EXACT DIFFERENTIATION VIA SLIDING MODE OBSERVER FOR SWITCHED SYSTEMS

H. Saadaoui*, M. Djemaï*, N. Manamanni, T. Floquet***, J-P. Barbot ***

* *Equipe Commande des Systèmes (ECS), ENSEA, 6 Av. du Ponceau, 95014 Cergy-Pontoise Cedex, France.*
{djemai,saadaoui,barbot}@ensea.fr
** *CReSTIC, University of Reims, Moulin de la Housse BP 1039, 51687 Reims cedex 2 - France,*
noureddine.manamanni@univ-reims.fr
*** *LAGIS, UMR CNRS 8146, Ecole Centrale de Lille, Cité Scientifique, BP 48, 59651 Villeneuve d'Ascq Cedex, France,* floquet@ec-lille.fr

Abstract: The main topic of this paper is the problem of observer synthesis for switched systems, which includes, as a specific case, the design of observers based on high order sliding mode technique. High order sliding mode is used to overcome the occurring chattering phenomena which induces some irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold. Moreover, in this paper, after presenting the general structure of the step by step differentiator, well show the step by step finite time convergence of the estimation error and the discrete state estimation. Two simulation examples illustrate the efficiency of the proposed approach.

Keywords: Hybrid system, Switched systems, Non linear observer, Sliding mode, Exact differentiator, Finite time convergence.

1. INTRODUCTION

Switched systems are a class of Hybrid Systems (HS) which consist of several subsystems that switch according to a given switching law (Antsaklis, 2000).

A rich and thorough bibliography deals with stability problems of switched systems, see (Branicky, 1998), (Liberzon, 2000), (Michel and Sun, 2003) and references therein. More recently, various researchers have studied observability and observer design for such systems. Some sufficient geometrical conditions to analyze the observability of hybrid dynamical systems were given in (Boutat et *al.*, 2004). These conditions are refined for the particular class of piecewise linear and nonlinear systems. The so-called extended joint observability matrix was proposed in (Vidal et *al.*, 2003), to analyze the observability of jump linear systems. In (Sontag, 1979), Sontag introduced a set of observability related definitions and examined the implications among the various concepts of observability. In the same way, other works deal with the hybrid observer design.

Indeed, in (Balluchi et *al.*, 2002) a methodology was presented for the design of dynamical observers of hybrid systems that reconstruct the discrete state and the continuous state from the knowledge of the continuous and discrete outputs. The design of linear observers for a class of linear hybrid systems was addressed in (De laSen and

Luo, 2000). Two observer prototypes based on the prediction errors were proposed.

Despite an abundant literature on the design of linear observers for hybrid systems, only few works are concerned with the design of nonlinear hybrid observers for hybrid systems (see for example (Lin et *al.*, 2002)and (Pettersson, 2005)).

Within this context, the problem of designing sliding mode observers for non linear hybrid systems without jump was discussed in (Djemaï et *al.*, 2005) and (Saadaoui et *al.*, 2005). Nevertheless, when the trajectory is in the neighborhood of the switching manifold, a chattering phenomena occurs and induces some irrelevant decision of switching between the subsystems. To overcome this problem, it is proposed to design high order sliding modes (Fridman and Levant, 1996).

Hence, the main purpose of this paper lies in nonlinear observer design for a class of HS. The considered class is assumed to be bounded state in finite time without, jumps and without Zeno phenomenon. The observer design is discussed by using a triangular input observer form introduced in (Barbot et *al.*, 1996) and (Drakunov and Utkin, 1995). The idea consists in using the step by step procedure. Another contribution of the paper lies in the convergence analysis of the estimation error in the general case.

The paper is organized as follows: Section 2 recalls some observability notions for hybrid systems. The high order sliding mode observer design and the convergence analysis are detailed in section 3. In section 4, two illustrative examples are discussed to show the performances of the developed algorithm.

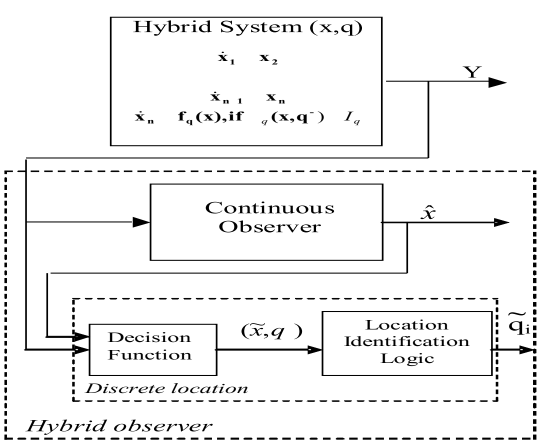

Fig. 1 : Hybrid observer structure

2. RECALLS ON OBSERVABILITY STUDY

Recently, many researchers have approached the study of observability of hybrid systems in general and switched systems in particular. In (Vidal et *al.*, 2003), was considered the case of autonomous switched systems. A definition of observability based on the concept of indistinguishability of continuous initial states and discrete states evolutions from the output in free evolution was given. In (Boutat et *al.*, 2004), both cases of linear and nonlinear switched systems were considered and some algebraic and geometrical conditions of observability for such class were stated.

In what follows is recalled the main result of (Boutat et *al.*, 2004), on the observability of the class of hybrid system considered in this paper. The proof of the theorem can be found in the cited reference. Let us consider the dynamical systems formed with two dynamics interconnected by a switching function:

$$\begin{cases} \dot{x} = F_1(x) \text{ and } y = h_1(x) \text{ if } \sigma(x) \leq 0 \\ \dot{x} = F_2(x) \text{ and } y = h_2(x) \text{ if } \sigma(x) > 0 \end{cases} \quad (1)$$

where $F_i(x)$ are smooth vector fields, $h_i(x)$ are smooth outputs and $\sigma(x)$ is a smooth switching function.

Assumption 1. We assume throughout this paper that:

a) All the evolution duration of each subsystem of (1) are measurable.
b) For $i = 1 : 2$ the codistribution:
$$\left\{dh_i, dL_{F_i}h_i, ..., dL_{F_i}^{(n-1)}h_i\right\}$$
has rank n, this implies that is locally weakly observable.

Assumption 1 a) means that systems with Zeno phenomenon are not considered.

Under assumption 1, if we know which of the subsystem evolves, one can conclude on the observability of the global system (1). Hence, when considering the observability coordinates (z^j, $j = 1:2$) defined by:

$$z_{i+1}^j = L_{F_j}^{(i)}h_j \qquad \text{for } 0 \leq i \leq n-1$$

where $L_{F_j}^{(i)}h_j$ is the i^{th} Lie derivative of h_j in the direction of F_j, and using the Fliess's observability canonical form, each subsystem of (1) can be written as:

$$\begin{cases} \dot{z}_i^1 = z_{i+1}^1 \qquad \text{for } i = 1 : n-1 \\ \dot{z}_n^1 = g_1(z_1^1, z_2^1, ..., z_n^1) \end{cases} \quad (2)$$

if $\sigma_1 := \phi^{-1}(z_1^1, z_2^1, ..., z_n^1) \leq 0$, and

$$\begin{cases} \dot{z}_i^2 = z_{i+1}^2 \qquad \text{for } i = 1 : n-1 \\ \dot{z}_n^2 = g_2(z_1^2, z_2^2, ..., z_n^2) \end{cases} \quad (3)$$

if $\sigma_2 := \phi^{-1}(z_1^2, z_2^2, ..., z_n^2) > 0$.

The approach to analyze the observability of (1), presented in (Boutat et *al.*, 2004), is based on the comparison of g_1 and g_2 on the one hand and σ_1 and σ_2 on the other hand. For this, we need to evaluate such functions in terms of the same variables. These variables are given naturally by the output y and its successive time derivatives $y^{(i)} = \frac{d^i y}{dt^i}$ for $i = 1 : n-1$.
Let us consider the two submanifolds:

$$\mathcal{M} = \{v \in \mathbb{R}^n \;/\; g_1(v) = g_2(v)\}$$
$$\mathcal{S} = \{v \in \mathbb{R}^n \;/\; \sigma_1(v) = \sigma_2(v)\}$$

and finally, the submanifold of common singularities of subsystems of system (1):

$$\mathcal{L} = \{x \in \mathbb{R}^n \;/\; F_1(x) = F_2(x) = 0\}$$

The main result is recalled in the following theorem.

Theorem 1. i) If $\mathcal{M}$ is a discrete set then system (1) is observable for any switch σ for which we have $\sigma(\mathcal{L}) \leq 0$ or else $\sigma(\mathcal{L}) > 0$.

ii) If dynamics (2) and (3) are transverse to $\mathcal{M}$ except on a discrete subset then the system is observable for any switch σ for which we have $\sigma(\mathcal{L}) \leq 0$ or else $\sigma(\mathcal{L}) > 0$.

iii) If $\mathcal{S} = \mathbb{R}^n$ then system (1) is observable.

The reader can refer to (Boutat et *al.*, 2004) for proof and more details. Some algebraic sufficient conditions on the observability of piecewise linear systems can also be found.

3. HYBRID OBSERVER

In this paper will be designed a step by step sliding mode observer. The idea consists in using the concept of equivalent vector (see (Drakunov, 1992) and (Drakunov and Utkin, 1995)) in an iterative way. Both systems (2) and (3) are in the so-called canonical observer form of a nonlinear autonomous system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = f_q(x) \quad if \quad \sigma_q(x) \in I_q \end{cases} \tag{4}$$

where $y = x_1$, $\sigma_q(x)$ for $q = 1, .., p$ is a switching function and I_q is a domain of validation of a subsystem q. The considered class of systems is assumed to be bounded state in finite time without jumps and does not concern Zeno phenomena.

The consideration of such class of systems (4) is not restrictive. In fact, most of lagrangian systems, for example, are written in the considered form.

Remark 1. The assumption of a canonical form (4) without jump is quite restrictive, because the diffeomorphism linked to the sub-system $q = 1$ is not generally the same than the diffeomorphism linked to the sub-system $q = 2$. Then at each switch from $q = 1$ to 2, the state in the observability canonical form (4) ((Fliess, 1990)) jump even if the state in the original (1) coordinates does not jump. This jump may be easily taken into account by a new structure of sliding mode which will be given in a forthcoming paper.

In (Djemaï et *al.*, 2005), a step by step first order sliding mode observer was mainly employed for the following reasons: the finite time convergence and the ability to take naturally into account the variable structure of the HS. Nevertheless, some difficulties occur due to the chattering phenomena. It induces some irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold. This problem was bypassed by using a low pass filter during the computation of the equivalent vector; unfortunately, this solution introduces a delay. In this work, a relevant solution for the case of switched systems is given. It consists in using exact and robust second order sliding mode differentiators (Super Twisting Algorithm, see (Fridman and Levant, 1996)).

The "Super Twisting Algorithm" (Figure 2) is given by the following structure:

$$\sum\nolimits_{obs} = \begin{cases} u(e_1) = u_1 + \lambda_1 |e_1|^{\frac{1}{2}} sign(e_1) \\ \dot{u}_1 = \alpha_1 sign(e_1) \end{cases} \tag{5}$$
$$\lambda_1, \alpha_1 > 0$$

where $e_1 = x_1 - \hat{x}_1$ and λ_1, α_1 are positive

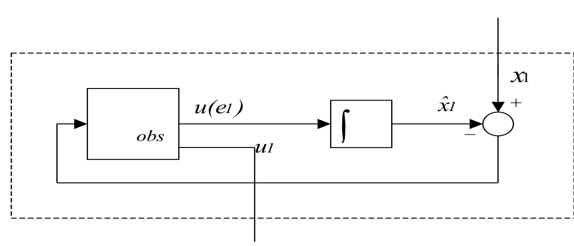

Fig. 2 : Super Twisting Algorithm Structure

parameters, and u_1 is the differentiator output where:

$$sign(e_1) = \begin{cases} +1 & \text{if} \quad e_1 > 0 \\ -1 & \text{if} \quad e_1 < 0 \\ \in [-1, 1] & \text{if} \quad e_1 = 0 \end{cases}$$

The step by step exact differentiator applied to (4), leads to the following form:

$$\begin{cases} \dot{\hat{x}}_1 = \tilde{x}_2 + \lambda_1 \left|\tilde{e}_1\right|^{1/2} sign(\tilde{e}_1) \\ \dot{\tilde{x}}_2 = \alpha_1 sign(e_1) \\ \dot{\hat{x}}_2 = E_1 \left[\tilde{x}_3 + \lambda_2 \left|\tilde{e}_2\right|^{1/2} sign(\tilde{e}_2)\right] \\ \dot{\tilde{x}}_3 = E_1 \alpha_2 sign(\tilde{e}_2) \\ \quad \vdots \\ \dot{\hat{x}}_{n-1} = E_{n-2} \left[\tilde{x}_n + \lambda_{n-1} \left|\tilde{e}_{n-1}\right|^{1/2} sign(\tilde{e}_{n-1})\right] \\ \dot{\tilde{x}}_n = E_{n-2}\alpha_{n-1} sign(\tilde{e}_{n-1}) \\ \dot{\hat{x}}_n = E_{n-1} \left[\tilde{\theta} + \lambda_n \left|\tilde{e}_n\right|^{1/2} sign(\tilde{e}_n)\right] \\ \dot{\tilde{\theta}} = E_{n-1}\alpha_n sign(\tilde{e}_n) \end{cases} \tag{6}$$

where $\tilde{e}_i = \tilde{x}_i - \hat{x}_i$, with $\tilde{x}_1 = x_1$ for $i = 1, .., n$, and the E_i for $i = 1, ...n-1$ are defined as

$$E_i = 0 \text{ if } \ \tilde{e}_i = \tilde{x}_i - \hat{x}_i \neq 0, \quad \text{else} \quad E_i = 1 \tag{7}$$

In practice

$$E_i = 0 \text{ if } \ \tilde{e}_i = \tilde{x}_i - \hat{x}_i > \epsilon, \quad \text{else} \quad E_i = 1$$

The structure of the step by step differentiator for a system of order n in canonical form is given in figure (3), where each bloc B_i for $i = 1, ..., n-1$, is only valid when $E_i = 1$.

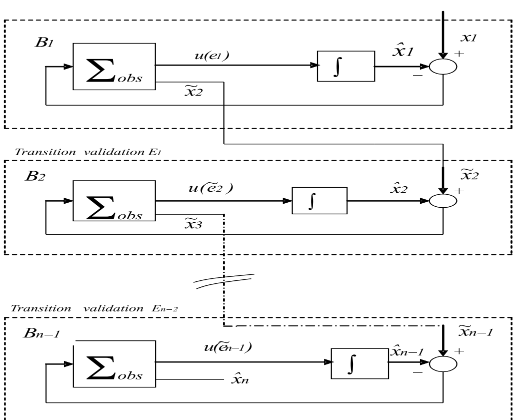

Fig. 3 : Structure of step by step differentiator

3.1 Convergence analysis

The convergence of the observation error is obtained in $(n-1)$ step in finite time. The idea consists in using the step by step observer such as described here after: The $(n-1)$ first steps consist in reconstructing the state vector and after that, under conditions of theorem 1, recovering in which state (location) q, the system evolves. Another feature of the differentiator (5) is the fact that the output u_1 does not depend directly on discontinuous functions but on an integrator output. So high frequency chattering, which can be very harmful for the system (see for example chaotic system known for its extreme sensitivity to noise (Djemaï et *al.*, 2005)), can be avoided. These properties are importants since the switching function can be obtained in a continuous way and without delays and chattering.

Theorem 2. Consider the system (4), assumed to be bounded state in finite time $t < \infty$, and the observer (6) based on the Super Twisting Algorithm (5). For any bounded initial conditions $x(0)$, $\hat{x}(0)$, there exists a choice of λ_i and α_i such that the state observer $\hat{x}$ converges in finite time $T_{fs} \ll \tau_q$ (dwell time(De Santis et *al.*, 2005)) to x and $\tilde{\theta}$ converge also in finite time to $f_q(x)$.

Proof. The proof is given in (Saadaoui et *al.*, 2005) for the case of $n = 2$ (see also (Levant , 1998) and (Davila et *al.*, 2005)). Figure 4 illustrates the finite time convergence behavior of the proposed observer. The demonstration is based on the error trajectory for each quadrant in the worst cases. In the case of $n > 2$, the convergence is ensured step by step following this order : $(\dot{e}_1 = e_2, e_1) \to (0, 0)$ in finite time T_1 in the first step. $(\dot{e}_2 = e_3, e_2) \to (0, 0)$ in finite time T_2 in the second step. And $(\dot{e}_i = e_i + 1, e_i) \to (0, 0)$ in finite time T_i in the step i. Finally, $(\dot{e}_{n-1} = e_n, e_{n-1}) \to (0, 0)$ in finite time T_{n-1} in the step $(n-1)$. The finite time convergence of the full state x is:

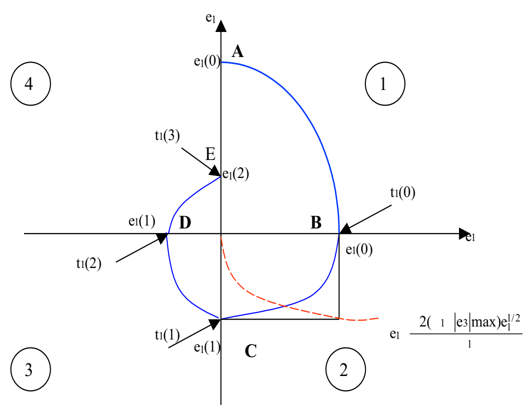

Fig. 4 : Majoring curve for finite time convergence observer

$$T_{fs} = \sum_{j=1}^{n-1} T_j \tag{8}$$

■

- Discrete time observer

This section, is concerned with the discrete part of the designed observer. Let us consider the system (4), the task of the discrete time observer is to locate which dynamic of the system is in evolution? In some cases, the knowledge of the system's output is sufficient to estimate the current location (i.e., if $\sigma(x) = \sigma(y)$). If this is not the case, some additional information obtained by using the continuous part of the observer, may be useful or are necessary to estimate the current location.

In our case, the discrete-observer receives as input: the observed state $\hat{x}$, the output y and the the information E_{n-1}. Its task is to provide an estimation $\hat{q}$ of the discrete location q of the hybrid plant at the current time. Contrarily to the general case; here, the continuous observer doesn't need to know the discrete location q. This is the main property of the canonical form (4). Indeed, the second order sliding mode observer (6) has to know only the output $y = x_1$ and also the $f_q(x)$ upper bound, noted g^+.

Thus, one can announce the following corollary:

Corollary 1. If the observer is sufficiently fast (i.e., $T_{fs} \ll \tau_q$), then for $t \geq T_{fs}$, one has $x = \hat{x}$ and $\tilde{\theta} = f_q(x)$. Then under conditions of theorem 1, the discrete state q is known.

Proof is a direct consequence of theorem 1 and 2.

4. SIMULATIONS AND COMMENTS

In (Djemaï et *al.*, 2005), when a low pass filter was used for $\tilde{x}_2$ and $\tilde{x}_3$ during the computation of the switching condition $\sigma(\tilde{x})$. The results showed a delay occurring for the switching decision and a chattering phenomenon. Indeed, the first order differentiator generates high levels of chattering even if its output is filtered through a low pass filter. A delay appears between the switching indicators S calculated on the basis of $\sigma(x)$ and S_o calculated on the basis of $\sigma(\tilde{x})$. When using sliding mode differentiators, the observer performances are presented, but the delay is completely removed and no chattering phenomenon occurs.

Example 1. Let us consider the following system put in the triangular input observer form:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3$$
$$\dot{x}_3 = \begin{cases} -\cos(30x_2) + 0.4 & \text{if} \quad x_2 < 0 \\ -40\cos(300x_3 + \pi/2) - 0.5 & \text{if} \quad x_2 \geq 0 \end{cases} \tag{9}$$

with $y = x_1$.

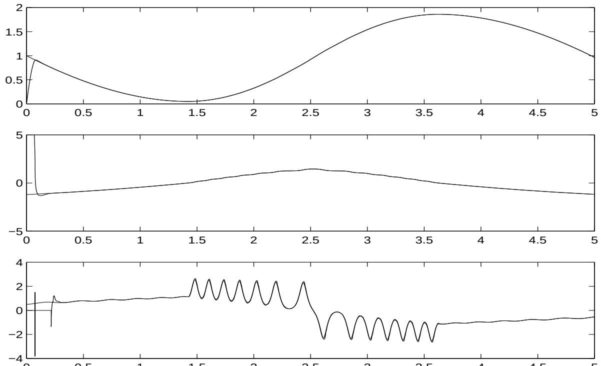

Fig. 5 : x_1 and $\hat{x}_1$, x_2 and $\tilde{x}_2$, and x_3 and $\tilde{x}_3$

The performances of the proposed hybrid observer are shown in figures 5 (estimated state, dashed line; system state, solid line).

Example 2. Let us consider the following switched systems:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \begin{cases} P_1(x) & if \quad \sigma(x) \geq 1 \\ P_2(x) & if \quad |\sigma(x)| < 1 \\ P_3(x) & if \quad \sigma(x) \leq -1 \end{cases}$$

with: $P_1(x) = -\frac{1800}{49}x_1 - \frac{55}{7}x_2 - \frac{25}{7}x_3 + \frac{2700}{49}$; $P_2(x) = \frac{100}{49}x_1 - \frac{36}{7}x_2 - \frac{6}{7}x_3$, and $P_3(x) = -\frac{1800}{49}x_1 - \frac{55}{7}x_2 - \frac{25}{7}x_3 + \frac{2700}{49}$. The switching condition: $\sigma(x) = -\frac{7}{100}(x_3 + x_2) - x_1$

The figure 6 highlights the efficiency of the proposed observer and shows the finite time step by step convergence.

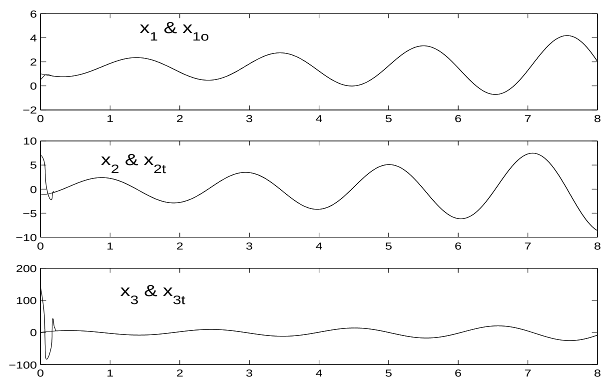

Fig. 6 : $(x_1, \hat{x}_1)$, $(x_2, \tilde{x}_2)$, and $(x_3, \tilde{x}_3)$

Figure 7 shows respectively the switching function $\sigma(x)$ and $\sigma(\tilde{x})$ and the corresponding switching indicator S, and S_o. There is no problem of delay between S (switching indicator calculated on the basis of the real states x_1, x_2, and x_3) and S_o (switching indicator calculated on the basis of the observed states x_1, $\tilde{x}_2$, and $\tilde{x}_3$). Also, it can be noted that when $\sigma(\tilde{x})$ is near -1 around $t = 5s$, there is no undesirable chattering phenomenon and no irrelevant decision of switching between the subsystems when the trajectory is in the neighborhood of the switching manifold as it was the case when classical sliding mode was used with a filter (see figure 8 and paper (Djemaï et *al.*, 2005)).

5. CONCLUSION

This paper has dealt with the design of non linear observer for hybrid systems. The considered systems concern switched non linear systems without jump and without zeno phenomenon. It was

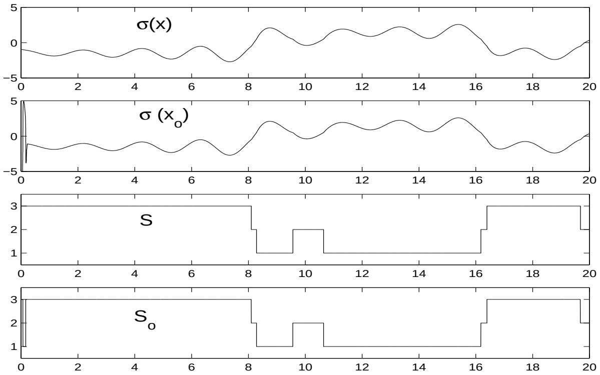

Fig. 7 : Switching surfaces $\sigma(x)$ and $\sigma(\tilde{x})$, and switching indicators S and So

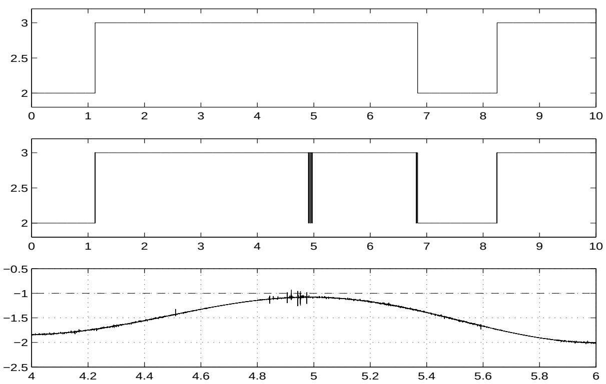
Fig. 8 : Switching indicators S and So obtained with classical sliding modes and zoom on $\sigma(\tilde{x})$

shown that the use of high order sliding mode technique; in our case exact differentiator of order 2 (Super twisting algorithm), leads to enhance the observer performances. Indeed, the robustness is guaranteed without introduction of any delays or chattering phenomena. Another contribution of the paper deals with the performed convergence analysis in the general case. We have showed that the finite time convergence of the full state is ensured and that the estimation of the discrete state is guaranteed.

REFERENCES

Antsaklis, P.J., editors (2000). *Proc. of the IEEE, Special Issue on Hybrid Systems: Theory and Applications, 43(2).*

Barbot, J.P., T. Boukhobza and M. Djemai. (1996). Sliding mode observer for triangular input form. *IEEE-CDC.*

Balluchi, A., L. Benvenuti, M.D. Di Benedetto and A.L. Sangiovanni-Vincentelly. (2002). Design of Observer for Hybrid Systems. *In LNCS 2289, pp. 77-89, Springer Verlag.*

Branicky, M.S., (1998). Multiple Lyapunov Fonctionsand Other Analysis Tools for Switched and Hybrid Systems. *IEEE Trans Auto. Cont., 43(2), pp. 461-474.*

Boutat, D., A. Benali and J.P. Barbot. (2004). About the observability of piecewise dynamical systems. *NOLCOS-2004. (CD-ROM).*

Davila, J., L. Fridman and A. Levant (2005). Second-Order Sliding-Mode Observer for Mechanical Systems.*IEEE Trans Auto. Cont.,* ***50****, pp. 41785-1790.*

Djemai, M., N. Manamanni and J. P. Barbot. (2005). Sliding Mode Observer For Triangular Input Hybrid System. *IFAC World congres Prague, 2005.*

De la Sen, M., and N. Luo. (2000). Design of linear observers for a class of linear hybrid systems. *Int. Journal of Systems Science,* ***31*** *No 9, pp. 1077-1090.*

De Santis, E., M.D. Di Benedetto, and G. Girasole. (2005). Digital Idle Speed Control of Automative Engines using Hybrid Models. *IFAC World Congress, Prague, July, 2005.*

Drakunov, S. (1992). Sliding mode observer based on equivalent control method. *IEEE-CDC, Tucson, USA, pp. 2368–2369.*

Drakunov, S., and V. Utkin. (1995). Sliding mode observer. tutorial. *IEEE-CDC, pp. 3376–3379.*

Fliess M. (1990). Generalized Controller Canonical Forms for Linear and Nonlinear Dynamics. *IEEE Trans on Automatic Control, 35, No 9, pp. 994–1000.*

Fridman, L., A. Levant. (1996), Sliding modes of higher order as a natural phenomenon in control theory. In Robust control via variable structure & Lyapunov techniques. *LNCIS 217. F. Garofalo, L. Glielmo Ed. Springer Verlag London, pp 107, 133. 1996.*

Liberzon, D.,(2003). On stabilization of linear systems with limited information. *IEEE Trans on Automatic Control, 48(2), Feb. 2003, pp. 304-307.*

Lin, L., Linawati, Lie Josa and E. Ambikarajah. (2002). A hybrid state estimation scheme for power systems. *Asian Pacific Conf.* ***1****, pp. 555-558.*

Levant A., Robust exact differentiation via sliding mode technique, Automatica, vol. 34, pp. 379-384, 1998.

Michel, A.N., and Y. Sun. (2003). Stability analysis of discontinuous dynamical systems determined by semigroups. *IEEE-CDC, Hawai, USA, pp. 1669-1674.*

Pettersson, S. (2005). Observer Design for Switched Systems using Multiple Quadratic Lyapunov Functions. *IEEE Int. Symp. Mediterrean Conf. on Cont. and Automation, pp. 262 - 267.*

Saadaoui, H., N. Manamani, M. Djemai, J. P. Barbot and T. Floquet. (2005). Exact differentiation and Sliding mode observer for switched Lagrangian systems . *To appear in Nonlinear Analysis: Theory, Methods & Applications, special issue: Hybrid Systems and Applications, pp. 1-20, 2005.*

Sontag, E.D. (1979). On the observability of polynomial systems : finite-time problems. *SIAM J. Cont. and Optim.,* ***17****(1), pp. 139-151.*

Vidal, R., A. Chiuso, S. Soatto and S.S. Sastry. (2003). Observability of linear hybrid systems. *In Hybrid Systems: Computation and Control, LNCS in Springer Verlag, Vol.* ***2623****, pp. 526-539.*