SEARCH FOR PERIOD-2 CYCLES IN A CLASS OF HYBRID DYNAMICAL SYSTEMS WITH AUTONOMOUS SWITCHINGS. APPLICATION TO A THERMAL DEVICE

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Abstract: Our purpose is to complete the analysis of a class of hybrid dynamical systems with autonomous switchings generated by a hysteresis phenomenon. Because we yet have found limit period-1 cycles in paper (Quémard *et al.*, 2005*b*) and because we deal with nonlinear equations systems, the question of the existence of more than one solution for them and so of the existence of cycles with more than one period is rather natural. Equations system for period-2 cycles is determined and a notion of stability is studied. A realistic application to a thermostat with anticipative resistance comes to illustrate theoretical results.

Keywords: Hybrid Dynamical System, Hysteresis, Stability, Period-2 Cycles, Thermostat with Anticipative Resistance, Chaotic Behaviors.

1. INTRODUCTION

In (Quémard *et al.*, 2005*b*), we presented the process for the determination of limit cycles for a hybrid dynamical systems (h.d.s) class with autonomous switchings (see also (Bensoussan and Menaldi, 1997), (Van Der Schaft and Schumacher, 1999), (Zaytoon, 2001)) and we gave the used original method based on formal calculus and on interval analysis for the numerical solution that we applied to a thermal device (Quémard *et al.*, 2005*a*).

Thanks to the relative simplicity of our model, the problem of finding those limit cycles was reduced to the problem of determining the solution (which was unique in (Quémard *et al.*, 2005*b*)) of a non trivial implicit equations system.

But, sometimes, such nonlinear systems can also display two or more simultaneous solutions which generally involve the existence of cycles with more than one period. The varying initial conditions can cause a system to choose one stationary solution rather than other (Zhusubaliyev and Mosekilde, 2003).

Here, we propose to highlight and study this phenomenon determining by the same reasoning than in (Quémard *et al.*, 2005*b*), equations system to solve in order to study period-2 cycles and we highlight those results with the application to a thermostat with anticipative resistance. A main contribution of this paper is to deal with a realistic thermal application of industrial interest whose model, in dimension three, is relatively simple and a little less complex than, for example, the one proposed in (Zhusubaliyev and Mosekilde, 2003), in dimension four.

So, firstly, we present the studied h.d.s class and the equations system to solve in order to obtain period-2 cycles. A study of stability is made using the not original but not usual too *point transformation method of Andronov*. Then, we confirm obtained results with an application to a thermostat with anticipative resistance. We present its mathematical model, temperatures variations simulations and a bifurcation diagram made with *Matlab* which confirm the existence of a period-2 cycle. The study of the sensitivity to initial conditions and of the two parameters variations is also illustrated. We conclude this paper underlining links between properties of such systems illustrated in the numerical study and necessary properties to satisfy in the domain of chaotic behaviors.

2. STUDIED H.D.S CLASS

In $\mathbb{R}^N$, we consider a basis which, in practice, will be either the canonical basis or an eigenvectors basis, generalized or not, which can be useful with calculuses for example. In relation to this basis, we consider the following h.d.s of order N:

$$\dot{X}(t) = AX(t) + q(\xi(t))B + C, \quad \xi(t) = LX(t), \tag{1}$$

where A is a square matrix of order N and X, B, C are columns matrices of order N and L is a row matrix of order N, all these matrices having real entries.

Moreover, we suppose that matrix A has negative and real but not null eigenvalues and that X and so ξ are continuous. Variable q is discrete, taking value 0 or 1 according to ξ. It responds to the hysteresis phenomenon described in Figure 1.

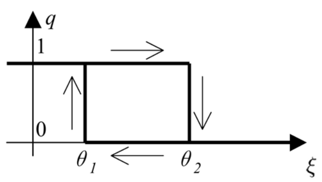

Fig. 1. Hysteresis variable

Values θ_1 and θ_2 are respectively the lower and the upper switching thresholds. Function $q(\xi)$, which is supposed right continuous, is given explicitly by:

$$\begin{cases} q(\xi(t)) = 1 - q(\xi(t_-)) \\ if \begin{cases} \xi(t_-) = \theta_1 \text{ and } q(\xi(t_-)) = 0 \\ \text{or } \xi(t_-) = \theta_2 \text{ and } q(\xi(t_-)) = 1 \end{cases} \\ q(\xi(t)) = q(\xi(t_-) \text{ otherwise} \end{cases}$$

In the first case, t is called switching time.

Finally, we introduce some notations which will be used later. Let $(U_n)_{n\in\mathbb{N}}$ be a suite. We set:

$$U_n^1 \triangleq U_{4n+1}\ ,\ U_n^2 \triangleq U_{4n-2}\ ,\ U_n^3 \triangleq U_{4n-1}\ ,\ U_n^4 \triangleq U_{4n}. \tag{2}$$

3. PERIOD-2 CYCLE EQUATIONS

In (Quémard *et al.*, 2005*b*), we have proved, as it seemed rather natural for such non linear systems (see (Girard, 2003), (Zhusubaliyev and Mosekilde, 2003)), the existence of limit period-1 cycles, that is to say the existence of periodic orbits with two different durations respectively between odd and even switching times.

Because we deal with non linear systems, it is also rather natural to think that there can exist more than one solution for such systems. That is why here, we present, following the same reasoning than in (Quémard *et al.*, 2005*b*), equations for period-2 cycles, that is to say for cycles with four different durations σ^1, σ^2, σ^3, σ^4 between switching times.

Let t_0 be a given initial time. Suite $t_1 < t_2 < ... < t_n < ...$ corresponds to the successive switching times in $]t_0, \infty[$ necessarily distinct since $X(t_{n+1}) \neq X(t_n)$. We suppose that following assumption $Hp(n)$ is implicitly satisfied in calculuses in which t_n appears.

Hypothesis *Hp(n) Instant t_n of the n^{th} switching (later to t_0) exists and is finite.*

Let us set in order to reduce notations $q_n \triangleq q(\xi(t_n))$, $\Delta q_n \triangleq q_n - q_{n-1}$. We have $q(\xi(t)) = q_n$ on $[t_n, t_{n+1}[$, $q_n \in \{0; 1\}$, $q_n = 1 - q_{n-1}$, $\Delta q_n = (-1)^{n-1}\Delta q_1$, $\Delta q_1 = \pm 1$ and with notations (2), we so have $q_n^1 = 1 - q_0$, $q_n^2 = q_0$, $q_n^3 = 1 - q_0$, $q_n^4 = q_0$.

Classical solution of (1) on $[t_n, t_{n+1}[$ gives:

$$X(t) = e^{(t-t_n)A}\Gamma_n - A^{-1}(q_nB + C), \tag{3}$$

with $\Gamma_n \in \mathbb{R}^N$ corresponding to the integration constants, functions of n. Applying the assumption of the state continuity at t_n, $X(t_n) = X(t_n-)$, equation (3) gives:

$$\forall n \geq 1 \quad \Gamma_n = e^{\sigma_n A}\Gamma_{n-1} + \Delta q_n A^{-1}B, \tag{4}$$

where we define $\forall n \geq 1$, $\sigma_n \triangleq t_n - t_{n-1} > 0$ the duration between two successive switching times t_{n-1} and t_n. For $n = 0$ and $t = t_0$, we obtain:

$$\Gamma_0 = X(t_0) + A^{-1}(q_0B + C). \tag{5}$$

Let $\forall n \geq 1$, $\xi_n = \xi(t_n)$. We remark that $\forall n \geq 1$, $\xi_n = q_n\theta_1 + q_{n-1}\theta_2$ and $\forall n \geq 2$, $\Delta\xi_n = -\Delta q_n\Delta\theta = (-1)^n\Delta q_1\Delta\theta$ with $\Delta\theta = \theta_2 - \theta_1$. Moreover, we have $\xi_n = LX(t_n-) = L(e^{\sigma_n A}\Gamma_{n-1} - A^{-1}(q_{n-1}B + C))$, and also $\xi_{n-1} = LX(t_{n-1}) = L(\Gamma_{n-1} - A^{-1}(q_{n-1}B + C))$. So, we obtain:

$$\begin{cases} \forall n \geq 2,\ L(e^{\sigma_n A} - I_N)\Gamma_{n-1} - \Delta\xi_n = 0 \\ \text{for } n = 1,\ L(e^{\sigma_1 A}\Gamma_0 - A^{-1}(q_0B + C)) - \xi_1 = 0. \end{cases} \tag{6}$$

As explained in (Quémard *et al.*, 2005*b*), it is from equations (4) and (6) that we can determine equations of cycles. Let us characterize a possible searched period-2 cycle with four different durations between switching times. With notations (2), let us set:

$$R_n \triangleq (\sigma_n^1, \Gamma_n^1, \sigma_n^2, \Gamma_n^2, \sigma_n^3, \Gamma_n^3, \sigma_n^4, \Gamma_n^4). \tag{7}$$

System of equations (4) and (6) $\forall n \geq 1$, is equivalent to system $H(R_n, R_{n+1}) = 0$, $\forall n$, $n \geq 1$ where $H = (H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8)^T$ is a function that we define by:

$$\begin{cases} H_1(R_n, R_{n+1}) = L(e^{\sigma^1_{n+1}A} - I_N)\Gamma^4_{n+1} + \Delta q_1 \Delta\theta \\ H_2(R_n, R_{n+1}) = \Gamma^1_{n+1} - e^{\sigma^1_{n+1}}\Gamma^4_{n+1} - \Delta q_1 A^{-1}B \\ H_3(R_n, R_{n+1}) = L(e^{\sigma^2_{n+1}A} - I_N)\Gamma^1_n - \Delta q_1 \Delta\theta \\ H_4(R_n, R_{n+1}) = \Gamma^2_{n+1} - e^{\sigma^2_{n+1}}\Gamma^1_n + \Delta q_1 A^{-1}B \\ H_5(R_n, R_{n+1}) = L(e^{\sigma^3_{n+1}A} - I_N)\Gamma^2_{n+1} + \Delta q_1 \Delta\theta \\ H_6(R_n, R_{n+1}) = \Gamma^3_{n+1} - e^{\sigma^3_{n+1}}\Gamma^2_{n+1} - \Delta q_1 A^{-1}B \\ H_7(R_n, R_{n+1}) = L(e^{\sigma^4_{n+1}A} - I_N)\Gamma^3_{n+1} - \Delta q_1 \Delta\theta \\ H_8(R_n, R_{n+1}) = \Gamma^4_{n+1} - e^{\sigma^4_{n+1}}\Gamma^3_{n+1} + \Delta q_1 A^{-1}B. \end{cases} \tag{8}$$

Let us suppose that R_n has a limit $R = (\sigma^1, \Gamma^1, \sigma^2, \Gamma^2, \sigma^3, \Gamma^3, \sigma^4, \Gamma^4)$. In those conditions, R is solution to system $H(R, R) = 0$, namely:

$$\begin{cases} L(e^{\sigma^1 A} - I_N)\Gamma^4 + \Delta q_1 \Delta\theta = 0 \\ \Gamma^1 - e^{\sigma^1}\Gamma^4 - \Delta q_1 A^{-1}B = 0 \\ L(e^{\sigma^2 A} - I_N)\Gamma^1 - \Delta q_1 \Delta\theta = 0 \\ \Gamma^2 - e^{\sigma^2}\Gamma^1 + \Delta q_1 A^{-1}B = 0 \\ L(e^{\sigma^3 A} - I_N)\Gamma^2 + \Delta q_1 \Delta\theta = 0 \\ \Gamma^3 - e^{\sigma^3}\Gamma^2 - \Delta q_1 A^{-1}B = 0 \\ L(e^{\sigma^4 A} - I_N)\Gamma^3 - \Delta q_1 \Delta\theta = 0 \\ \Gamma^4 - e^{\sigma^4}\Gamma^3 + \Delta q_1 A^{-1}B = 0. \end{cases} \tag{9}$$

From the second equation of (9) and using the 4^{th},the 6^{th} and the 8^{th} equations of the same system, we obtain:

$$\Gamma^1 = \Delta q_1 (I_N - e^{\sigma^1 A} + e^{(\sigma^1+\sigma^4)A} - e^{(\sigma^1+\sigma^3+\sigma^4)A})$$

$$(I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B.$$

Identically, the 4^{th},the 6^{th} and the 8^{th} equations of system (9) become:

$$\Gamma^2 = -\Delta q_1 (I_N - e^{\sigma^2 A} + e^{(\sigma^1+\sigma^2)A} - e^{(\sigma^1+\sigma^2+\sigma^4)A})$$

$$(I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B.$$

$$\Gamma^3 = \Delta q_1 (I_N - e^{\sigma^3 A} + e^{(\sigma^2+\sigma^3)A} - e^{(\sigma^1+\sigma^2+\sigma^3)A})$$

$$(I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B.$$

$$\Gamma^4 = -\Delta q_1 (I_N - e^{\sigma^4 A} + e^{(\sigma^3+\sigma^4)A} - e^{(\sigma^2+\sigma^3+\sigma^4)A})$$

$$(I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B.$$

Reinjecting those expressions for Γ^i, $i = 1, .., 4$, in the 1^{st}, 3^{rd}, 5^{th} and 7^{th} equations of (9), we can deduce:

$$\begin{cases} L(I_N - e^{\sigma^1 A}) \\ (I_N - e^{\sigma^4 A} + e^{(\sigma^3+\sigma^4)A} - e^{(\sigma^2+\sigma^3+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0 \\ \\ L(I_N - e^{\sigma^2 A}) \\ (I_N - e^{\sigma^1 A} + e^{(\sigma^1+\sigma^4)A} - e^{(\sigma^1+\sigma^3+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0 \\ \\ L(I_N - e^{\sigma^3 A}) \\ (I_N - e^{\sigma^2 A} + e^{(\sigma^1+\sigma^2)A} - e^{(\sigma^1+\sigma^2+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0 \\ \\ L(I_N - e^{\sigma^4 A}) \\ (I_N - e^{\sigma^3 A} + e^{(\sigma^2+\sigma^3)A} - e^{(\sigma^1+\sigma^2+\sigma^3)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0. \end{cases} \tag{10}$$

But system (10) is not independant because there exists a linear combination using the four equations (the 1^{st} equation minus the 2^{nd} is equal to the 4^{th} equation minus the 3^{rd}). So, here, it just remains three independant equations for four unknowns. The 4^{th} equation will come from the condition of initial switching defined by equation (6) and using equation (4) in the case $n = 1$ and given by:

$$L(\Gamma^1 - A^{-1}((1-q_0)B + C)) - (1-q_0)\theta_1 - q_0\theta_2 = 0.$$

And so, we obtain the final system:

$$\begin{cases} F_1 = L(I_N - e^{\sigma^1 A}) \\ (I_N - e^{\sigma^4 A} + e^{(\sigma^3+\sigma^4)A} - e^{(\sigma^2+\sigma^3+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0 \\ \\ F_2 = L(I_N - e^{\sigma^3 A}) \\ (I_N - e^{\sigma^2 A} + e^{(\sigma^1+\sigma^2)A} - e^{(\sigma^1+\sigma^2+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B + \Delta\theta = 0 \\ \\ F_3 = L(e^{\sigma^2 A}(I_N - e^{\sigma^1 A} + e^{(\sigma^1+\sigma^4)A}) \\ -e^{\sigma^4 A}(I_N - e^{\sigma^3 A} + e^{(\sigma^2+\sigma^3)A})) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B = 0 \\ \\ F_4 = \Delta q_1 L(I_N - e^{\sigma^1 A} + e^{(\sigma^1+\sigma^4)A} - e^{(\sigma^1+\sigma^3+\sigma^4)A}) \\ (I_N - e^{(\sigma^1+\sigma^2+\sigma^3+\sigma^4)A})^{-1}A^{-1}B \\ -LA^{-1}((1-q_0)B + C) - (1-q_0)\theta_1 - q_0\theta_2 = 0 \end{cases} \tag{11}$$

As proved in theorem 1 in (Quémard *et al.*, 2005*b*) for a period-1 cycle, suite $(\Gamma_n)_{n\geq 0}$ such that $\Gamma_{4p+1} \triangleq \Gamma^1$ for $p \geq 0$, $\Gamma_{4p-2} \triangleq \Gamma^2$ for $p \geq 1$, $\Gamma_{4p-1} \triangleq \Gamma^3$ for $p \geq 1$, $\Gamma_{4p} \triangleq \Gamma^4$ for $p \geq 1$ and suite $(\sigma_n)_{n\geq 0}$ such that $\sigma_{4p+1} \triangleq \sigma^1$ for $p \geq 0$, $\sigma_{4p-2} \triangleq \sigma^2$ for $p \geq 1$, $\sigma_{4p-11} \triangleq \sigma^3$ for $p \geq 1$, $\sigma_{4p} \triangleq \sigma^4$ for $p \geq 1$ define here a trajectory which is a period-2 cycle for $t \geq t_1$.

4. STABILITY STUDY

To study the period-2 cycles stability, we use, like for the period-1 cycles (see (Quémard *et al.*, 2005*b*)), an adapted *point transformation method of Andronov* (Meerov *et al.*, 1979) which is equivalent to the classical idea of the Poincaré map extended to h.d.s (Girard, 2003) what is highlighted in an appendix in (Quémard *et al.*, 2005*b*).

This method consists in a linearization of the problem in the neighbourhood of a fixed point and in an application of the $\mathcal{Z}$-transformation.

From (9), we have $H(R,R) = 0$ with $R = (\sigma^1, \Gamma^1, \sigma^2, \Gamma^2, \sigma^3, \Gamma^3, \sigma^4, \Gamma^4)$. We linearize the problem in the neighbourhood of R replacing $H(R_n, R_{n+1})$ made explicit in (8) by relation:

$$\frac{\partial H}{\partial R_n}(R,R)T_n + \frac{\partial H}{\partial R_{n+1}}(R,R)T_{n+1} = 0, \quad (12)$$

where $T_n = R_n - R$ and $T_{n+1} = R_{n+1} - R$.

Let us set $U \triangleq \frac{\partial H}{\partial R_n}(R,R)$ and $V \triangleq \frac{\partial H}{\partial R_{n+1}}(R,R)$. Let T^* be the z-transform of suite $(T_n)_{n\in\mathbb{N}}$ (We give an arbitrary value to $\sigma_0 = t_1 - t_0$ in order to have $T_0 = R_0 - R$ well defined). We have $\mathcal{Z}((T_{n+1})_{n\in\mathbb{N}}) = z(T^* - T_0)$. We obtain $UT^* + zV(T^* - T_0) = 0$ and so $T^* = (z^{-1}U + V)^{-1}VT_0$. Let us set $\Delta \triangleq |z^{-1}U + V| = \frac{1}{z^{|K|}}|U + zV|$ where $|K| = 4(N+1)$. Some calculuses lead to:

$$|U + zV| = z^{3N+5}$$

$$.det \begin{pmatrix} 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ M_1 & zI_n & 0 & 0 & 0 & 0 & 0 & M_2 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & M_3 & M_4 & I_N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & M_5 & M_6 & I_N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & 0 & 0 & M_7 & M_8 & I_N \end{pmatrix}$$

$$= z^{3N+5}\Delta',$$

where $M_1 = -Ae^{\sigma^1 A}\Gamma^4$, $M_2 = -e^{\sigma^1 A}$, $M_3 = -e^{\sigma^2 A}$, $M_4 = -Ae^{\sigma^2 A}\Gamma^1$, $M_5 = -e^{\sigma^3 A}$, $M_6 = -e^{\sigma^3 A}\Gamma^2$, $M_7 = -e^{\sigma^4 A}$ and $M_8 = -e^{\sigma^4 A}\Gamma^3$. We can deduce $\Delta = \frac{1}{z^{N-1}}\Delta'$. We remark that the coefficient of z^N in Δ' is null and so, Δ' is an at most $N-1$ degree polynomial. The stability notion corresponds here to the convergence of suite $(R_n)_{n\in\mathbb{N}}$ defined by (7) towards R, that is to say of suite $(T_n)_{n\in\mathbb{N}}$ towards zero. We can say that a cycle is asymptotically stable (see (Meerov *et al.*, 1979)) if all roots of determinant Δ' have moduli less than one, is unstable if there exists at least one root whose modulus is superior to one and we cannot conclude about the cycle stability if there exists one root whose modulus is equal to one and if the others have moduli less than one.

5. PRESENTATION OF A THERMOSTAT WITH ANTICIPATIVE RESISTANCE MODEL

Here, we apply all those previous results to a thermal application treated in (Cébron, 2000) for a problem of optimal control. It deals with a thermostat with anticipative resistance, common in the industrial market (Cyssau, 1990) which controls a convector located in the same room. Figure 2 gives a representation of the physical system and notations used further.

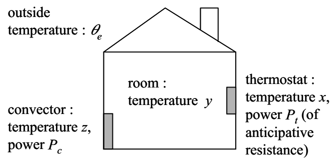

Fig. 2. Thermal process

The functioning principle of such thermostats is the following: powers P_c and P_t are active when $q(x) = 1$ and inactive when $q(x) = 0$. If initially $q = 1$, as P_t is active, the desired temperature is reached by the thermostat temperature x before the room temperature y which makes q changes its value from 1 to 0. Nevertheless, because of the thermal inertia of the room fluid, the room temperature y can eventually continue to increase before decreasing but surely less than without anticipative resistance since active time for P_c is reduced . With this principle, cycle times for the room temperature are shorter and so, as we can wait and verify by simulation or experience, the room temperature variations are reduced. This fact is of industrial interest for energy saving.

Here, we are in dimension $N = 3$ with $X = (x,y,z)^T$ and we consider the $\mathbb{R}^3$ canonical basis. A power assessment and Newton law (Saccadura, 1998) give the following set of equations:

$$\begin{cases} m_t C_t \dot{x} = -\dfrac{x-y}{R_t} + q(x)P_t \\ m_p C_p \dot{y} = -\dfrac{y-z}{R_c} - \dfrac{y-\theta_e}{R_m} \\ m_c C_c \dot{z} = -\dfrac{z-y}{R_c} + q(x)P_c \end{cases} \quad (13)$$

which is in the form (1). We choose the following realistic numerical values: $q_0 = 1$, $R_t = 1.5$ $K.W^{-1}$, $R_c = 1.35$ $K.W^{-1}$, $R_m = 0.9$ $K.W^{-1}$, $Q_t = m_t C_t = 50$ $J.K^{-1}$, $Q_c = m_c C_c = 732.5$ $J.K^{-1}$, $P_c = 50$ W, $Q_p = m_p C_p = 5000$ $J.K^{-1}$, $\theta_e = 281$ K, $\theta_1 = 293$ K, $\theta_2 = 294$ K, $P_t = 0.8$ W.

Simulations with *Matlab* enable us to obtain time (Figures 3 and 4) and phase (Figure 5) plots.

In Figures 3 and 4, large variations, medium variations and small variations represent respectively temperature variations for the convector, for the thermostat and for the room. Stars and crosses highlight the different switching times. We can

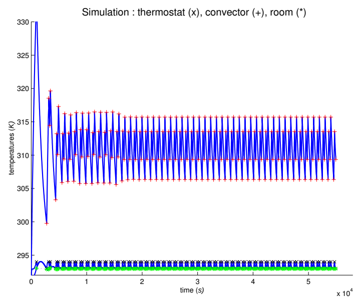

Fig. 3. Time plots

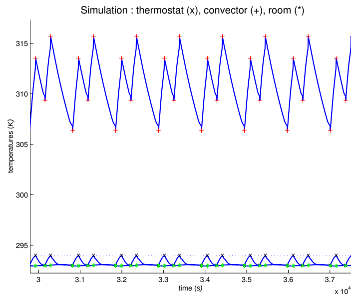

Fig. 4. Time plots (zoom)

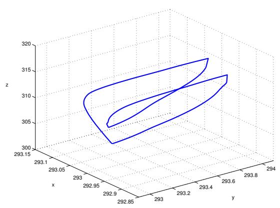

Fig. 5. Phase plots

observe that those variations do not look like those shown in Figure 6 and obtained for other chosen values in (Quémard *et al.*, 2005*b*) and in (Quémard *et al.*, 2005*a*) to illustrate a period-1 cycle.

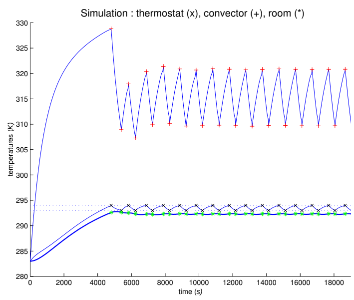

Fig. 6. Illustration of a period-1 cycle

Indeed, it seems that there exists two maxima and two minima for the room and the convector temperatures. The phase plot in space (x, y, z) (Figure 5) confirms this, showing that the system completes two full rotations before repeats itself. This is an illustration of a period-2 cycle (there are four different durations between the different switchings) that one can find in the thermostat with anticipative resistance system.

The following numerical study enables to confirm the existence of period-2 cycles for this studied class of h.d.s computing the possible different durations between two successive switching times.

6. NUMERICAL STUDY

6.1 Determination of the period-2 cycle

We solve system $F = (F_1, F_2, F_3, F_4)^T$ (11) with four unknowns σ^1, σ^2, σ^3, σ^4 programming in *Matlab* a classical algorithm of Newton (Dennis and Schnabel, 1983) taking as initial conditions, values of σ^1, σ^2, σ^3, σ^4 given by simulation: $\sigma^1 = 132.8371\ s$, $\sigma^2 = 529.7769\ s$, $\sigma^3 = 142.3932\ s$, $\sigma^4 = 226.8447\ s$. We also choose those following stop conditions $F^T F < 10^{-15}$ and $i \leq 1000$ (i being the number of iterations) in order that the algorithm can stop if there is no convergence. After 754 iterations, we obtain those following results: $\sigma^1 = 142.1941\ s$, $\sigma^2 = 523.5236\ s$, $\sigma^3 = 157.7412\ s$, $\sigma^4 = 301.2583\ s$ which confirm the existence of four different durations and so the existence of period-2 cycles.

Then, with *Maple* and using the found numerical values for σ^i, $i = 1, .., 4$ given above, we determine the period-2 cycle values:

$$X_{inf1} \simeq (293.00\ 292.9596\ 306.3574)^T,$$
$$X_{inf2} \simeq (293.00\ 292.9696\ 308.7245)^T,$$
$$X_{sup1} \simeq (294.00\ 292.9979\ 315.6553)^T,$$
$$X_{sup2} \simeq (294.00\ 292.9503\ 314.3314)^T,$$

which are well in accordance with Figures 3 and 4.

Moreover, *Maple* is also used for stability study. We compute roots z_i, $i = 1, 2$ of determinant Δ' made explicit in Section 4 and we obtain $z_1 = 0.73569$, $z_2 = 0.00345$ whose moduli are all less than one. So, we can conclude positively with the *point transformation method of Andronov* about the stability of this period-2 cycle.

6.2 Study of the period-2 cycle

From system (11), we can observe that, if we exchange values of σ^1 and σ^3 on the one hand and values of σ^2 and σ^4 on the other hand, we obtain exactly the same system than previously since it is just a question of considered initial discrete state in the description of the orbit. So, in this case, we can conclude that $\sigma^1 = 157.7412\ s$, $\sigma^2 = 301.2583\ s$, $\sigma^3 = 142.1941\ s$, $\sigma^4 = 523.5236\ s$ is also a solution of system (11) which describes the same stable period-2 cycle than before.

Moreover, if now, we set $\sigma^1 = \sigma^3$ and $\sigma^2 = \sigma^4$, equations F_1 and F_2 are identical and F_3 is reduced to zero. So, in this case, it amounts to the system of two equations for two unknowns solved in (Quémard *et al.*, 2005*b*) to find a period-1 cycle. Solving $F_1 = 0$ and $F_4 = 0$, we obtain

$\sigma^1 = \sigma^3 = 147.8814$ s and $\sigma^2 = \sigma^4 = 406.8390$ s which also is a solution for system (11). The application of the *point transformation method of Andronov* with *Maple* enables us to conclude negatively about the stability of this period-1 cycle (determinant Δ' has two roots -1.07178 and -0.05866).

So we can conclude that there is a period-doubling which arises with the lost of stability of the period-1 cycle. As we show computing σ^i, $i = 1, .., 4$, the period-1 cycle coninues to exist but now as a saddle cycle (see (Zhusubaliyev and Mosekilde, 2003), (Guckenheimer and Holmes, 1991), (Peitgen *et al.*, 1992)).

With initial conditions chosen above, the Newton algorithm, which is very efficient to estimate zeros of a non linear function, converges towards the period-2 cycle presented above. Nevertheless, with other initial conditions, it can converge towards the period-1 cycle. So, to highlight the sensitivity to initial conditions of this Newton algorithm (see (Zhusubaliyev and Mosekilde, 2003), (Peitgen *et al.*, 1992), (Baker and Gollub, 1990), it is interesting to locate all initial values for which the method converges towards a same zero. Using *Matlab*, we can find a solution to this problem colouring with the same colour all initial points leading to a same zero.

For a question of computing swiftness for the computer and of visibility on figures, we restrict us to plottings in dimension two, setting fixed values for example for σ^1 and σ^3. We plot coulour points in the plane (σ^2, σ^4). For $\sigma^1 = \sigma^3 = 148$ s, we obtain Figure 7, for $\sigma^1 = 135$ s, $\sigma^3 = 148$ s, we obtain Figure 8.

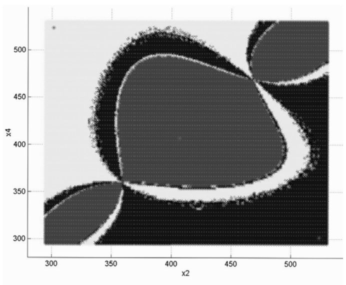

Fig. 7. Influence of initial conditions for fixed values for $\sigma^1 = \sigma^3 = 148$ s

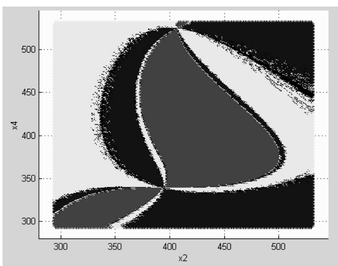

Fig. 8. Influence of initial conditions for fixed values for $\sigma^1 = 135$ s, $\sigma^3 = 148$ s

Light grey set and black set represent all initial points which lead to the period-2 cycle differentiating zero $\sigma^1 = 157.7412$ s, $\sigma^2 = 301.2583$ s, $\sigma^3 = 142.1941$ s, $\sigma^4 = 523.5236$ s (light grey set), and zero $\sigma^1 = 142.1941$ s, $\sigma^2 = 523.5236$ s, $\sigma^3 = 157.7412$ s, $\sigma^4 = 301.2583$ s (dark set). Finally, dark grey set represents all initial points for which the Newton algorithm converges towards the period-1 cycle. Those figures bring out the different fractal attraction basins and the sensitivity to initial conditions principally near the boundaries of the different attraction basins.

6.3 Influence of parameters variations

Vary some parameters can be very useful in order to study the hybrid dynamical system (1). Yet, the only one-parameter variation enables to highlight the period-doubling observed previously. Here, we choose to vary the resistance of the convector R_c. Thus, keeping the same initial conditions and plotting at each iteration of the Newton algorithm σ^2 and σ^4 (we prefer those durations instead of σ^1 and σ^3 because their difference is more important and so we can have a better visibility) while parameter R_c is varying, we obtain in Figure 9 a diagram of bifurcation for our system. It underlines that the curve divides into two branches at approximatively $R_c = 1.21$. Calculuses of roots of Δ' enable us to improve this value approximation to $R_c = 1.20844$ since it is from this value for R_c that one of the real root crosses the boundary of stability at -1 and from this, the period-2 orbit appears.

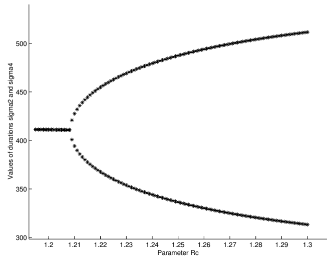

Fig. 9. Bifurcation diagram for thermostat system with the one parameter R_c variation

Now, we can vary simultaneously two parameters. Here, we choose to change values of thermal parameters R_c and P_c. The remaining thermal parameters are supposed to be constant and have the same numerical values than in section 5. At each value for R_c and P_c, from system (11) and using a Newton algorithm, we can determine if numerical values lead to a limit cycle and if it is the case, we can define the nature of the found cycles. Here, we limit our study only detecting period-1 cycles and period-2 cycles but in the future, it would be very interesting to detect cycles with more than two periods if they exist. The division of parameters space for our system (1) into domains of different modes of oscillations is shown in Figure 10. Sets of light grey points

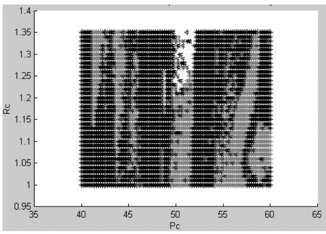

Fig. 10. Influence of parameters variations to the system mode of oscillations

represent sets of values for R_c and P_c which lead to period-2 cycles, those of grey points are values for which the system gives period-1 cycles and finally, sets of black points define R_c and P_c values which lead neither to period-1 cycles nor to period-2 cycles.

7. CONCLUSION

This work completes paper (Quémard *et al.*, 2005*b*) beginning to answer to questions formulated at the end of the study of limit period-1 cycles. Indeed, it establishes that the studied class of h.d.s (1) with autonomous switchings generated by a hysteresis phenomenon can also admit period-2 cycles with four different durations between successive switchings. Some formal and numerical characterizations of this type of cycles are given in this paper. Theoretical results are confirmed by the good adequation with simulations using *Matlab*.

The considered application for that is here interesting because it deals with a thermal device of industrial interest and because its model is a tridimensional model a little less complicated that the one studied in (Zhusubaliyev and Mosekilde, 2003) and from which, some chaotic behaviors have been observed. Nevertheless, a limit of the numerical study is to only detect period-1 and period-2 cycles. Then, some properties like the sensitivity of the model to initial conditions and like the period-doubling with the diagram of bifurcation are illustrated. They concern necessary conditions in the domain of chaotic behaviors. Moreover, Figure 11, which presents a simulation of temperatures variations with *Matlab* for another set of values for the studied thermostat with anticipative resistance model, highlights irregular oscillations and strange behaviors particularly for the room temperature.

So, even if numerical errors which can come from the computer have not to be neglected, the possibility to obtain chaotic behaviors for our h.d.s class is reinforced with yet the existence of multiple cycles and of some necessary properties for that. So, it is an interesting open question which would also show how complexity can arise from simple models.

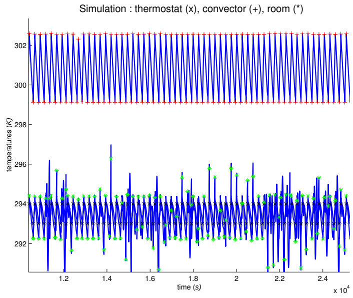

Fig. 11. Illustration of irregular oscillations for some particular values

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