CONVERGENT DESIGN OF SWITCHED LINEAR SYSTEMS

R.A. van den Berg A.Y. Pogromsky J.E. Rooda

Eindhoven University of Technology
Department of Mechanical Engineering
P.O.Box 513, 5600 MB Eindhoven, The Netherlands.
[*r.a.v.d.berg, a.pogromsky, j.e.rooda*] *@tue.nl*

Abstract: This paper deals with the design of switching rules for switched linear systems with inputs, in such a way that the resulting closed-loop system is exponentially convergent. Two types of switching rules are addressed, that is state-based and observer-based rules. The developed theory is illustrated by two examples.

Keywords: Convergent systems, switched systems, continuous time systems, exponential stability, performance evaluation, observer.

1. INTRODUCTION

A switched linear system is a hybrid/nonlinear system which consists of several linear subsystems and a switching rule that decides which of the subsystems is active at each moment in time. These systems have been a subject of growing interest in the last decades, see e.g. (Liberzon and Morse, 1999; DeCarlo *et al.*, 2000) and references therein. Because of the combination of multiple linear systems/controllers, a well-tuned switched linear system can achieve better performance then a single linear system, or can achieve certain control goals that cannot be realized by linear systems (Morse, 1996; Narendra and Balakrishnan, 1997; Feuer *et al.*, 1997).

Besides these extended possibilities that switched linear systems have with respect to linear systems, the design of such a switched system also brings along difficulties. For example, if all the linear subsystems of a switched system are stable, this does not automatically guarantee the stability of that switched system. A good example of this apparent contradiction is given in (Branicky, 1998). Another property that a linear time invariant (LTI) system with asymptotically stable homogeneous part has, but is not natural for a nonlinear/hybrid system, is that any solution of an LTI system with a bounded input converges to a unique solution that depends only on the input. Nonlinear/hybrid system that *do* possess this property are referred to as convergent. Solutions of convergent system "forget" their initial conditions and after some transient depend only on the system input, which can be a command or reference signal.

Convergency of nonlinear/hybrid systems is an interesting property, since it results in a limit solution that is independent of the initial conditions of the system. This is useful in for example synchronization problems (Pogromsky *et al.*, 2002). Another possible area of interest is the performance analysis of nonlinear systems. For general nonlinear systems simulation-based analysis is quite impossible, since all possible initial conditions need to be evaluated in order to obtain a reliable analysis. For a convergent system, however, this problem does not exist, since all initial conditions lead to the same limit solution. Therefore simulation can be used to analyse and optimize performance of convergent systems. This motivates studies related to the design of convergent systems.

The property that all solutions of a system "forget" their initial conditions and converge to some steady-state solution has been addressed in a number of publications, e.g. (Fromion *et al.*, 1996; Lohmiller and Slotine, 1998; Fromion *et al.*, 1999; Pavlov *et al.*, 2004; Angeli, 2002; Pavlov *et al.*, 2005*b*). In this paper, the focus lies on the convergent design of switched linear systems using only the switching rule as "design variable". Two different cases are considered. First, the case is considered in which the switching rule is based on static state feedback. Secondly, the case is considered in which full state information is not available. In this case a switching rule is discussed that is based on an observer.

The outline of this paper is as follows. In Section 2 a basic definition on stability is recalled that is required in the remainder of this article. Section 3 presents various definitions and properties of convergent systems. In Section 4 the design of a switching rule is discussed that makes the closed-loop switched linear system convergent. The main results of this section are presented in two theorems which give sufficient conditions under which such a switching rule can be found. Two examples are provided in Section 5 to illustrate these theorems. Section 6 concludes the paper.

2. PRELIMINARIES

In this article exponential stability will be considered. For the sake of completeness, this definition is given here.

Definition 1. A solution $\mathbf{x}(t, t_0, \bar{\mathbf{x}}_0)$ of a system $\dot{\mathbf{x}} = f(\mathbf{x}, t)$, defined for all $t \in (t_*, +\infty)$, is said to be *exponentially stable* if there exist positive δ, C, β such that $||\mathbf{x}_0 - \bar{\mathbf{x}}_0|| < \delta$ implies

$$||\mathbf{x}(t, t_0, \mathbf{x}_0) - \mathbf{x}(t, t_0, \bar{\mathbf{x}}_0)|| \leq Ce^{-\beta(t-t_0)}||\mathbf{x}_0 - \bar{\mathbf{x}}_0||$$

3. CONVERGENT SYSTEMS

In this section definitions and properties of convergent systems are presented. Those systems are very closely related to systems with globally exponentially stable solutions and the definitions presented here extend those given by Demidovich (Demidovich, 1967).

The following class of systems is considered

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{w}(t)) \tag{1}$$

with state $\mathbf{x} \in \mathbb{R}^n$ and input $\mathbf{w} \in \overline{\mathbb{PC}}_m$. Here, $\overline{\mathbb{PC}}_m$ is the class of bounded (for all $t \in \mathbb{R}$) piecewise continuous inputs $\mathbf{w}(t) : \mathbb{R} \to \mathbb{R}^m$. Assume that the function $f(\mathbf{x}, \mathbf{w})$ satisfies some regularity conditions to ensure the existence of a Filippov solution, see e.g. (Filippov, 1988), p.76.

Definition 2. System (1) is said to be *exponentially convergent* if there is a solution $\bar{\mathbf{x}}(t) = \mathbf{x}(t, t_0, \bar{\mathbf{x}}_0)$ satisfying the following conditions for every input $\mathbf{w}(t) \in \overline{\mathbb{PC}}_m$: (i) $\bar{\mathbf{x}}(t)$ is defined and bounded for all $t \in (-\infty, +\infty)$, (ii) $\bar{\mathbf{x}}(t)$ is globally exponentially stable for every input $\mathbf{w}(t) \in \overline{\mathbb{PC}}_m$.

The solution $\bar{\mathbf{x}}(t)$ is called a *limit solution*. As follows from the definition of convergency, any solution of a convergent system "forgets" its initial condition and converges to some limit solution which is independent of the initial conditions. For exponentially convergent systems this limit solution $\bar{\mathbf{x}}(t)$ is unique, i.e. it is the only solution defined and bounded for all $t \in (-\infty, +\infty)$ (Pavlov *et al.*, 2005*a*).

For system (1) consider a scalar continuously differentiable function $V(\mathbf{x})$. Define a time derivative of this function along solutions of system (1) as follows

$$\dot{V} = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \dot{\mathbf{x}}(t, t_0, \mathbf{x}_0) \quad \text{a.e.}$$

Definition 3. System (1) is called quadratically convergent if there exists a positively definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a number $\alpha > 0$ such that for any input $\mathbf{w} \in \overline{\mathbb{PC}}_m$ for the function $V(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_1 - \mathbf{x}_2)^T \mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2)$ it holds that

$$\dot{V}(\mathbf{x}_1, \mathbf{x}_2, t) \leq -\alpha V(\mathbf{x}_1, \mathbf{x}_2). \tag{2}$$

Lemma 4. (Pavlov *et al.*, 2005*a*) If system (1) is quadratically convergent, then it is exponentially convergent.

The proof of this lemma is based on the following result, which will be also used in the sequel.

Lemma 5. (Yakubovich, 1964) Consider system (1) with a given input $\mathbf{w}(t)$ defined for all $t \in \mathbb{R}$. Let $\mathcal{D} \subset \mathbb{R}^n$ be a compact set which is positively invariant with respect to dynamics (1).Then there is at least one solution $\bar{\mathbf{x}}(t)$, such that $\bar{\mathbf{x}}(t) \in \mathcal{D}$ for all $t \in (-\infty, +\infty)$.

Note that for convergent nonlinear systems performance can be evaluated in almost the same way as for linear systems. Due to the fact that the limit solution of a convergent system only depends on the input and is independent of the initial conditions, performance evaluation of one solution (i.e. one arbitrary initial state) for a certain input suffices, whereas for general nonlinear systems all initial states need to be evaluated to obtain a reliable analysis. This means that for convergent systems simulation becomes a reliable analysis tool and for example 'Bode-like' plots can be drawn to analyse the system performance. An example of simulation based performance analysis can be found in Section 5.1.

4. CONVERGENCY OF SWITCHED SYSTEMS

Consider the switched dynamical system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{w}(t) \\ \mathbf{y}(t) &= \mathbf{C}_i\mathbf{x}(t) \end{aligned} \quad i = 1, \ldots, k \tag{3}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{w}(t) \in \overline{\mathbb{PC}}_m$ is the input, and $\mathbf{y}(t) \in \mathbb{R}^l$ is the output. These dynamics for example represent the system in Figure 1. Suppose the collection of matrices $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$, $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$, and $\{\mathbf{C}_1, \ldots, \mathbf{C}_k\}$ is given, and $\mathbf{A}_i$ is Hurwitz for all $i = 1, \ldots, k$. This implies for the system in Figure 1 that the plant and all linear controllers are already fixed. The general problem is to find a *switching rule* such that the closed-loop system is exponentially convergent. In this section, two kinds of switching rules are discussed. First, a switching rule is addressed that is based on static state feedback, i.e. $i = \sigma(\mathbf{x}, \mathbf{w})$. Secondly, the case is considered in which not the entire state can be measured, but just some output $\mathbf{y}$. For this case a switching rule is discussed that is based on an observer.

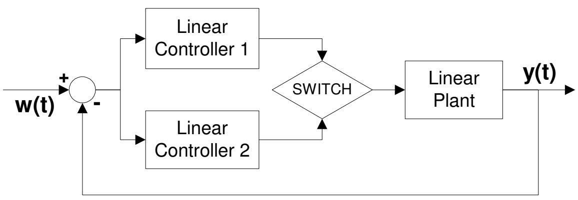

Fig. 1. Switched linear system.

4.1 Switching rule based on state feedback

Suppose a common Lyapunov matrix $\mathbf{P} = \mathbf{P}^T > 0$ exists that satisfies the following inequalities

$$\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i < 0, \quad i = 1, \ldots, k. \tag{4}$$

Consider the following switching rule

$$\sigma(\mathbf{x}, \mathbf{w}) = \arg \min_i \{\mathbf{x}^T\mathbf{Z}_{ix}\mathbf{x} + \mathbf{x}^T\mathbf{Z}_{iw}\mathbf{w}\} \tag{5}$$

in which $\mathbf{Z}_{iw} = 4\mathbf{P}\mathbf{B}_i$ and $\mathbf{Z}_{ix}$ are matrices to be defined.

Theorem 6. If there exist a solution $\mathbf{P} = \mathbf{P}^T > 0$ of (4) and $\mathbf{Z}_{1x}, \ldots, \mathbf{Z}_{kx}$ such that

$$\mathbf{Z}_{ix} \neq \mathbf{Z}_{jx} \text{ and/or } \mathbf{Z}_{iw} \neq \mathbf{Z}_{jw} \quad \forall i, j \leq k, \; i \neq j \tag{6}$$

and for some $\varepsilon > 0$

$$\begin{aligned} &\begin{bmatrix} \mathbf{P}\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P} & -(\mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_j) \\ -(\mathbf{A}_j^T\mathbf{P} + \mathbf{P}\mathbf{A}_i) & \mathbf{P}\mathbf{A}_j + \mathbf{A}_j^T\mathbf{P} \end{bmatrix} \\ &\quad + \begin{bmatrix} -(\mathbf{Z}_{ix} - \mathbf{Z}_{jx}) & 0 \\ 0 & \mathbf{Z}_{ix} - \mathbf{Z}_{jx} \end{bmatrix} \\ &\quad \leq -\varepsilon \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \forall i, j \leq k, \; i \neq j \end{aligned} \tag{7}$$

then switching rule (5) with matrices $\mathbf{Z}_{1x}, \ldots, \mathbf{Z}_{kx}$ makes system (3) quadratically convergent.

Proof: Let $\mathbf{P}$ be a common Lyapunov matrix for the collection $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ and consider the Lyapunov function candidate

$$V(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x}_1 - \mathbf{x}_2)^T\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) \tag{8}$$

If $\sigma(\mathbf{x}_1, \mathbf{w}) = \sigma(\mathbf{x}_2, \mathbf{w})$ the inequality

$$\dot{V} \leq -\alpha V, \quad \alpha > 0$$

is obviously satisfied. Let $\sigma(\mathbf{x}_1, \mathbf{w}) = p$ and $\sigma(\mathbf{x}_2, \mathbf{w}) = q$.

$$\begin{aligned} \dot{V} &= \mathbf{x}_1^T(\mathbf{A}_p^T\mathbf{P} + \mathbf{P}\mathbf{A}_p)\mathbf{x}_1 + \mathbf{x}_2^T(\mathbf{A}_q^T\mathbf{P} + \mathbf{P}\mathbf{A}_q)\mathbf{x}_2 \\ &\quad - \mathbf{x}_1^T(\mathbf{A}_p^T\mathbf{P} + \mathbf{P}\mathbf{A}_q)\mathbf{x}_2 - \mathbf{x}_2^T(\mathbf{P}\mathbf{A}_p + \mathbf{A}_q^T\mathbf{P})\mathbf{x}_1 \\ &\quad + 2\mathbf{x}_1^T\mathbf{P}(\mathbf{B}_p - \mathbf{B}_q)\mathbf{w} + 2\mathbf{x}_2^T\mathbf{P}(\mathbf{B}_q - \mathbf{B}_p)\mathbf{w} \end{aligned} \tag{9}$$

The switching rule (5) implies the following constraint functions for mode p

$$S_1(\mathbf{x}, \mathbf{w}) = \mathbf{x}_1^T(\mathbf{Z}_{px} - \mathbf{Z}_{qx})\mathbf{x}_1 + \mathbf{x}_1^T(\mathbf{Z}_{pw} - \mathbf{Z}_{qw})\mathbf{w} \leq 0$$

and for mode q

$$S_2(\mathbf{x}, \mathbf{w}) = \mathbf{x}_2^T(\mathbf{Z}_{qx} - \mathbf{Z}_{px})\mathbf{x}_2 + \mathbf{x}_2^T(\mathbf{Z}_{qw} - \mathbf{Z}_{pw})\mathbf{w} \leq 0.$$

The system is quadratically convergent if for some $\varepsilon > 0$

$$\dot{V} \leq -\varepsilon \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

for all $(\mathbf{x}, \mathbf{w})$ that satisfy $S_1(\mathbf{x}, \mathbf{w}) \leq 0$ and $S_2(\mathbf{x}, \mathbf{w}) \leq 0$. Using the $\mathcal{S}$-procedure, the previous condition is satisfied if the following inequality holds

$$\dot{V} - S_1 - S_2 \leq -\varepsilon \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{I}_n & -\mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

This inequality is equivalent to (7). □

Remark 7. Note that (7) is an LMI with design variables $\mathbf{P}$ and $\mathbf{Z}_{1x}, \ldots, \mathbf{Z}_{kx}$, which can be solved efficiently using available LMI toolboxes.

4.2 Observer-based switching rule

Consider the observer for system (3)

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_i\hat{\mathbf{x}}(t) + \mathbf{B}_i w(t) + \mathbf{L}_i\mathbf{C}_i(\mathbf{x} - \hat{\mathbf{x}}) \tag{10}$$

with $i = 1, \ldots, k$, $\hat{\mathbf{x}}$ the estimate of state $\mathbf{x}$ and $\mathbf{L}_i \in \mathbb{R}^{n \times l}$ the observer gain matrix. Assume that a common Lyapunov matrix exists such that (4) is satisfied. Now consider the observer-based switching rule

$$\sigma(\hat{\mathbf{x}}, \mathbf{w}) = \arg \min_i \{\hat{\mathbf{x}}^T \mathbf{Z}_{ix} \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{Z}_{iw} \mathbf{w}\} \quad (11)$$

in which $\mathbf{Z}_{iw} = 4\mathbf{P}\mathbf{B}_i$ and $\mathbf{Z}_{ix}$ are matrices to be defined.

Theorem 8. If there exist a solution $\mathbf{P} = \mathbf{P}^T > 0$ of (4) and $\mathbf{Z}_{1x}, \ldots, \mathbf{Z}_{kx}$ such that conditions (6) and (7) are satisfied, *and* if there exist a $\mathbf{P}_2 = \mathbf{P}_2^T > 0$ and $\mathbf{L}_i$ for $i = 1, \ldots, k$, such that for all $i = 1, \ldots, k$

$$(\mathbf{A}_i - \mathbf{L}_i\mathbf{C}_i)^T \mathbf{P}_2 + \mathbf{P}_2(\mathbf{A}_i - \mathbf{L}_i\mathbf{C}_i) < 0 \quad (12)$$

then switching rule (5) with matrices $\mathbf{Z}_{1x}, \ldots, \mathbf{Z}_{kx}$ makes system (3) exponentially convergent.

Proof: First it is proven that the state $\mathbf{x}(t)$ of system (3) either lies in a positive invariant compact set or converges exponentially in time to this set. Consider the Lyapunov function

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P} \mathbf{x}$$

Since there exists a common $\mathbf{P}$ such that (4) is satisfied, it follows that

$$\dot{V}(\mathbf{x}) \leq -\alpha V + \beta^* |\mathbf{x}||\mathbf{w}| \leq -\alpha V + \beta\sqrt{V}$$

for some positive constants α, β^*, and β, and bounded input $\mathbf{w} \in \overline{\mathbb{PC}}_m$. Note that there exists a level set

$$\Omega = \left\{ \mathbf{x} \;\middle|\; V(\mathbf{x}) \leq \frac{\beta^2}{\alpha^2} \right\}$$

outside of which $\dot{V} < 0$. This implies that all initial $V(\mathbf{x}(0))$ within this level set remain within the set. All $V(\mathbf{x}(0))$ outside this set converge exponentially to this set as can be seen from

$$\dot{V} \leq -\alpha V + \beta\sqrt{V} \leq -\frac{1}{2}\alpha \left(V - \frac{\beta^2}{\alpha^2} \right)$$

Since V is a quadratic function of $\mathbf{x}(t)$, it can be concluded that $\mathbf{x}(t)$ also converges exponentially to the positively invariant compact set Ω.

Secondly it is proven that the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ decreases exponentially towards zero as $t \to \infty$ if (12) holds for all $i = 1, \ldots, k$. Since both the observer (10) and system (3) use the same switching rule (11) the error dynamics become

$$\dot{\mathbf{e}} = \begin{cases} (\mathbf{A}_1 - \mathbf{L}_1\mathbf{C}_1)\mathbf{e} & \text{for } \sigma(\hat{\mathbf{x}}, \mathbf{w}) = 1 \\ \vdots & \\ (\mathbf{A}_k - \mathbf{L}_k\mathbf{C}_k)\mathbf{e} & \text{for } \sigma(\hat{\mathbf{x}}, \mathbf{w}) = k \end{cases}$$

If there exists a common Lyapunov matrix $\mathbf{P}_2$ for all $(\mathbf{A}_i - \mathbf{L}_i\mathbf{C}_i)$, $i = 1, \ldots, k$, i.e. condition (12) is satisfied, then the equilibrium point $\mathbf{e} = 0$ is globally exponentially stable.

Finally consider the Lyapunov function and its derivative given in respectively (8) and (9). Let $\sigma(\hat{\mathbf{x}}_1, \mathbf{w}) = p$ and $\sigma(\hat{\mathbf{x}}_2, \mathbf{w}) = q$. The observer-based switching rule (11) implies the following constraint functions for mode p

$$S_1(\hat{\mathbf{x}}, \mathbf{w}) = \hat{\mathbf{x}}_1^T(\mathbf{Z}_{px} - \mathbf{Z}_{qx})\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_1^T(\mathbf{Z}_{pw} - \mathbf{Z}_{qw})\mathbf{w} \leq 0$$

and for mode q

$$S_2(\hat{\mathbf{x}}, \mathbf{w}) = \hat{\mathbf{x}}_2^T(\mathbf{Z}_{qx} - \mathbf{Z}_{px})\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_2^T(\mathbf{Z}_{qw} - \mathbf{Z}_{pw})\mathbf{w} \leq 0.$$

Substituting $\hat{\mathbf{x}}_i$ by $\mathbf{x}_i - \mathbf{e}_i$ gives

$$S_1(\hat{\mathbf{x}}_1, \mathbf{w}) = S_1(\mathbf{x}_1, \mathbf{w}) + S_1(\mathbf{e}_1, \mathbf{w}) - f(\mathbf{e}_1, \mathbf{x}_1),$$
$$S_2(\hat{\mathbf{x}}_2, \mathbf{w}) = S_2(\mathbf{x}_2, \mathbf{w}) + S_2(\mathbf{e}_1, \mathbf{w}) + f(\mathbf{e}_2, \mathbf{x}_2),$$

with

$$f(\mathbf{e}_i, \mathbf{x}_i) = \mathbf{x}_i^T(\mathbf{Z}_{qx} - \mathbf{Z}_{px})\mathbf{e}_i + \mathbf{e}_i^T(\mathbf{Z}_{qx} - \mathbf{Z}_{px})\mathbf{x}_i.$$

Subsequently, the $\mathcal{S}$-procedure is applied to obtain

$$\dot{V} - S_1(\hat{\mathbf{x}}_1, \mathbf{w}) - S_2(\hat{\mathbf{x}}_2, \mathbf{w}) \leq -\alpha_1 V + g(..)$$

with

$$g(..) = -S_1(\mathbf{e}_1, \mathbf{w}) - S_2(\mathbf{e}_1, \mathbf{w}) + f(\mathbf{e}_1, \mathbf{x}_1) - f(\mathbf{e}_2, \mathbf{x}_2)$$

Since $\mathbf{e}_i(t)$ tends exponentially towards zero as $t \to \infty$, $\mathbf{x}_i(t)$ lies in Ω or converges exponentially in time towards this set for $i = 1, 2$, and $\mathbf{w}(t)$ is bounded, function g tends exponentially towards zero as a function of time. Thus, using the switching rule (11) the following inequality is true

$$\dot{V} \leq -\alpha_1 V + \gamma e^{-\alpha_2 t}$$

where α_1, α_2, γ are some positive constants. This implies that $V(\mathbf{x}_1(t) - \mathbf{x}_2(t))$ reduces exponentially towards zero as $t \to \infty$ and therefore that system (3) is exponentially convergent. This completes the proof. □

Remark 9. Since there exists a common $\mathbf{P}$ for all $\mathbf{A}_i$, $i = 1, \ldots, k$, condition (12) can always be met (take $\mathbf{L}_i = 0$).

5. TWO EXAMPLES

The theory presented in the previous section is now illustrated my means of two examples. For both examples the system in Figure 1 is considered, of which the dynamics is given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_i \mathbf{x} + \mathbf{B}_i w(t), \quad i = 1, 2 \\ y &= \mathbf{C}\mathbf{x} \end{aligned} \quad (13)$$

with $\mathbf{x}(t) \in \mathbb{R}^3$ the state, $w(\cdot) \in \overline{\mathbb{PC}}_1$ the input, and

$$\mathbf{A}_1 = \begin{bmatrix} -5 & -8 & 3 \\ 10 & -2 & 0 \\ 9 & -1 & -6 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 14 \\ -6 \\ 7 \end{bmatrix},$$
$$\mathbf{A}_2 = \begin{bmatrix} -8 & -5 & -8 \\ 13 & -8 & 2 \\ -2 & 1 & -4 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 20 \\ -16 \\ 8 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

In the first example this system is made quadratically convergent using a state dependent switching rule. For the obtained convergent system the performance is analyzed and compared to the performance of the corresponding linear systems. In the second example an observer-based switching rule is used to render a system exponentially convergent, when only the output y can be measured.

5.1 Performance of a convergent switched system

Consider system (13) with the given matrices. Using an LMI toolbox the following common Lyapunov matrix can be found

$$\mathbf{P} = \begin{bmatrix} 0.1973 & -0.0179 & 0.0073 \\ -0.0179 & 0.1653 & -0.0149 \\ 0.0073 & -0.0149 & 0.1932 \end{bmatrix} > 0$$

such that conditions (6) and (7) are satisfied, using $\mathbf{Z}_{ix} = \mathbf{A}_i^T\mathbf{P} + \mathbf{P}\mathbf{A}_i,\ i = 1, 2$. Switching rule (5) thus makes the system (13) quadratically convergent,

$$\begin{aligned} \dot{V} &\le -6.6643(\mathbf{x}_1 - \mathbf{x}_2)^T\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) \\ &\le -6.6643V. \end{aligned}$$

Subsequently, the fact that

$$\begin{aligned} (\mathbf{x}_1 - \mathbf{x}_2)^T\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) &\ge \lambda_{\min}(\mathbf{P})(\mathbf{x}_1 - \mathbf{x}_2)^2 \\ (\mathbf{x}_1 - \mathbf{x}_2)^T\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) &\le \lambda_{\max}(\mathbf{P})(\mathbf{x}_1 - \mathbf{x}_2)^2 \end{aligned}$$

with $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ respectively the minimum and maximum eigenvalue of $\mathbf{P}$, leads to the following upper bound

$$\begin{aligned} |\mathbf{x}_1(t) - \mathbf{x}_2(t)| &\le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\ |\mathbf{x}_1(0) - \mathbf{x}_2(0)|\ e^{\frac{-6.6643}{2}t} \\ &\le 1.3885\ |\mathbf{x}_1(0) - \mathbf{x}_2(0)|\ e^{-3.3321t}. \end{aligned} \tag{14}$$

In order to analyse the performance of this switched system, only one solution of the system needs to be evaluated, since the limit solution of this (convergent) system is independent of its initial conditions. In Figure 2 the performance of the switched system is compared with the performance of the two corresponding linear systems, i.e., $\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1 w(t)$ and $\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2 w(t)$. The performance measure applied here is the relative tracking error of the limit solution

$$\sqrt{\frac{\int_{t_l}^{t_l+T} (w(t) - y(t))^2\, dt}{\int_{t_l}^{t_l+T} w(t)^2 dt}}, \tag{15}$$

where T is a time period that is long enough to obtain a good average of the tracking error and t_l is a moment in time for which all considered solutions are close enough to the limit solution. The time t_l is in this example determined visually, but a bound can be calculated as well using (14).

The performance is evaluated for the following input signals

$$w(t) = \sin(bt), \quad b \in [10^{-2}, 10^3].$$

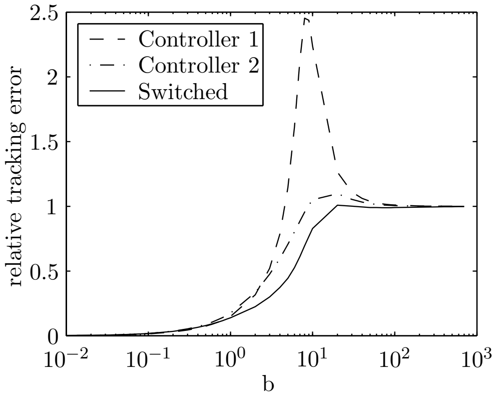

Fig. 2. Performance of switched system

From Figure 2 it can be concluded that for the considered performance measure (15) the switched system performs better than the linear systems for the input range $b \in [10^0, 10^2]$. This means that besides improvement of transient behaviour (see e.g. (Feuer *et al.*, 1997)), the use of switched control instead of linear control can sometimes provide better stationary behaviour.

5.2 Convergency using observer-based switching

In this example the effect of observer-based switching as opposed to state-based switching is shown. Consider again system (13) with the given matrices and consider (10) with gain matrices

$$\mathbf{L}_1 = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 5 \\ 10 \\ -10 \end{bmatrix}$$

which are chosen in such a way that condition (12) is satisfied for some $\mathbf{P}_2$. Thereby all conditions for Theorem 8 are satisfied, which implies that that system (13) with observer-based switching is exponentially convergent for any initial condition $\mathbf{x}(0)$, any initial estimation error $\mathbf{e}(0) = \mathbf{x}(0) - \hat{\mathbf{x}}(0)$ and any input $w \in \overline{\mathbb{PC}}_1$. In Figure 3 the convergency of the system output y is visualized for $\mathbf{x}(0) = [0; 0; 0]$, $w = \sin(5t)$, and several initial estimation errors $\mathbf{e}(0) = \{[10; 10; 10], [100; -100; 0], [-100, 0, 100]\}$ (respectively the dashed ,dash-dotted, and dotted line). Furthermore, the output of the system with state-based switching is plotted (solid line) to make a comparison with the observer-based switching. Note that only the transient solution is influenced by the choice of switching, the limit solution is identical for both types of switching. Therefore, if the performance analysis of Section 5.1 would

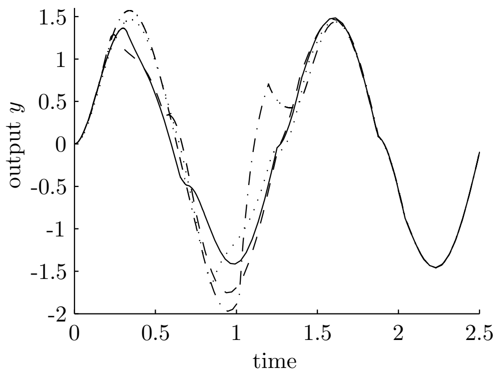

Fig. 3. Observer-based vs. state-based switching

be repeated for the same system but now with observer-based switching, then the results would be identical to those in Figure 2.

6. CONCLUSION

In this paper the following design problem for switched linear systems has been considered: under which conditions is it possible to design a switching rule that makes the resulting closed-loop system convergent? Two cases have been considered: state feedback and output feedback. Sufficient conditions have been found that guarantee the existence of a state-based switching rule which renders the closed-loop system quadratically convergent. For the case with output feedback, the switching rule is based on the state of an observer of the system, and sufficient conditions have been found for the existence of a switching rule that makes the closed-loop system exponentially convergent. By means of an example it has been illustrated that a simulation-based performance evaluation is feasible for a convergent switched system. In the example it has also been indicated that switched control can provide better limit behaviour than linear control.

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