GLOBAL INPUT-TO-STATE STABILITY AND STABILIZATION OF DISCRETE-TIME PIECE-WISE AFFINE SYSTEMS

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Abstract: This paper presents sufficient conditions for global Input-to-State (practical) Stability (ISpS) and stabilization of discrete-time, possibly discontinuous, Piece-Wise Affine (PWA) systems. Piece-wise quadratic candidate ISpS (ISS) Lyapunov functions are employed for both analysis and synthesis purposes. This enables us to obtain sufficient conditions based on linear matrix inequalities, which can be solved efficiently. One of the advantages of using the ISpS framework is that the additive disturbance inputs are explicitly taken into account in the analysis and synthesis procedures, and the results apply to PWA systems in their full generality, i.e. non-zero affine terms are allowed in the regions in the partition whose closure contains the origin.

Keywords: Piece-wise affine systems, Hybrid systems, Robust stability, Stabilization methods.

1. INTRODUCTION

Several results on nominal stability analysis of Piece-Wise Affine (PWA) systems are available in the literature, see for example (Mignone *et al.*, 2000), (Ferrari-Trecate *et al.*, 2002), (Feng, 2002) for results in discrete-time. These works employ the Lyapunov stability framework and consider Piece-Wise Quadratic (PWQ) candidate Lyapunov functions. Recently, in (Lazar and Heemels, 2006) the authors showed that nominally exponentially stable discrete-time PWA systems can have zero robustness to arbitrarily small additive disturbances, mainly due to the absence of a continuous Lyapunov function. Therefore, in discrete-time, it is important that disturbances are taken into account when analyzing stability of PWA systems, since robustness is relevant for practical applications.

Robust stability results for discrete-time PWA systems were presented in (Ferrari-Trecate *et al.*, 2002, Section 3), which deals with Linear Matrix Inequalities (LMI) based l_2-gain analysis for PWA systems; and in (Grieder, 2004, Chp. 8.5), where it was observed that, if a robust positively invariant set can be calculated for a nominally asymptotically stable PWA system, then *local* robust convergence is ensured. For *continuous-time* input-to-state stability (Sontag, 1989) results for switched systems and hybrid systems we refer the reader to the recent works (Vu *et al.*, 2005) and (Cai and Teel, 2005). However, to the best of the authors' knowledge, a global robust stability analysis methodology for *discrete-time* PWA systems that can be used for both analysis and synthesis purposes is missing from the literature.

As such, we consider discrete-time PWA systems subject to *unbounded* additive disturbance inputs and we employ the Input-to-State (practical) Stability (ISpS) framework (Sontag, 1989), (Jiang, 1993) in order to obtain *global* robust stability results. For simplicity

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and clarity of exposition, only PWQ candidate ISpS (ISS) Lyapunov functions are considered, but the results can also be extended *mutatis mutandis* to piece-wise polynomial or piece-wise affine candidate functions. The paper consists of two parts: the first part deals with ISpS (ISS) analysis, while the second part provides techniques for input-to-state stabilizing controllers synthesis. In both sections the sufficient conditions for ISpS (ISS) are expressed in terms of LMIs, which can be solved efficiently (Boyd *et al.*, 1994).

One of the advantages of using the ISpS (ISS) framework for studying robust stability of discrete-time PWA systems is that the additive disturbance inputs are explicitly taken into account in the analysis and synthesis procedures. Also, the ISpS framework enables us to obtain robust stability results for PWA systems in their full generality, i.e. non-zero affine terms are allowed in the regions in the state-space partition whose closure contains the origin. Note that this situation is often excluded in other works. In this paper we develop a new LMI technique for dealing with non-zero affine terms, which does not rely on a system transformation and the *S*-procedure, e.g. as done in (Ferrari-Trecate *et al.*, 2002, Remark 3). This technique makes it possible to obtain LMI based sufficient conditions for input-to-state stabilizing controllers synthesis as well, and not just for analysis.

1.1 Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c}$ to denote the set $\{k \in \mathbb{Z}_+ \mid k \geq c\}$ for some $c \in \mathbb{Z}_+$. Let $\|\cdot\|$ denote the Euclidean norm. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| \triangleq \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its induced Euclidean norm. For a positive definite matrix $Z \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and the largest eigenvalue of Z, respectively. For a sequence $\{z_p\}_{p\in\mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$ let $\|\{z_p\}_{p\in\mathbb{Z}_+}\| \triangleq \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$. Let $z_{[k]}$ denote the truncation of $\{z_p\}_{p\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k],p} = z_p,\ p \leq k$. For a set $\mathscr{P} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{P}$ the boundary of $\mathscr{P}$, by $\mathrm{int}(\mathscr{P})$ its interior and by $\mathrm{cl}(\mathscr{P})$ its closure. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathscr{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot,k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$.

2. INPUT-TO-STATE STABILITY AND PROBLEM STATEMENT

Consider the discrete-time autonomous perturbed nonlinear system described by

$$x_{k+1} = G(x_k, v_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $v_k \in \mathbb{R}^{d_v}$ is an unknown *disturbance input* and $G : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero disturbance input, meaning that $G(0,0) = 0$.

Next, we define the notions of Input-to-State practical Stability (ISpS) (Jiang, 1993), (Jiang *et al.*, 1996) and Input-to-State Stability (ISS) (Sontag, 1989), (Jiang and Wang, 2001) for the discrete-time perturbed nonlinear system (1).

Definition 1. The system (1) is said to be *globally ISpS* if there exist a $\mathscr{KL}$-function β, a $\mathscr{K}$-function γ and a non-negative constant d such that, for each $x_0 \in \mathbb{R}^n$ and all $\{v_p\}_{p\in\mathbb{Z}_+}$ with $v_p \in \mathbb{R}^{d_v}$ for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2}$$

If the above condition holds for $d = 0$, the system (1) is said to be *globally ISS*.

In what follows we state a *discrete-time* version of the *continuous-time* ISpS sufficient conditions of Proposition 2.1 of (Jiang *et al.*, 1996). This result will be used throughout the paper to establish ISpS and ISS for the particular case of PWA systems. For the proof we refer the reader to (Lazar *et al.*, 2006).

Theorem 2. Let d_1, d_2 be non-negative constants, let a, b, c, λ be positive constants with $c \leq b$ and let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ and $\sigma \in \mathscr{K}$. Furthermore, let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1 \tag{3a}$$

$$V(G(x,v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2 \tag{3b}$$

for all $x \in \mathbb{R}^n$ and all $v \in \mathbb{R}^{d_v}$. Then it holds that:

(i) The system (1) is globally ISpS;

(ii) If inequalities (3) hold for $d_1 = d_2 = 0$, the system (1) is globally ISS.

Definition 3. A function V that satisfies the hypothesis of Theorem 2 is called an *ISpS (ISS) Lyapunov function*.

Remark 4. The hypothesis of Theorem 2 allows that both G and V are discontinuous. If inequality (3a) holds for $d_1 = 0$, then the hypothesis of Theorem 2 *only* implies continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$.

In this paper we focus on perturbed discrete-time, possibly discontinuous, PWA systems of the form

$$x_{k+1} = G(x_k, v_k) \triangleq A_j x_k + f_j + D_j v_k \text{ if } x_k \in \Omega_j, \tag{4}$$

where $A_j \in \mathbb{R}^{n\times n}$, $f_j \in \mathbb{R}^n$, $D_j \in \mathbb{R}^{n\times d_v}$ for all $j \in \mathscr{S}$ and $\mathscr{S} \triangleq \{1, 2, \ldots, s\}$ is a *finite set* of indexes. The collection $\{\Omega_j \mid j \in \mathscr{S}\}$ defines a partition of $\mathbb{R}^n$, meaning that $\cup_{j\in\mathscr{S}}\Omega_j = \mathbb{R}^n$ and $\mathrm{int}(\Omega_i)\cap\mathrm{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron. Let $\mathscr{S}_0 \triangleq \{j \in \mathscr{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$, $\mathscr{S}_1 \triangleq \{j \in \mathscr{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$ and let $\mathscr{S}_{\mathrm{aff}} \triangleq \{j \in \mathscr{S} \mid f_j \neq 0\}$, $\mathscr{S}_{\mathrm{lin}} \triangleq \{j \in \mathscr{S} \mid f_j = 0\}$, so that $\mathscr{S}_0 \cup \mathscr{S}_1 = \mathscr{S}_{\mathrm{aff}} \cup \mathscr{S}_{\mathrm{lin}} = \mathscr{S}$.

The aim of this paper is to derive sufficient conditions for global ISpS and global ISS, respectively, of system (4). In order to do so, we consider PWQ candidate ISpS (ISS) functions of the form

$$V : \mathbb{R}^n \to \mathbb{R}_+,\ V(x) = x^\top P_j x \ \text{ if } \ x \in \Omega_j, \qquad (5)$$

where P_j, $j \in \mathscr{S}$, are positive definite and symmetric matrices. It is easy to observe that V satisfies condition (3a) with $\alpha_1(\|x\|) \triangleq \min_{j\in\mathscr{S}} \lambda_{\min}(P_j)\|x\|^2$, $\alpha_2(\|x\|) \triangleq \max_{j\in\mathscr{S}} \lambda_{\max}(P_j)\|x\|^2$ and $d_1 = 0$.

3. ANALYSIS

In this section we present LMI based sufficient conditions for global ISpS (ISS) of system (4). Let Q be a known positive definite and symmetric matrix and let γ_1, γ_2 be known positive numbers with $\gamma_1\gamma_2 > 1$. For any $(j,i) \in \mathscr{S}\times\mathscr{S}$ consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} \Xi_{ji} & -A_j^\top P_i & -A_j^\top P_i \\ -P_iA_j & \gamma_1 P_i & -P_i \\ -P_iA_j & -P_i & \gamma_2 P_i \end{pmatrix} > 0, \qquad (6)$$

where

$$\Xi_{ji} \triangleq P_j - A_j^\top P_i A_j - E_j^\top U_{ji} E_j - Q - M_{ji}.$$

The matrix E_j, $j \in \mathscr{S}$, defines the cone $\mathscr{C}_j \triangleq \{x \in \mathbb{R}^n \mid E_j x \geq 0\}$ that satisfies $\Omega_j \subseteq \mathscr{C}_j$. The role of these matrices is to introduce an S-procedure relaxation (Johansson and Rantzer, 1998). The unknown variables in (6) are the matrices P_j, $j \in \mathscr{S}$, which are required to be positive definite and symmetric, the matrices U_{ji}, $(j,i) \in \mathscr{S}\times\mathscr{S}$, which are required to have non-negative elements, and the matrices M_{ji}, $(j,i) \in \mathscr{S}_{\mathrm{aff}}\times\mathscr{S}$, which are required to be positive definite and symmetric. For all $(j,i) \in \mathscr{S}_{\mathrm{lin}}\times\mathscr{S}$ we take $M_{ji} = 0$. For any $(j,i) \in \mathscr{S}_{\mathrm{aff}}\times\mathscr{S}$, define

$$\mathscr{E}_{ji} \triangleq \{x \in \mathbb{R}^n \mid x^\top M_{ji} x < (1+\gamma_1) f_j^\top P_i f_j\}.$$

Theorem 5. Let system (4), the matrix $Q > 0$ and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ be given. Suppose that the LMIs

$$\Delta_{ji} > 0, \quad (j,i) \in \mathscr{S}\times\mathscr{S} \qquad (7)$$

are feasible. Then, it holds that:

(i) The system (4) is globally ISpS;

(ii) If[2] $(\cup_{i\in\mathscr{S}}\mathscr{E}_{ji}) \cap \Omega_j = \emptyset$ for all $j \in \mathscr{S}_{\mathrm{aff}}$, then system (4) is globally ISS;

(iii) If system (4) is Piece-Wise Linear (PWL), i.e. $\mathscr{S}_{\mathrm{lin}} = \mathscr{S}$, then system (4) is globally ISS.

[2] Note that this implies $\mathscr{S}_0 \subseteq \mathscr{S}_{\mathrm{lin}}$.

PROOF. The proof consists in showing that V, as defined in (5), is an ISpS (ISS) Lyapunov function.

(i) As by the hypothesis $\Delta_{ji} > 0$ for all $(j,i) \in \mathscr{S}\times\mathscr{S}$, it follows that:

$$\begin{pmatrix} x^\top & f_j^\top & (D_j v)^\top \end{pmatrix} \Delta_{ji} \begin{pmatrix} x \\ f_j \\ D_j v \end{pmatrix} \geq 0,$$

for all $x \in \Omega_j$, $(j,i) \in \mathscr{S}\times\mathscr{S}$ and all $v \in \mathbb{R}^{d_v}$. The above inequality yields:

$$\begin{aligned}&(A_jx + f_j + D_jv)^\top P_i (A_jx + f_j + D_jv) - x^\top P_j x \\ &\leq -x^\top Q x + (1+\gamma_2)(D_jv)^\top P_i (D_jv) - x^\top E_j^\top U_{ji} E_j x + \\ &(1+\gamma_1) f_j^\top P_i f_j - x^\top M_{ji} x \leq -\lambda_{\min}(Q)\|x\|^2 + \\ &(1+\gamma_2)\max_{i\in\mathscr{S}} \lambda_{\max}(P_i) \max_{j\in\mathscr{S}} \|D_j\|^2\|v\|^2 + \\ &(1+\gamma_1)\max_{i\in\mathscr{S}} \lambda_{\max}(P_i) \max_{j\in\mathscr{S}} \|f_j\|^2. \qquad (8)\end{aligned}$$

Hence,

$$V(A_jx + f_j + D_jv) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2$$

for all $x \in \Omega_j$, $(j,i) \in \mathscr{S}\times\mathscr{S}$ and all $v \in \mathbb{R}^{d_v}$, where

$$\begin{aligned}\alpha_3(\|x\|) &\triangleq \lambda_{\min}(Q)\|x\|^2, \\ \sigma(\|v\|) &\triangleq (1+\gamma_2)\max_{i\in\mathscr{S}} \lambda_{\max}(P_i) \max_{j\in\mathscr{S}} \|D_j\|^2\|v\|^2, \\ d_2 &\triangleq (1+\gamma_1)\max_{i\in\mathscr{S}} \lambda_{\max}(P_i) \max_{j\in\mathscr{S}} \|f_j\|^2.\end{aligned}$$

From (6) we also have that for all $(j,i) \in \mathscr{S}\times\mathscr{S}$, $\Delta_{ji} > 0 \Rightarrow \Xi_{ji} > 0 \Rightarrow x^\top(P_j - Q)x \geq 0$ for all $x \in \Omega_j$. Then, it follows that for all $j \in \mathscr{S}$ and all $x \in \Omega_j$:

$$\lambda_{\min}(Q)\|x\|^2 \leq x^\top Q x \leq x^\top P_j x \leq \max_{j\in\mathscr{S}} \lambda_{\max}(P_j)\|x\|^2,$$

which yields $\lambda_{\min}(Q) \triangleq c \leq b \triangleq \max_{j\in\mathscr{S}} \lambda_{\max}(P_j)$. Hence, the function V defined in (5) satisfies the hypothesis of Theorem 2 with $d_1 = 0$ and $d_2 = (1+\gamma_1)\max_{i\in\mathscr{S}} \lambda_{\max}(P_i)\max_{j\in\mathscr{S}} \|f_j\|^2$. Then, the statement follows from Theorem 2.

(ii) To establish global ISS, we need to prove that in the above setting, we obtain $d_2 = 0$ under the additional hypothesis. For $j \in \mathscr{S}_{\mathrm{lin}}$, if $x \in \Omega_j$ we obtain $d_2 = 0$ due to $f_j = 0$. For any $j \in \mathscr{S}_{\mathrm{aff}}$, if $x \in \Omega_j$ it holds that $x \notin \cup_{i\in\mathscr{S}}\mathscr{E}_{ji}$. This yields:

$$(1+\gamma_1) f_j^\top P_i f_j - x^\top M_{ji} x \leq 0,$$

and thus, from the first inequality in (8) it follows that the function V defined in (5) satisfies the hypothesis of Theorem 2 with $d_1 = d_2 = 0$. Then, the statement follows from Theorem 2.

(iii) This is a special case of part (ii). □

The matrix Q gives the gain of the $\mathscr{K}$-function α_3 and is related to the decrease of the state norm, and hence, to the transient behavior. If ISpS (ISS) is the only goal, Q can be chosen less positive definite to reduce conservativeness of the LMI (7). The numbers γ_1, γ_2 and the matrices $\{P_j \mid j \in \mathscr{S}\}$ yield the constant

$d_2 = (1+\gamma_1)\max_{i\in\mathscr{S}}\lambda_{\max}(P_i)\max_{j\in\mathscr{S}}\|f_j\|^2$ and the gain of the $\mathscr{K}$-function

$$\sigma(s) = (1+\gamma_2)\max_{i\in\mathscr{S}}\lambda_{\max}(P_i)\max_{j\in\mathscr{S}}\|D_j\|^2 s^2.$$

Note that a necessary condition for feasibility of the LMI (7) is $\gamma_1\gamma_2 > 1$. As it would be desirable to obtain a constant d_2 and gain of the function σ as small as possible, one has to make a trade-off in choosing γ_1 and γ_2. One could add a cost criterion to (7) and specify γ_1, γ_2 as unknown variables in the resulting optimization problem, which might solve the trade-off. Although in this case (7) is a bilinear matrix inequality (i.e. due to $\gamma_1 P_i$, $\gamma_2 P_i$), since the unknowns γ_1, γ_2 are scalars, this problem can be solved efficiently via semi-definite programming solvers (software), e.g. (Sturm, 2001), (Löfberg, 2002), by setting lower and upper bounds for γ_1, γ_2 and doing bisections.

Remark 6. If the disturbance inputs are bounded, which is a reasonable assumption in practice, it can be proven that ISpS implies global ultimate boundedness. This means that the ISpS property also implies the usual robust stability (convergence) property, e.g. as the one defined in (Grieder, 2004, Chp. 8.5), while the result of Theorem 5 part (i) applies to a more general class of PWA systems.

4. SYNTHESIS

In this section we address the problem of input-to-state (practically) stabilizing controllers synthesis for perturbed discrete-time non-autonomous PWA systems:

$$x_{k+1} = g(x_k, u_k, v_k) \triangleq A_j x_k + B_j u_k + f_j + D_j v_k \quad \text{if } x_k \in \Omega_j, \tag{9}$$

where $u_k \in \mathbb{R}^m$ is the control input and $B_j \in \mathbb{R}^{n\times m}$ for all $j \in \mathscr{S}$. The nomenclature in (9) is similar with the one used in Section 2 for system (4).

In this paper we take the control input as a PWL state-feedback control law of the form:

$$u_k \triangleq h(x_k) \triangleq K_j x_k \quad \text{if} \quad x_k \in \Omega_j, \tag{10}$$

where $K_j \in \mathbb{R}^{m\times n}$ for all $j \in \mathscr{S}$. The aim is to calculate the feedback gains $\{K_j \mid j \in \mathscr{S}\}$ such that the PWA closed-loop system (9)-(10) is globally ISpS and ISS, respectively. For this purpose we make use again of PWQ candidate ISpS (ISS) Lyapunov functions of the form (5).

For any $(j,i) \in \mathscr{S}\times\mathscr{S}$, consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} \Delta_{ji}^{11} & \Delta_{ji}^{12} \\ \Delta_{ji}^{21} & \Delta_{ji}^{22} \end{pmatrix} > 0, \tag{11}$$

where

$$\Delta_{ji}^{11} \triangleq \begin{pmatrix} Z_j & * & * \\ -(A_jZ_j + B_jY_j) & \gamma_1 Z_i & -Z_i \\ -(A_jZ_j + B_jY_j) & -Z_i & \gamma_2 Z_i \end{pmatrix},$$

the term $*$ denotes $-(A_jZ_j + B_jY_j)^\top$ and, for $j \in \mathscr{S}_{\text{aff}}$

$$\Delta_{ji}^{22} \triangleq \text{diag}\left(\left[\begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}, \begin{pmatrix} N_{ji} & 0 & 0 \\ 0 & N_{ji} & 0 \\ 0 & 0 & N_{ji} \end{pmatrix}\right]\right),$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} \triangleq \begin{pmatrix} (A_jZ_j + B_jY_j)^\top & 0 & 0 & Z_j & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while for $j \in \mathscr{S}_{\text{lin}}$,

$$\Delta_{ji}^{22} \triangleq \text{diag}\left(\left[\begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}\right]\right),$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} \triangleq \begin{pmatrix} (A_jZ_j + B_jY_j)^\top & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The operator $\text{diag}([L_1, \ldots, L_n])$ denotes a diagonal matrix of appropriate dimensions with the matrices $L_1, \ldots, L_n$ on the main diagonal, and the element 0 denotes everywhere a zero matrix of appropriate dimensions. The unknown variables in (11) are the matrices $Z_j \in \mathbb{R}^{n\times n}$, $j \in \mathscr{S}$, which are required to be positive definite and symmetric, the matrices $Y_j \in \mathbb{R}^{m\times n}$, $j \in \mathscr{S}$, and the matrices N_{ji}, $(j,i) \in \mathscr{S}_{\text{aff}}\times\mathscr{S}$, which are required to be positive definite and symmetric. The matrix Q is a known positive definite and symmetric matrix and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ play the same role as described in Section 3. For any $(j,i) \in \mathscr{S}_{\text{aff}}\times\mathscr{S}$, define

$$\mathscr{E}_{ji} \triangleq \{x \in \mathbb{R}^n \mid x^\top N_{ji}^{-1} x < (1+\gamma_1) f_j^\top P_i f_j\}.$$

Theorem 7. Let system (9), the matrix $Q > 0$ and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ be given. Suppose that the LMIs

$$\Delta_{ji} > 0, \quad (j,i) \in \mathscr{S}\times\mathscr{S} \tag{12}$$

are feasible and let $\{Z_j, Y_j \mid j \in \mathscr{S}\}$ and $\{N_{ji} \mid (j,i) \in \mathscr{S}_{\text{aff}}\times\mathscr{S}\}$ be a solution. For all $j \in \mathscr{S}$ let $P_j \triangleq Z_j^{-1}$ and let $K_j \triangleq Y_j Z_j^{-1}$. For all $(j,i) \in \mathscr{S}_{\text{lin}}\times\mathscr{S}$ take $M_{ji} = 0$. For all $(j,i) \in \mathscr{S}_{\text{aff}}\times\mathscr{S}$ take $M_{ji} = N_{ji}^{-1}$. Then, it holds that:

(i) The closed-loop system (9)-(10) is globally ISpS;

(ii) If $(\cup_{i\in\mathscr{S}}\mathscr{E}_{ji}) \cap \Omega_j = \emptyset$ for all $j \in \mathscr{S}_{\text{aff}}$, then the closed-loop system (9)-(10) is globally ISS;

(iii) If system (9) is PWL, i.e. $\mathscr{S}_{\text{lin}} = \mathscr{S}$, then the closed-loop system (9)-(10) is globally ISS.

PROOF. By applying the Schur complement (Boyd *et al.*, 1994) to (12), for any $(j,i) \in \mathscr{S}\times\mathscr{S}$ we obtain

$$\Delta_{ji}^{11} - \Delta_{ji}^{21\top}\Delta_{ji}^{22\,-1}\Delta_{ji}^{21} > 0,$$

which yields the equivalent matrix inequality:

$$\Phi_{ji} \triangleq \begin{pmatrix} \Gamma_{ji} & * & * \\ -(A_jZ_j+B_jY_j) & \gamma_1 Z_i & -Z_i \\ -(A_jZ_j+B_jY_j) & -Z_i & \gamma_2 Z_i \end{pmatrix} > 0, \quad (13)$$

where the term $*$ denotes $-(A_jZ_j+B_jY_j)^\top$ and

$$\Gamma_{ji} \triangleq Z_j - (A_jZ_j+B_jY_j)^\top Z_i^{-1}(A_jZ_j+B_jY_j) - Z_jQZ_j - Z_jN_{ji}^{-1}Z_j.$$

By pre- and post-multiplying (13) with $\begin{pmatrix} Z_j^{-1} & 0 & 0 \\ 0 & Z_i^{-1} & 0 \\ 0 & 0 & Z_i^{-1} \end{pmatrix}$ and by substituting Z_j^{-1} with P_j, $Y_jZ_j^{-1}$ with K_j and N_{ji}^{-1} with M_{ji} turns inequality (13) into the equivalent matrix inequality:

$$\begin{pmatrix} \Xi_{ji} & * & * \\ -P_i(A_j+B_jK_j) & \gamma_1 P_i & -P_i \\ -P_i(A_j+B_jK_j) & -P_i & \gamma_2 P_i \end{pmatrix} > 0,$$

for all $(j,i) \in \mathscr{S}\times\mathscr{S}$, where the term $*$ denotes $-(A_j+B_jK_j)^\top P_i$ and

$$\Xi_{ji} \triangleq P_j - (A_j+B_jK_j)^\top P_i(A_j+B_jK_j) - Q - M_{ji}.$$

Then, it follows that the LMI (7) is feasible for the closed-loop system (9)-(10) for all $(j,i) \in \mathscr{S}\times\mathscr{S}$. The rest of the proof is analogous to the proof of Theorem 5. □

5. ILLUSTRATIVE EXAMPLE

In this example we illustrate the result of Theorem 7 part (ii). Let

$$A(T_s) \triangleq \begin{pmatrix} 1 & T_s & \frac{T_s^2}{2!} & \frac{T_s^3}{3!} \\ 0 & 1 & T_s & \frac{T_s^2}{2!} \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B(T_s) \triangleq \begin{pmatrix} \frac{T_s^4}{4!} \\ \frac{T_s^3}{3!} \\ \frac{T_s^2}{2!} \\ T_s \end{pmatrix}$$

denote the dynamics corresponding to a discrete-time quadruple integrator, i.e. $x_{k+1} = A(T_s)x_k + B(T_s)u_k$, obtained from a continuous-time quadruple integrator via a sampled-and-hold device with sampling period $T_s > 0$. Let x_i, $i = 1,2,3,4$, denote the i-th component of the state vector. Let $\mathbb{X} \triangleq \{x \in \mathbb{R}^4 \mid -2 < x_4 < 2\}$, let $\Omega_1 \triangleq \{x \in \mathbb{R}^4 \mid x_4 \geq 2\}$ and let $\Omega_4 \triangleq \{x \in \mathbb{R}^4 \mid x_4 \leq -2\}$. Let $\Omega_2 \triangleq \{x \in \mathbb{X} \mid x_4 \geq 0\}$ and $\Omega_3 \triangleq \{x \in \mathbb{X} \mid x_4 < 0\}$. Consider now the following perturbed piecewise affine system:

$$x_{k+1} = \begin{cases} A_1x_k + B_1u_k + f_1 + D_1v_k & \text{if} \quad x_k \in \Omega_1 \\ A_2x_k + B_2u_k + f_2 + D_2v_k & \text{if} \quad x_k \in \Omega_2 \\ A_3x_k + B_3u_k + f_3 + D_3v_k & \text{if} \quad x_k \in \Omega_3 \\ A_4x_k + B_4u_k + f_4 + D_4v_k & \text{if} \quad x_k \in \Omega_4, \end{cases} \quad (14)$$

where $A_1 = A_4 = A(1.2), B_1 = B_4 = B(1.2)$, $A_2 = A(0.9), B_2 = B(0.9)$, $A_3 = A(0.8), B_3 = B(0.8)$, $f_2 = f_3 = 0$, $f_1 = f_4 = [0.1 0.1 0.1 0.1]^\top$ and $D_1 = D_2 = D_3 = D_4 = [1 1 1 1]^\top$. The LMIs (12) were solved[3] for $Q = 0.01I_4$, $\gamma_1 = 2$ and $\gamma_2 = 4$, yielding the following weights of the PWQ ISS Lyapunov function $V(x) = x^\top P_j x$ if $x \in \Omega_j$, $j = 1,2,3,4$, feedbacks $\{K_j \mid j = 1,2,3,4\}$ and matrix M:

$$P_1 = P_4 = \begin{bmatrix} 0.3866 & 0.7019 & 0.5532 & 0.1903 \\ 0.7019 & 1.5632 & 1.3131 & 0.4688 \\ 0.5532 & 1.3131 & 1.2255 & 0.4552 \\ 0.1903 & 0.4688 & 0.4552 & 0.1955 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.3574 & 0.6052 & 0.4420 & 0.1407 \\ 0.6052 & 1.2725 & 0.9894 & 0.3278 \\ 0.4420 & 0.9894 & 0.8812 & 0.3046 \\ 0.1407 & 0.3278 & 0.3046 & 0.1328 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.3779 & 0.6410 & 0.4597 & 0.1453 \\ 0.6410 & 1.3414 & 1.0298 & 0.3390 \\ 0.4597 & 1.0298 & 0.9007 & 0.3118 \\ 0.1453 & 0.3390 & 0.3118 & 0.1334 \end{bmatrix},$$

$$K_1 = K_4 = \begin{bmatrix} -0.3393 & -1.1789 & -1.8520 & -1.7028 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.5584 & -1.7607 & -2.4729 & -2.0012 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.6814 & -2.0895 & -2.8249 & -2.1705 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0156 & 0.0075 & 0.0023 & 0.0005 \\ 0.0075 & 0.0212 & 0.0082 & 0.0016 \\ 0.0023 & 0.0082 & 0.0146 & 0.0044 \\ 0.0005 & 0.0016 & 0.0044 & 0.0081 \end{bmatrix}.$$

One can easily establish that the hypothesis of Theorem 7 part (ii) is satisfied, i.e. $\mathscr{E}_{1i} \cap \Omega_1 = \emptyset$ and $\mathscr{E}_{4i} \cap \Omega_4 = \emptyset$ for all $i = 1,2,3,4$, by observing that

$$\begin{aligned} \min_{x\in\Omega_1} x^\top Mx &= \min_{x\in\Omega_4} x^\top Mx \\ &= 0.4340 \\ &> 0.3221 = \max_{i=1,2,3,4} (1+\gamma_1) f_1^\top P_i f_1 \\ &= \max_{i=1,2,3,4} (1+\gamma_1) f_4^\top P_i f_4. \end{aligned}$$

Hence, system (14) in closed-loop with (10) is globally ISS. The gain of the σ function corresponding to $\gamma_2 = 4$ is 15.8772. This yields an ISS gain equal to 42.52 for system (14)-(10) via the relation $\gamma(s) = \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right) = 42.52s$ (see (Lazar *et al.*, 2006) for details), where $\rho = \frac{c}{b} \in [0,1)$ and γ is the $\mathscr{K}$-function from (2). The closed-loop states trajectories obtained for initial state $x_0 = [6 6 4 4]^\top$ are plotted in Figure 1 together with the additive disturbance input history. The disturbance input was randomly generated in the interval $[0\ 1]$ until sampling time 60 and then it was set equal to zero. As guaranteed by Theorem 7, the closed-loop system (14)-(10) is globally ISS, which ensures asymptotic stability in the Lyapunov sense when the disturbance inputs converges to zero, as it can be observed in Figure 1.

[3] For simplicity we used a common matrix N for all possible mode transitions that can occur when the state is in mode one or mode four, i.e. $N = N_{11} = N_{12} = N_{13} = N_{44} = N_{42} = N_{43}$, which yields $M = N^{-1}$.

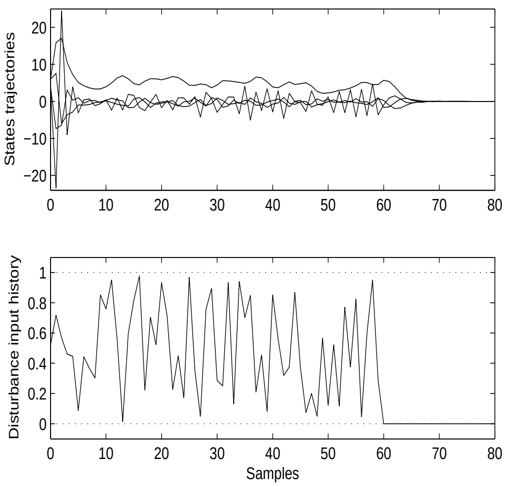

Fig. 1. States trajectories and disturbance input histories for the closed-loop system (14)-(10).

6. CONCLUDING REMARKS

In this paper we presented LMI based sufficient conditions for global input-to-state (practical) stability and stabilization of discrete-time perturbed, possibly discontinuous, PWA systems. The importance of these results cannot be overstated since, recently, in (Lazar and Heemels, 2006) the authors showed that nominally exponentially stable discrete-time PWA systems can have zero robustness to arbitrarily small additive disturbances and hence, special precautions must be taken when implementing stabilizing controllers for PWA systems in practice.

State and input constraints have not been considered in order to obtain global ISpS (ISS) results. However, the usual LMI techniques (Boyd *et al.*, 1994) for specifying state and/or input constraints can be added to the sufficient conditions presented in this paper, resulting in local ISpS (ISS) of constrained PWA systems. Also, a local (i.e. in some subset of $\cup_{j\in\mathscr{S}_0}\Omega_j$) ISS result is obtained under the hypothesis of Theorem 5 (Theorem 7) part (ii), in the case when $\mathscr{E}_{ji}\cap\Omega_j \neq \emptyset$ for some $(j,i)\in\mathscr{S}_{\text{aff}}\times\mathscr{S}$.

The future work deals with extensions to PWA systems affected by parametric uncertainties and the use of norm based candidate ISS Lyapunov functions.

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