ROBUST CONTROL STRATEGIES FOR MULTI–INVENTORY SYSTEMS WITH AVERAGE FLOW CONSTRAINTS

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Abstract: In this paper we consider multi–inventory systems in presence of uncertain demand. We assume that i) demand is unknown but bounded in an assigned compact set and ii) the control inputs (controlled flows) are subject to assigned constraints. Given a long–term average demand, we select a nominal flow that feeds such a demand. In this context, we are interested in a control strategy that meets at each time all possible current demands and achieves the nominal flow in the average. We provide necessary and sufficient conditions for such a strategy to exist and we characterize the set of achievable flows. Such conditions are based on linear programming and thus they are constructive. In the special case of a static flow (i.e. a system with 0–capacity buffers) we show that the strategy must be affine. The dynamic problem can be solved by a linear-saturated control strategy (inspired by the previous one). We provide numerical analysis and illustrating examples.

Keywords: Inventory control, Robust control, Bounded disturbances, Manufacturing systems, Linear programming.

1. INTRODUCTION

Multi–inventory systems (Hadley and Whitin, 1963) are met in several different contexts, such as manufacturing (Boukas *et al.*, 1995; Chase and Ramadge, 1992), network routing (Iftar and Davison, 1990), communications (Ephremides and Verdú, 1989), water distribution (Larson and Keckler, 1969), logistics and traffic control (Moreno and Papageorgiou, 1995). Hence, their control is of relevant economic interest. The control concerns storage and processing operations and aims at meeting the external demand of finished products (Forrester, 1961).

In this work we simultaneously consider the two following aspects.

- *Instantaneous fluctuations* — These are assumed unknown due to the large number of unpredictable factors that influence the demand. The control must face all possible variations, within prescribed limits, in order to meet the demand.
- *Long term information* — The long–term average demand, henceforth also called nominal

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demand, should be faced, in the average, by the nominal flow, whenever possible.

Therefore we are seeking for a stabilizing strategy capable of balancing the flow in the long run. The main results of the paper are reported next.

- We first consider static strategies (i.e. we assume 0–capacity buffers). We provide necessary and sufficient conditions for the existence of a strategy which is able to meet all the possible demands and assures the desired flow average, whenever the demand meets its nominal average. Such a static strategy is affine
- We characterize the set of all flows corresponding to the nominal demand which can be achieved in the average.
- We show that the very conditions, valid in the static case, are sufficient for the existence of a dynamic strategy, based on the feedback of the buffer levels. The proposed feedback strategy is a linear-saturated dynamic control.

2. PROBLEM FORMULATION

Consider the following continuous time system

$$\dot{x}(t) = Bu(t) - w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u(t) \in \mathbb{R}^m$ is the controlled flow vector, B is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is an exogenous (uncontrolled) input, typically modeling demand, whose value is externally determined. To model backlog $x(t)$ may be less than zero.

We assume that u and w are subject to the next constraints

$$u(t) \in \mathcal{U} = \{u : u^- \leq u \leq u^+\}, \tag{2}$$

where u^- and u^+ are assigned vectors and the expression is to be intended component-wise. We assume that w is constrained as follows

$$w(t) \in \mathcal{W}, \tag{3}$$

where $\mathcal{W}$ is a polytope. We also introduce the following assumptions.

Assumption 1. Matrix B has full row rank.

Given a vector function of time $f : \mathbb{R}^+ \to \mathbb{R}^n$ we introduce the following notation

$$Av[f] = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\ dt. \tag{4}$$

Function $Av[f]$ will be referred to as the deterministic average of f, henceforth the *average*, and we will always assume that such a value exists whenever considered.

Assumption 2. The set $\mathcal{W}$ includes $\bar{w} = Av[w]$ in its relative interior [2].

We will consider static and dynamic stabilizing policies for the system according to the following definitions.

Definition 3. The function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a *static balancing strategy* if for $u(t) = \Phi(w(t))$,

$$Bu(t) = w(t),$$

and $u(t) \in \mathcal{U}$, for all $w(t) \in \mathcal{W}$, for all $t \geq 0$.

Definition 4. Given $\epsilon > 0$ and a reference value $\bar{x}$, an *ϵ-stabilizing strategy* is a feedback control for which there exists a continuous positive function $\phi(t)$, monotonically decreasing and converging to 0 as $t \to \infty$ such that for all $w(t) \in \mathcal{W}$ and for all $x(0)$, the conditions $u(t) \in \mathcal{U}$ and

$$\|x(t) - \bar{x}\| \leq \max\{\|x(0)\|\phi(t), \epsilon\}$$

hold true.

As a preliminary result, we introduce the following basic conditions (Blanchini *et al.*, 2000).

Theorem 5. For the considered system

i there exists a static balancing strategy as in Definition 3 if and only if

$$\mathcal{W} \subseteq B\mathcal{U}; \tag{5}$$

ii there exists a feedback stabilizing strategy as in Definition 4 if and only if

$$\mathcal{W} \subseteq int\{B\mathcal{U}\}. \tag{6}$$

Henceforth, we assume that the appropriate necessary and sufficient condition is met (depending on which kind of strategy we are considering). Assume to apply either a balancing or an ϵ-stabilizing strategy. As a consequence, $x(t)$ remains constant or bounded. Then, by integrating (1) we have that, necessarily,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [Bu(t) - w(t)]\, dt = \lim_{T\to\infty} \frac{1}{T}\ [x(t) - x(0)] = 0,$$

which implies that the average value of w is equal to the average value of Bu

$$B\ Av[u(t)] = Av[w(t)]. \tag{7}$$

[2] we mean that $\bar{w}$ is an interior point of $\mathcal{W}$ with respect to the smallest linear subspace including it, for instance given a vector $v \neq 0$, 0 is in the relative interior of a segment joining v and $-v$

Formally, the problem is the following.

Problem 6. Assume that the average $\bar{w} \in \mathcal{W}$ is given. Consider the feasible flow $\bar{u} \in \mathcal{U}$ such that

$$B\bar{u} = \bar{w}.$$

Provide a yes–no answer to the question: does there exist a static balancing (or dynamic ϵ–stabilizing) strategy such that whenever $Av[w] = \bar{w}$ then $Av[u] = \bar{u}$? In the case of a positive answer we will say that $\bar{u}$ is achievable.

In the following sections we will solve constructively the problem for both static and dynamic strategies.

3. STATIC STRATEGIES

In this section we consider the case in which the controlled flow is a function of the demand w so that $Bu(t) = w(t)$.

For the simple notations we work under the following assumption.

Assumption 7. The nominal average "demand" is zero, i.e. $\bar{w} = Av[w] = 0 \in \mathcal{W}$.

Then we can translate the problem by writing the new model

$$\dot{x}(t) = B(u(t) - u_0) - [w(t) - \bar{w}] = B\delta u(t) - \delta w(t)$$

and by translating the constraints as

$$u^- - u_0 \leq \delta u(t) \leq u^+ - u_0, \qquad \delta w(t) \in \mathcal{W} - \bar{w}.$$

where $Av[\delta w] = 0$.

Theorem 8. Under Assumption 1 and 2 let condition (5) be satisfied. Then there exists a static balancing strategy that achieves the average $Av[u] = 0$ whenever $Av[w] = 0$ if and only if there exists a "tall" matrix D $m \times n$ such that

$$BD = I \tag{8}$$

$$u^- \leq Dw^{(i)} \leq u^+, \quad i = 1, \ldots, s. \tag{9}$$

where $w^{(i)}$ are the vertices of $\mathcal{W}$. Moreover, if such necessary and sufficient conditions are satisfied, then the static strategy is linear

$$u(t) = Dw(t). \tag{10}$$

PROOF. See the proof in (Bauso *et al.*, 2006). □

The previous theorem allows us to check a single candidate $\bar{u}$ we fixed to zero. We can now characterize the *set of achievable average flows*, namely the set of all vectors such that $Av[w] = 0$ implies $Av[u] = \bar{u} \in \mathcal{U}$.

Corollary 9. The set of all achievable average flows, provided that a suitable static balancing strategy is applied, is made up by all the vectors $\bar{u} \in ker[B]$ such that there exists a matrix D, $m \times n$, with

$$BD = I \tag{11}$$

$$u^- \leq Dw^{(i)} + \bar{u} \leq u^+, \quad i = 1, \ldots, s. \tag{12}$$

In this case the static strategy is affine

$$u = Dw + \bar{u}.$$

PROOF. It follows immediately from the theorem by applying the translation $u - \bar{u}$. □

We have seen that as long as a strategy achieving the average exists, this has to be linear (or affine taking into account possible translations on w).

4. DYNAMIC STRATEGIES

Here we show how to achieve an average flow by a dynamic stabilizing strategy. show, in the next subsection, that conditions (8) and (9) are sufficient for the existence of a dynamic ϵ–stabilizing strategy of the form

$$\begin{aligned} \dot{y}(t) &= f(y(t), x(t), w(t)) \\ u(t) &= g(y(t), x(t), w(t)). \end{aligned} \tag{13}$$

To provide results about necessity of (8) and (9) we need to better characterize the class of dynamic strategies by additional assumptions (see, e.g., (Bauso *et al.*, 2006)).

4.1 Sufficiency of the conditions

Let assumptions (8) and (9) be satisfied and consider the corresponding matrix D. Equation (8) means that D is a right inverse of B and it is a standard property of linear algebra that this is equivalent to the existence of two matrices C and F which "square" B and D producing two matrices inverse to each other, namely such that

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{14}$$

Consider the following augmented system

$$\begin{aligned} \dot{x}(t) &= Bu(t) - w(t) \\ \dot{y}(t) &= Cu(t). \end{aligned} \tag{15}$$

The additional dynamic variable $\dot{y}(t) = Cu(t)$ has the goal of keeping trace of the load unbalancing with respect to the desired average 0.

The first step is to show that under (8) and (9), the extended system (15) satisfies the stabilizability conditions (6) as well (in the extended state–space), precisely for all $w \in \mathcal{W}$ there exists $u \in \mathcal{U}$ such that

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} u,$$

or equivalently that, for all $w \in \mathcal{W}$, there exists $u \in \mathcal{U}$ such that

$$u = \left[\, D \;\; F \,\right] \begin{bmatrix} w \\ 0 \end{bmatrix} = Dw.$$

The existence of such u is an immediate consequence of (9). Indeed, it is easy to verify that, if $\mathcal{W} \in int\{B\mathcal{U}\}$, then the u which corresponds to w is in the interior of the extended set. Then the problem can be solved as follows.

- Determine D such that (8) and (9) are satisfied.
- Determine C and F such that (14) is satisfied.
- Design a control which stabilizes (15).

Observe that Theorem 5 applied to the extended system (15) guarantees the existence of such a stabilizing control.

Here we propose a new strategy based on a variable transformation. In the following we exploit (for the first time) the structure of the set $\mathcal{U}$. Consider the new variable $z(t)$ defined as

$$z(t) = \left[\, D \;\; F \,\right] \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} z(t)$$

This variable satisfies the equation

$$\dot{z}(t) = u(t) - Dw(t). \tag{16}$$

The new system (16) is decoupled in its state variable, precisely it is equivalent to

$$\dot{z}_i(t) = u_i(t) - D_i w(t), \tag{17}$$

where we have denoted by D_i the ith row of D and where $u_i^- \le u_i \le u_i^+$. Denote by

$$\rho_i^- = \min_{w \in \mathcal{W}} \; D_i w,$$
$$\rho_i^+ = \max_{w \in \mathcal{W}} \; D_i w,$$

The stabilizability conditions are equivalent to the fact that for all $w \in \mathcal{W}$

$$u_i^- < \rho_i^- < \rho_i^+ < u_i^+.$$

Henceforth, without restriction, we consider the single–buffer case, namely the scalar system

$$\dot{z}(t) = u(t) - r(t),$$

with

$$\rho^- \le r(t) \le \rho^+, \quad u^- \le u(t) \le u^+.$$

Define the saturated control (see Fig. 1)

$$u(t) = sat_{[u^-,u^+]}(-\kappa z(t)) \tag{18}$$

with $\kappa > 0$ and where

$$sat_{[\alpha,\beta]}(\zeta) = \begin{cases} \beta, & \text{if } \; \zeta > \beta, \\ \zeta, & \text{if } \; \alpha \le \zeta \le \beta, \\ \alpha, & \text{if } \; \zeta < \alpha. \end{cases}$$

We will use the same notation (18) for the multi–input control derived applying the formula component–by–component. Note that this control

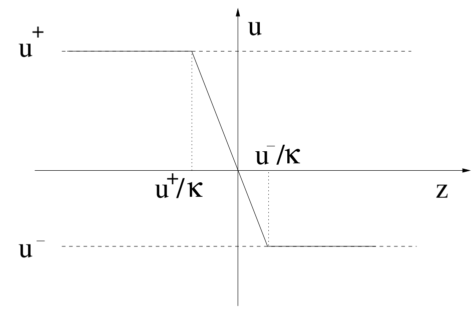

Fig. 1. The function (18)

function is Lipschitz continuous. For $\kappa \to \infty$, the control (18) converges to the bang bang control

$$bb_{[u^-,u^+]}(\zeta) = \begin{cases} u^+, & \text{if } \; \zeta > 0, \\ 0, & \text{if } \; \zeta = 0, \\ u^-, & \text{if } \; \zeta < 0, \end{cases}$$

which is of the type considered in (Blanchini *et al.*, 2000).

Theorem 10. The variable $z(t)$ with the control (18) converges to the interval $[-u^+/\kappa, -u^-/\kappa]$ (which includes 0 as an interior point). Therefore the global system converges to the corresponding hyper–box (i.e. that delimited by $-u_i^+/\kappa \le z_i \le -u_i^-/\kappa$, $i = 1, 2, \ldots, m$).

PROOF. The proof derives from the fact that, for $z \ge -u^-/\kappa$, we have that the control is saturated to its lower level $u = u^-$, then

$$\dot{z} = u^- - r \le u^- - \rho^- < 0. \tag{19}$$

Conversely for $z \le -u^+/\kappa$ we have that $u = u^+$, then

$$\dot{z} = u^+ - r \ge u^+ - \rho^+ > 0. \tag{20}$$

Therefore $z(t)$ reaches the interval in finite time and is ultimately confined in it. □

As a consequence of the previous theorem we have that, choosing κ large enough, we can bound z in an arbitrarily small interval. Therefore we achieve ϵ–stability. We have now to show that the controller so obtained satisfies the average requirement. Indeed variable $z(t)$ remains bounded so

arcs	1	2	3	4	5	6	7	8	9
upper bounds	3	2	3	3	3	3	3	5	5

Table 1. Controlled flows constraints

nodes	1	2	3	4	5
upper bounds	0	2	3	2	2
averages	0	1	2	1	1

Table 2. Demand bounds

$\|z(t) - z(0)\| \leq \xi$. By integrating (16) we have that

$$\frac{1}{T}\int_0^T u(t)dt - \frac{1}{T}\int_0^T Dw(t)dt = \frac{z(T)-z(0)}{T} \to 0$$

as $T \to \infty$. This yields

$$Av[u] = Av[Dw],$$

that is all we need to claim that sufficiency of (8) and (9) is proved.

Example 11. Let us solve Problem 6 for the system depicted in Fig. 2 (B is then the incidence matrix of the network). Table 1 summarizes the

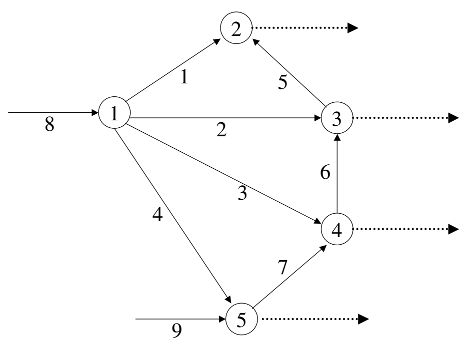

Fig. 2. Example of a system with 5 nodes and 9 arcs.

controlled upper flows constraints (the lower constraints are all set to 0) whereas Table 2 the demand bounds and the long–term average demands. Now, given the nominal demand $\bar{w} = [0\ 1\ 2\ 1\ 1]$ and the nominal balancing flow $\bar{u} = [1\,1\,1\,0\,0\,1\,1\,3\,2]' \in \mathcal{U}$ (which is $\bar{w} = B\bar{u}$) we have to determine whether $\bar{u}$ is an achievable average flow, namely, it is such that if $Av[w] = \bar{w}$ then $Av[u] = \bar{u}$. A possible matrix $D = \hat{D}$ satisfying (8)(9) is

$$\hat{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ -0.1 & 0 & 0.5 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0 \\ 0.6 & 1 & 1 & 0 & 0 \\ 0.4 & 0 & 0 & 1 & 1 \end{bmatrix}. \qquad (21)$$

We simulate the system with dynamic strategy (18) (we initialize $x(0) = 0$, $y(0) = 0$, and set $\kappa = 4$). Fig. 3 displays the average flow $Av[\delta u(t)]$ and the variable $z(t)$. In agreement with the expected results, the simulated average flow $Av[u]$ tends to the prescribed average flow $\bar{u}$ and the variable z converges to the interval $[-\delta u^+/\kappa, -\delta u^-/\kappa]$ Fig. 4 shows that the fluctuations of the buffer lengths are confined within a pre-specified neighborhoods of 0. (we remind that $x = 0$ is the desired buffer level and thus negative values do not necessarily imply backlog).

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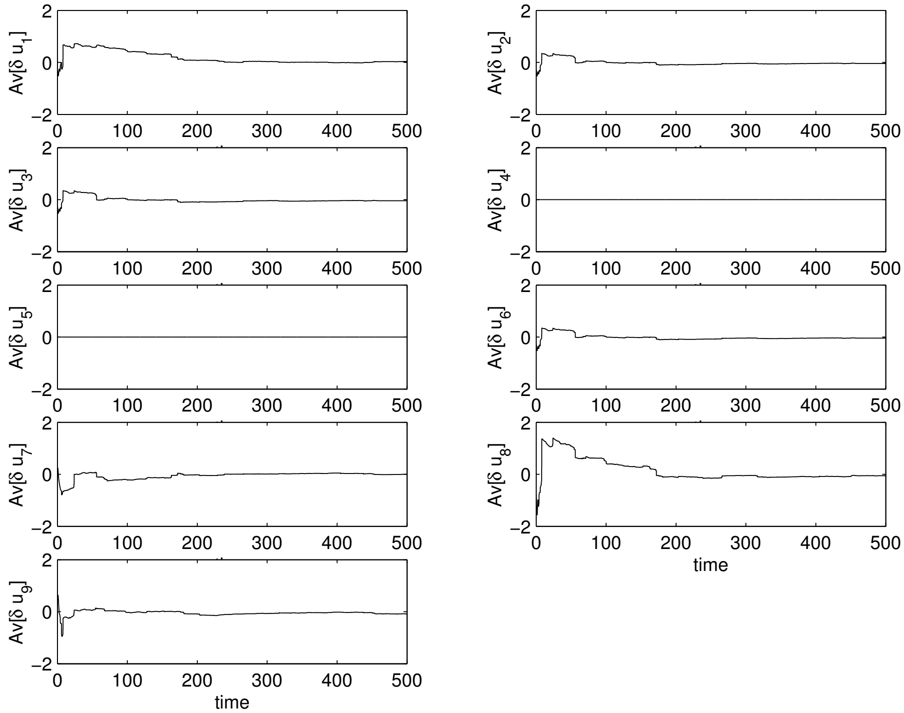

Fig. 3. The average flow $Av[\delta u]$ with dynamic strategy (18) and $\kappa = 4$.

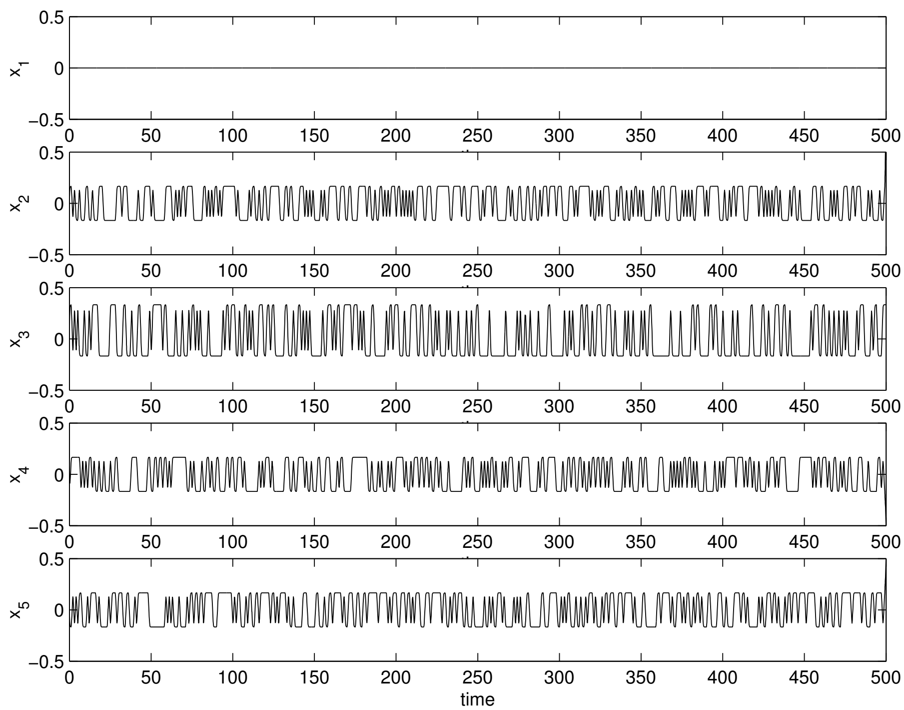

Fig. 4. The buffer length x with dynamic strategy (18) and $\kappa = 4$.