to solve the problem. As a special but very important class of switched systems, switched linear systems provide an attractive framework which bridges the gap between linear systems and the highly complex and/or uncertain systems. Switched linear systems are relatively easy to handle as many powerful tools from linear analysis are applicable to cope with these systems. Such systems are enough accurate to represent many practical engineering systems with complex dynamics. The study of switched linear systems provides additional insights to some long–standing problems, such as robust, adaptive, and intelligent control, gain scheduling, or multi–rate digital control. The recent results in switched systems has benefited many real world systems such as power systems, automotive control, air traffic control, network and congestion control.

The importance of multi–controller switched schemes is underlined in large scale complex systems when implementing low–cost low–order local controllers. It motivates the development of new control design methods which include the solution using multi–controller switched schemes mainly for large scale complex systems. Symmetric composite systems represent an important class of these systems.

1.1 *Prior work*

There are available many various results on switched systems (Leonessa *et al.* 2000), (Liberzon 2003),

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(Sun and Ge 2005), (Cheng 2004), (Pettersson 2003), (Varaiya 1993), (Wong and Brockett 1982), but there are only few results on uncertain switched systems. (Ji and Wang 2005) and (Savkin and Evans 2002) present results on norm bounded uncertainties. One of important problems in uncertain switched systems is the design of switching rules which guarantee quadratic stability and performance. Such switching rules must be independent of uncertainties. A state-dependent switching rule satisfying this requirement which is called the min-projection strategy presents (Ji and Wang 2005) and (Pettersson 2003). (Ji and Wang 2005) deals with robust H_∞ control for switched state feedback and static output switched feedback. All these references deal with a centralized switching rule.

Motivation for studying symmetric composite systems arises in very different application areas. Real world system examples can be found in parallel systems such as flow splitting parallel reactors with combined precooling (Hovd and Skogestad 1994), electric power systems operating in parallel (Bakule and Lunze 1988), (Lunze 1992), industrial manipulators with several degrees of freedom (Vukobratovic and Stokic 1992), flexible structures (Trächtler 1991), space crystal furnace (Ebert 1999), homogeneous interconnected systems such as seismic cables (El-Sayed and Krishnaprasad 1981). More exhaustive survey of other applications present (Hovd and Skogestad 1994), (Yang and Zhang 1996), and (Bakule 2005). Low–order control design for delay–less uncertain parameter symmetric composite system state space models consider (Hovd and Skogestad 1994), (Bakule and Rodellar 1996), and (Wang and Zhang 2000).

One of new open research directions is the inclusion of switching issues into the formulation and solution of the output switching controllers for a class of uncertain switching symmetric composite systems. The paper extends the results on the low–order control design for symmetric composite systems in (Bakule and Rodellar 1996), (Yang and Zhang 1995), (Bakule 2003). While (Bakule and Rodellar 1996), (Yang and Zhang 1995), and (Bakule 2003) include models with rank-one uncertainties, (Bakule 2005) considers norm bounded uncertainties. It essentially simplifies the solution.

To the author's best knowledge, the problem of low–order switching static output controller design with H_∞ norm bound γ including decentralized state–dependent switching rule for uncertain switching symmetric composite systems has not been solved up to now.

1.2 *Outline of the paper*

This paper presents complexity-reduced procedure for decentralized switching output controller design guaranteeing the level of disturbance attenuation γ together with decentralized state–dependent switching rule for uncertain switching discrete–time symmetric composite systems. The original system is reduced to a low–order control design problem preserving structural properties of the overall problem. Its order equals to a subsystem's order of the given global system. The switching static output controller is designed for this reduced system using the well known LMI for the selected switching rule. It is proved that when such switching controller is implemented into each subsystem together with the local switching rule, then the resulting decentralized switching controller quadratically stabilizes the overall system with a given bound of disturbance attenuation.

2. PROBLEM FORMULATION

Notation. In this paper $\|w(t)\|_2 = \sqrt{\int_0^\infty w^T(t)w(t)\,dt}$ denotes the 2-norm for $w : [0,\infty] \mapsto \mathbb{R}^p$ belonging to the space $L_2^p[0,\infty]$, provided that $\|w(t)\|_2 < \infty$.

Consider an uncertain switching linear symmetric composite system consisting of N subsystems, where the ith structural subsystem is described as follows

$$\begin{aligned} x_i(t+1) &= (A_r + \Delta A_i(t))x_i(t) + B_r u_i(t) + B_w w_i(t) \\ &\quad + s_{zi}(t) \\ v_i(t) &= C_v x_i(t) + D_v u_i(t) \\ y_i(t) &= C x_i(t) \qquad\qquad i = 1,\ldots,N \end{aligned} \tag{1}$$

where x_i, u_i, w_i, s_{zi}, v_i, and y_i are n-, m-, m_w-, p_s-, p_v-, and p_y-dimensional vectors of the subsystem states, control inputs, exogenous inputs, interconnection inputs, controlled and measured outputs, respectively. The continuous function $r = r(x_i,t) : \Re^n \times \Re^+ \rightarrow \{1,\ldots,\kappa\}$ is the switching rule for the ith subsystem to be designed for all i. $r(x_i,t) = k$ means that the kth switching subsystem is activated for the ith structural subsystem. It is evident that there are N identical switching rules where each subsystem has assigned one local state–dependent switching rule operating independently from other rules. Interconnections are described in the form

$$s_{zi} = \sum_{j=1}^{N} L_{ij} y_{zj} \tag{2}$$

where y_{zj} is the p_z-dimensional vector of the interconnection output from the subsystem j which is related to the state vector in the form

$$y_{zj} = C_z x_j \tag{3}$$

The interconnection matrices L_{ij} have the following structure

$$L_{ii} = 0 \qquad L_{ij} = L_q + \Delta L_{qij}(t) \qquad (i \neq j) \tag{4}$$

$A_r, B_r, B_w, C, C_v, D_v$, and L_q, are constant nominal matrices. $\Delta A_i(t)$ and $\Delta L_{qij}(t)$ are norm bounded uncertainties which admit the following structure

$$\begin{aligned} \Delta A_i(t) &= D_A F_{Ai}(t) E_A \\ \Delta L_{qij}(t) &= D_L F_{Lij}(t) E_L \end{aligned} \tag{5}$$

$D_A, ..., E_L$ are constant matrices. Uncertainties are lumped in unknown Lebesgue measurable functions $F_{(*)}$ satisfying the bounds $F_{(*)}^T F_{(*)} \leq I$ for all $t \geq 0$. I denotes a unit matrix of appropriate dimensions.

Introduce the following inequality which is related to the system (1)–(5)

$$\|v(t)\|_2 \leq \gamma \|w(t)\|_2 \qquad (6)$$

where $v(t) = (v_1^T(t), ..., v_N^T(t))^T$ and analogously $w(t) = (w_1^T(t), ..., w_N^T(t))^T$.

Denote for (1)–(5) $x(t) = (x_1^T(t), ..., x_N^T(t))^T$, $u(t) = (u_1^T(t), ..., u_N^T(t))^T$, and $\bar{r}(x,t) = (r(x_1,t), ..., r(x_N,t))$. It means that the switching in the global system is realized at N different and independent places.

Definition 1. Consider the system (1)–(5). This system is *quadratically stabilizable via switched output feedback* if there exist a switching rule $\bar{r}(x,t)$ and an associated static output feedback $u_i = K_{r(x_i,t)} y_i$ with $K_k, k \in \Lambda, \Lambda = \{1, ..., \kappa\}$ for all i independently on all admissible uncertainties and such that the resulting closed–loop system with this feedback and $w(t) = 0$ is quadratically stable.

Definition 2. Consider the system (1)–(5). This system is *quadratically stabilizable with H_∞ disturbance attenuation via switched output feedback* if it is quadratically stabilizable via switched output feedback and under zero initial conditions the relation (6) holds for any non-zero $w(t)$ and for all admissible uncertainties.

The goal is to find a global decentralized static output feedback switching controller and a decentralized switching rule quadratically stabilizing the system (1)–(5) for any admissible uncertainties as follows

$$\begin{aligned} u_i &= K_{r(x_i,t)} y_i \\ r(x_i,t) &= \arg\min_{k\in\Lambda}(x_i^T \Omega_k x_i) \qquad i = 1, \dots, N \end{aligned} \qquad (7)$$

where x_i is the n-dimensional controller state of the subsystem i. K_k is the constant matrix of the feedback controller acting at the kth switching mode with $k \in \Lambda$. The switching rule is selected as a quadratic form, where Ω_k is a constant matrix corresponding to the kth switching mode to be determined for each k. The set of these matrices is supposed independent of i. Notice that the set Λ of gain matrices to be determined is identical for all subsystems, thus taking advantage of the symmetric structure of the large scale composite system to reduce the control design complexity.

2.1 *The Problem*

Given a system (1)–(5) and a positive number γ. The goal is to derive a complexity-reduced procedure for designing a static output feedback decentralized switching memory–less controller and a decentralized switching rule (7) for the system (1)–(5) such that the closed–loop system (1)–(5), (7) is quadratically stable with H_∞ disturbance attenuation bound γ satisfying (6) for all admissible uncertainties.

3. MAIN RESULTS

This section presents a constructive procedure for the design of matrices K_k and Ω_k for all $k \in \Lambda$.

First, a low–order control design problem is constructed.

$$\Delta A_a(t) = D_a F_a(t) E_a \qquad (8)$$

where $D_a, ..., E_a$ are constant block matrices given by decomposing the matrix $\frac{N}{2} L_q C_z$ into the form

$$\frac{N}{2} L_q C_z = D_a E_a \qquad (9)$$

$F_a(t)$ in (8) is unknown Lebesgue measurable function satisfying $F_a(t)^T F_a(t) \leq I$ for all $t \geq 0$.

Define the n-dimensional system using the uncertainties (8) as follows

$$\begin{aligned} x_m(t+1) &= (A_{mr} + \Delta A_m(t)) x_m + B u_m(t) + B_w w_m(t) \\ v_m(t) &= C_v x_m(t) + D_v u_m(t) \\ y_m &= C x_m \end{aligned} \qquad (10)$$

where the nominal matrices are defined by the expressions

$$A_{mr} = A_r + (\frac{N}{2} - 1) L_q C_z \qquad (11)$$

with the uncertainties given as follows

$$\begin{aligned} \Delta A_m(t) &= D_A F_A(t) E_A + (\frac{N}{2} - 1) D_L F_L(t) E_L \\ &+ D_a F_a(t) E_a = D_{Am} F_{Am}(t) E_{Am} \end{aligned} \qquad (12)$$

$F_{(.)}(t)$ in (12) are unknown Lebesgue measurable functions satisfying standard norm bounded conditions.

Introduce the following inequality which is related to the system (10)–(12) for a given $\gamma > 0$

$$\|v_m(t)\|_2 \leq \gamma \|w_m(t)\|_2. \qquad (13)$$

Consider a static output switching stabilizing memory–less controller with the switching rule for the control problem (10)–(12) in the form

$$\begin{aligned} u_m &= K_{r(x_m,t)} y_m \\ r(x_m,t) &= \arg\min_{k\in\Lambda}(x_m^T \Omega_k x_m) \end{aligned} \qquad (14)$$

The matrices K_k for all $k \in \Lambda$ can be determined by the procedure as follows. Suppose that the matrix C in (10) has a full rank.

Theorem 3. Consider the system (10)–(12) and a positive constant γ as given in (13). The switched static controller (14) quadratically stabilizes the closed–loop system (10)–(12), (14) with H_∞ disturbance attenuation γ for all admissible uncertainties if there exist a matrix $Q > 0$, matrices N_k, V_k for all $k \in \Lambda$, and a scalar $\eta > 0$ such that for some scalars $\alpha_1, \ldots, \alpha_\kappa$ the following LMI

$$R(A_{mr}) = \begin{pmatrix} W & \Gamma & 0 & \Upsilon & \Xi \\ \bullet & \Phi & \Psi & 0 & 0 \\ \bullet & \bullet & \Theta & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \eta I \end{pmatrix} < 0 \qquad (15)$$

and the relation

$$CQ = V_k C \qquad k \in \Lambda \qquad (16)$$

are satisfied. The blocks in (15) mean $W = \sum_{k=1}^{\kappa} \alpha_k Q$, $\Gamma = (\sqrt{\alpha_1} Q A_{m1}^T + C^T N_1^T B_1, \ldots, \sqrt{\alpha_\kappa} Q A_{m\kappa}^T + C^T N_\kappa^T B_\kappa)$, $\Upsilon = (\sqrt{\alpha_1} Q C_v^T + C^T N_1^T D_v, \ldots, \sqrt{\alpha_\kappa} Q C_v^T + C^T N_\kappa^T D_v)$, $\Xi = (\sqrt{\alpha_1} Q E_{Am}^T, \ldots, \sqrt{\alpha_\kappa} Q E_{Am}^T)$, $\Phi = \mathrm{diag}(-Q + \eta D_{Am} D_{Am}^T, \ldots, -Q + \eta D_{Am} D_{Am}^T)$, $\Psi = \mathrm{diag}(B_w, \ldots \ldots, B_w)$, $\Theta = \mathrm{diag}(-\gamma^2 I, \ldots, -\gamma^2 I)$.

The output feedback gain matrices are given by

$$K_k = \frac{1}{\sqrt{\alpha}_k N_k V_k^{-1}} \qquad (17)$$

and the switching rule in (14) has the form

$$\Omega_k = (A_{mk} + B_k K_k C)^T (Q - \gamma^{-2} B_w B_w^T - \eta D_{Am} D_{Am}^T)^{-1} (A_{mk} + B_k K_k C) + \eta^{-1} E_{Am}^T E_{Am} - Q^{-1} + (C_v + D_v K_k C)^T (C_v + D_v K_k C) \qquad (18)$$

Remark. Theorem 3 is based on Lemma 4 and Theorem 3 in (Ji and Wang 2005). This result is convenient for direct computations of the switching static output controller with the given switching rule. The following theorem states the main result.

Theorem 4. Given the switching symmetric composite system (1)–(5) and a positive constant γ. Construct the reduced control design system (10)–(12). Select the controller matrices K_k and Ω_k in (14) satisfying (15) and (15) for the system (10)–(12). Then implementing the matrices K_k, Ω_k into (7), the global closed–loop overall system (1)–(5), (7) is quadratically stable with H_∞ disturbance attenuation γ by (6).

The proof is given in the Appendix.

Remark. A common Lyapunov function is used in Theorem 3. However, the methodology presented by Theorem 4 can be directly extended on the case of multiple Lyapunov functions to reduce the conservatism caused by a single Lyapunov function, see e.g. (Pettersson 2003).

4. CONCLUSION

The paper contributes by a new complexity–reduced control design method for low–order switching H_∞ static output feedback controllers guaranteeing the level of disturbance attenuation for a class of switching uncertain discrete–time symmetric composite systems. The structural properties of this class of large scale systems are used for the construction of low–order switching design system. The switching H_∞ output control together with the selected switching rule designed for this switching design model is consequently implemented as local identical switching controllers including local switching rules into the given original system. The procedure ensures the quadratic stability of the global switched closed–loop system with given bound of disturbance attenuation.

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APPENDIX

The Appendix presents the proof of Theorem 4. It requires to introduce selected preliminaries. Denote the global system description of the system (1)–(5) as follows

$$\begin{aligned} x(t+1) &= (\overline{A}_r + \Delta\overline{A}(t))x(t) + \overline{B}u(t) + \overline{B}_w w(t) \\ v(t) &= \overline{C}_v x(t) + \overline{D}_v u(t) \\ y(t) &= \overline{C}x(t) \end{aligned} \tag{19}$$

where x, u, w, v, and y are $n\times N$-, $m\times N$-, $m_w\times N$-, $p_v\times N$, and $p_y\times N$-dimensional vectors of the subsystem states, control inputs, exogenous inputs, controlled and measured outputs, respectively. The continuous function $r = r(x,t) : \Re^{n\times N}\times\Re^+ \to \{1,...,\overline{\kappa}\}$ is the switching rule for the global system. The nominal matrices are defined as follows

$$\begin{aligned} &\overline{A}_r = (\overline{A}_{rij}) \quad \overline{A}_{rii} = A_r \quad \overline{A}_{ij} = L_{ij}C_z \\ &\overline{B}_r = \text{diag}(B_r,...,B_r) \quad \overline{B}_w = \text{diag}(B_w,...,B_w) \\ &\overline{C}_v = \text{diag}(C_v,...,C_v) \quad \overline{D}_v = \text{diag}(D_v,...,D_v) \\ &\overline{C} = \text{diag}(C,...,C) \end{aligned} \tag{20}$$

The uncertainty terms have the form

$$\Delta\overline{A}(t) = \overline{D}_A\overline{F}_A(t)\overline{E}_A \tag{21}$$

The constant matrices are defined as follows

$$\begin{aligned} \overline{D}_A =& \text{diag}(\overline{D}_1,...,\overline{D}_N) \\ &\overline{D}_i = (D_L...D_L \;\; D_A \;\; C_L...D_L) \\ \overline{E}_A =& \text{diag}(\overline{E}_1 ...,\overline{E}_N) \\ &\overline{E}_i = (E_L...E_L \;\; E_A \;\; E_L...E_L) \end{aligned} \tag{22}$$

D_A is located at the ith position in $\overline{D}_i$. The uncertainty structure is lumped in uncertainty functions in the form

$$\begin{aligned} \overline{F}_A(t) =& \text{diag}(F_{A1},...,F_{AN}) \\ &F_{Ai} = \text{diag}(F_{Li1},... \\ &\quad ...,F_{Lii-1},F_{Ai},F_{Lii+1},...,F_{LiN}) \end{aligned} \tag{23}$$

Uncertainties $\overline{F}_A(t)$ are unknown Lebesgue measurable functions satisfying standard norm bounded conditions.

Consider a static output switching stabilizing full order controller with the switching rule for the system (19)–(23) in the form

$$\begin{aligned} u &= \overline{K}_{\overline{r}(x,t)}y \\ \overline{r}(x,t) &= \arg\min_{l\in\overline{\Lambda}}(x^T\overline{\Omega}x) \end{aligned} \tag{24}$$

where $\overline{\Lambda} = \{1,..,\kappa^N\}$. $\overline{\Omega} = \text{diag}(\Omega,...,\Omega)$ with $\Omega \in \{\Omega_1,...,\Omega_\kappa\}$ which are selected according to the switching rule. To simplify the notation, the symbol Ω is used for the set of available matrices in the switching rule for any structural subsystem. It is identical for all these subsystems.

The matrices $\overline{K}_l = \text{diag}(K_{r(x_1,t)},...,K_{r(x_N,t)})$ for all $l \in \overline{\Lambda}$ can be determined by the procedure as follows.

Definition 5. Consider the system (19)–(23). This system is *d-quadratically stabilizable with H_∞ disturbance attenuation via switched output feedback* if it is quadratically stabilizable with H_∞ disturbance attenuation via switched output feedback with a block diagonal gain matrix and decentralized switching rule. Each local gain has its own switching rule.

Theorem 6. Consider the system (19)–(23) and a positive constant γ as given in (6). The switched static controller (24) d-quadratically stabilizes the closed–loop system (19)–(24) with H_∞ disturbance attenuation γ for all admissible uncertainties if there exist a block diagonal matrix $\overline{Q} > 0$, block diagonal matrices

$\overline{N}_l, \overline{V}_l$ for all $l \in \overline{\Lambda}$ with $n \times n$ diagonal blocks, and a scalar $\eta > 0$ such that for some scalars $\alpha_1, ..., \alpha_{\overline{\kappa}}$ the following LMI

$$\overline{R}(\overline{A}_r) = \begin{pmatrix} \overline{W} & \overline{\Gamma} & 0 & \overline{\Upsilon} & \overline{\Xi} \\ \bullet & \overline{\Phi} & \overline{\Psi} & 0 & 0 \\ \bullet & \bullet & \overline{\Theta} & 0 & 0 \\ \bullet & \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & \bullet & \eta I \end{pmatrix} < 0 \qquad (25)$$

and the relation

$$\overline{CQ} = \overline{V}_l \overline{C} \qquad l \in \overline{\Lambda} \qquad (26)$$

are satisfied. The blocks in (25) mean $\overline{W} = \sum_{l=1}^{\overline{\kappa}} \alpha_l \overline{Q}$, $\overline{\Gamma} = (\sqrt{\alpha_1}\overline{QA}_1^T + \overline{C}^T\overline{N}_1^T\overline{B}_1, ..., \sqrt{\alpha_{\overline{\kappa}}}\overline{QA}_{\overline{\kappa}}^T + \overline{C}^T N_{\overline{\kappa}}^T \overline{B}_{\overline{\kappa}})$, $\overline{\Upsilon} = (\sqrt{\alpha_1}\overline{QC}_v^T + \overline{C}^T\overline{N}_1^T\overline{D}_v, \ldots, \sqrt{\alpha_{\overline{\kappa}}}\overline{QC}_v^T + \overline{C}^T\overline{N}_{\overline{\kappa}}^T\overline{D}_v)$, $\overline{\Xi} = (\sqrt{\alpha_1}\overline{QE}_A^T, \ldots, \sqrt{\alpha_{\overline{\kappa}}}\overline{QE}_A^T)$, $\overline{\Phi} = \mathrm{diag}(-\overline{Q} + \eta\overline{D}_A\overline{D}_A^T, \ldots, -\overline{Q} + \eta\overline{D}_A\overline{D}_A^T)$, $\overline{\Psi} = \mathrm{diag}(\overline{B}_w, \ldots, \overline{B}_w)$, $\overline{\Theta} = \mathrm{diag}(-\gamma^2 I, \ldots, -\gamma^2 I)$.

The output feedback gain matrices are given by

$$\overline{K}_l = \frac{1}{\sqrt{\alpha_l}\overline{N}_l\overline{V}_l^{-1}} \qquad (27)$$

and the switching rule in (24) has the form

$$\begin{aligned}\overline{\Omega}_l = {} & (\overline{A}_l + \overline{B}_1\overline{K}_l\overline{C})^T(\overline{Q} - \gamma^{-2}\overline{B}_w\overline{B}_w^T - \eta\overline{D}_A\overline{D}_A^T)^{-1} \\ & \overline{A}_l + \overline{B}_l\overline{K}_l\overline{C}) + \eta^{-1}\overline{E}_A^T\overline{E}_A - \overline{Q}^{-1} + (\overline{C}_v \\ & + \overline{D}_v\overline{K}_l\overline{C})^T(\overline{C}_v + \overline{D}_v\overline{K}_l\overline{C})\end{aligned} \qquad (28)$$

Consider a real $sn \times sn$ matrix $T(n,s)$ in the form

$$T(n,1) = I,$$
$$T(n,s) = \begin{pmatrix} I & 0 & \ldots & 0 & I \\ 0 & I & \ldots & 0 & I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & I \\ -I & -I & \ldots & -I & I \end{pmatrix}, \qquad s > 1, \qquad (29)$$

where I denotes here $n \times n$ identical matrix. Denote

$$\begin{aligned}\overline{T}(i) &= \mathrm{diag}[T(n, N-i)I, ..., I], \quad i = 0, ..., N-1, \\ G &= \overline{T}(0)\overline{T}(1) \quad \ldots \quad \overline{T}(N-1).\end{aligned} \qquad (30)$$

The following theorem presents the way how to use the structural properties of symmetric composite systems to construct reduced-order systems with equivalent dynamic properties (Yang and Zhang 1995).

Theorem 7. Consider the matrix $\overline{A}$ in the system (20)–(23) and any given $J = \mathrm{diag}[J_o, ..., J_o]$, where J, J_o are $Nn \times Nn$, $n \times n$ matrices, respectively. Then it holds

$$\begin{aligned} G^{-1}\overline{A}G &= \mathrm{diag}(A_s, ..., A_s, A_c), \\ G^T\overline{A}G &= \mathrm{diag}(2A_s, ..., 6A_s ... N(N-1)A_s, NA_c), \\ G^{-1}J(G^{-1})^T &= \mathrm{diag}(\frac{1}{2}J_o, \frac{1}{6}J_o, ..., \frac{1}{N(N-1)}J_o), \\ G^TJG &= \mathrm{diag}(2J_o, ..., N(N-1)J_o, NJ_o) \end{aligned} \qquad (31)$$

Proof of Theorem 4. Consider the system (19)–(23) with $A_s = A - L_qC_z$, $A_c = A_s + NL_qC_x$ and two particular cases of the uncertainties (8) such as $\Delta A_a(t) = \frac{N}{2}L_qC_z$ and $\Delta A_a(t) = -\frac{N}{2}L_qC_z$. It leads using (31) to the relations

$$A_{mr} + \Delta A_m(t) = A_s + \Delta A_a(t) + \Delta A(t) = A_c - \Delta A(t) \qquad (32)$$

Suppose that the gain matrices K_k in (17) satisfy the conditions (15)–(16) in Theorem 4 for a given matrix $Q > 0$ and a constant γ. Then by Theorem 4 it is sufficient to show that these gain matrices when implemented into the decentralized controller (7) together with the switching rule lead to quadratic stability with H_∞ disturbance attenuation γ satisfying (6) for the closed–loop system (1)–(5), 7). The notion of the quadratic stability with H_∞ disturbance attenuation γ of the system (1)–(6) is equivalent to the notion of feasible solution by Theorem 4.

Denote $\overline{G} = \mathrm{diag}(G, G, G, G, G)$, where G defined by (31). Now, we get by using Theorem 7 when applying only standard operations on all terms of the inequality (25) in Theorem 6

$$\begin{aligned}&\overline{G}^T\overline{R}(\overline{A}_r)\overline{G} \\ &= \mathrm{diag}[2R(A_s), ..., N(N-1)R(A_s), NR(A_c)]\end{aligned} \qquad (33)$$

If $R(A_{mr}) < 0$ holds by Theorem 3, then also $R(A_s) < 0$ and $R(A_c) < 0$ hold because the uncertainties in (10), (12) include both systems with the matrices A_s, A_c as special cases. The matrix $\overline{G}_s$ is nonsingular. Therefore, the relation $\overline{G}^T\overline{R}(\overline{A}_r)\overline{G} < 0$ holds.

The last item concerns the relation between the switching rules (28) and (18). The way of reasoning is analogous to the transformation of LMIs in (33). It results in the relation

$$\begin{aligned}&\overline{G}^T\overline{\Omega}(\overline{A}_r)\overline{G} \\ &= \mathrm{diag}[2\Omega(A_s), ..., N(N-1)\Omega(A_s), N\Omega(A_c)]\end{aligned} \qquad (34)$$

which are included in $\Omega(A_{mr})$ as indicated in (18). It leads to N independent switching rules where one switching rule is assigned to one structural subsystems. The necessity to select scalars α_l in Theorem 6 reduces accordingly to κ identical scalars for each structural subsystem.

Thereby, the closed loop system (1)–(5), (7) is d-quadratically stabilized with H_∞ disturbance attenuation γ satisfying (6). Q.E.D.