ON THE FINITE-TIME STABILIZATION OF A NONLINEAR UNCERTAIN DYNAMICS VIA SWITCHED CONTROL

Giorgio Bartolini * **Alessandro Pisano** * **Elio Usai** *

* *Department of Electrical and Electronic Engineering, University of Cagliari (ITALY).*
{giob,pisano,eusai}@diee.unica.it

Abstract: This paper concerns the finite-time control problem for a class of nonlinear uncertain SISO dynamics with relative degree three. We assume that the measurements are corrupted by uncertain nonlinearities so that the *sign* of the phase variables is the only reliable information available for feedback. The stabilizing properties of a switching control scheme commuting between two unstable structures are demonstrated. Constructive proof and computer simulations are provided.

Keywords: Finite-time control, Nonlinear uncertain systems, Switching controllers

1. INTRODUCTION

A usual approach to the output-feedback control of nonlinear systems consists in combining an observer and a controller such that the latter stabilizes a suitable estimated output having relative degree one (Atassi and Khalil, 1999; Teel and Praly, 1995; Oh and Khalil, 1997; Krishnamurthy *et al.*, 2001).

This approach may fail, especially when the observers are used as differentiators, if sensors are not linear and sufficiently accurate. In fact, sensor saturation and, more seriously, uncertain nonlinearities of sensor devices, limit and often prevent the plane use of the previously mentioned methods even in a linear context (Tao and Kokotovic, 1996; Cao *et al.*, 2003; Zuo, 2005). It is well known (Anosov, 1959) that even the simple triple integrator cannot be stabilized if the only information about the output is its sign.

In this paper we consider the finite-time stabilization problem for an output variable, says s, whose third time derivative is affected by the control variable u. Taking into account the well-known Anosov's result about the stabilization problem for a triple integrator (Anosov, 1959), we assume that the measurement of the phase variables $(s, \dot{s}, \ddot{s})$ is corrupted by an uncertain memory-less nonlinearity preserving the sign of the measured quantity.

We shall demonstrate in this work that a proper switching logic between two Variable Structure Controllers (VSC) can guarantee the semi-global convergence of a class of nonlinear uncertain dynamics towards the set $(s, \dot{s}, \ddot{s}) = (0, 0, 0)$.

The first VSC, $u = -k_0 sign(s)$, is referred to as the "Anosov Unstable" (AU),and the second, $u = -k_1 sign(\dot{s}) - k_2 sign(\ddot{s})$, is a peculiar realization of the Twisting algorithm (Levant, 1993), that uses $\dot{s}$ and $\ddot{s}$, instead of s and $\dot{s}$, as feedback signals. In this note it is named the "Modified Twisting" (M-TW) algorithm.

These two algorithms are strongly inadequate if they act alone. Actually, the Anosov unstable algorithm causes oscillations of the phase variables

(Anosov, 1959) while the Modified Twisting steers the system at some point $P_e \equiv (s_e, 0, 0)$ of the $\dot{s} = \ddot{s} = 0$ axis (Levant, 1993), with, in general, $s_e \neq 0$.

The paper is organized as follows: Section II contains the problem formulation. Section III contains the constructive derivation of the main result, summarized in Theorem 1. Section IV reports some simulation examples, and the concluding remarks are given in Section V. Some details related to the convergence proof are discussed in the Appendix.

2. PROBLEM FORMULATION

Consider the following class of nonlinear uncertain dynamics in regular form (Utkin, 1992)

$$\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, \xi) \quad (1)$$

$$\dot{\xi} = \begin{bmatrix} \xi_2 \\ \xi_3 \\ f(\mathbf{x}, \xi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(\mathbf{x}, \xi) \end{bmatrix} u(t) \quad (2)$$

where $\mathbf{x} \in R^m$ is the internal dynamics state-vector, $\xi = [\xi_1, \xi_2, \xi_3] \equiv [s, \dot{s}, \ddot{s}] \in R^3$ is the vector collecting the output $s \in R$ and its first and second derivative, $u \in R$ is the scalar plant control input.

The control aim is to define a control input u guaranteeing that vector ξ is steered to the origin in finite time. Assume what follows:

A1 The internal dynamics (1) are input-to-state stable (ISS) with linear gain (Sontag, 1998).

A2 There are known positive constants F_0, F_1, F_2, G_m, G_M such that

$$|f(\mathbf{x}, \xi)| \leq F_0 + F_1 \|\mathbf{x}\| + F_2 \|\xi\| \quad (3)$$

$$0 < G_m \leq g(\mathbf{x}, \xi) \leq G_M \quad (4)$$

Assumption $A1$ implies that there exist constants K_1, K_2, $K_3 \in R^+$ such that, for every $\mathbf{x}_0 \in R^m$ and every bounded and continuous $\xi(t)$, the unique maximal solution of the initial value problem (1), with $\mathbf{x}(t_0) = \mathbf{x}_0$, has interval of existence R^+ and, for any $t \in [t_1, t_2] \subseteq [t_0, \infty)$, satisfies the following condition

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_1)\| + K_1 \sup_{t_1 \leq t \leq t_2} |s| + K_2 \sup_{t_1 \leq t \leq t_2} |\dot{s}| + K_3 \sup_{t_1 \leq t \leq t_2} |\ddot{s}| \quad (5)$$

A3 Constants K_1, K_2, K_3 in (5) are known

A4 The initial conditions $\mathbf{x}(t_0)$, $\xi(t_0)$ belong to known compact domains X_0 and Ω_0, respectively:

$$\mathbf{x}(t_0) \in X_0, \qquad \xi(t_0) \in \Omega_0 \quad (6)$$

Let signals s, $\dot{s}$, $\ddot{s}$ be measured by sensors featuring a nonlinear uncertain characteristics (see Fig. 1) such that

$$s_m = h_1(s) \quad \dot{s}_m = h_2(\dot{s}) \quad \ddot{s}_m = h_3(\ddot{s}) \quad (7)$$

with the subscript m standing for "measured".

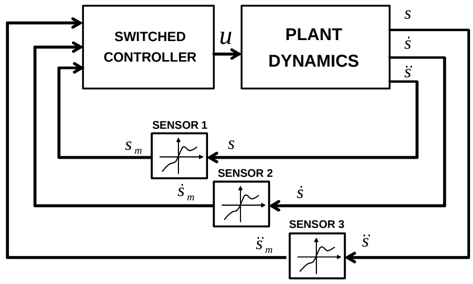

Fig. 1. Block scheme of the control system with nonlinear measurement characteristics

A5 The uncertain sensor characteristics satisfy the following relationship

$$x h_i(x) > 0 \quad \forall x \neq 0, \quad h_i(0) = 0, \quad i = 1, 2, 3 \quad (8)$$

Assumption A5 means that the sign of the measured signals matches the sign of the "true" ones. Nothing is required regarding the slope of the measurement nonlinearity.

3. MAIN RESULT

The finite-time stabilization problem for the uncertain system (1)-(2) satisfying Assumptions A1-A5 can be solved by means of a switched controller implementing a switching policy between the following two "Structures":

$$\begin{array}{c}\textbf{Anosov Unstable} \text{ (AU)}: \\ u = -U_0 sign(s)\end{array} \quad (9)$$

$$\begin{array}{c}\textbf{Modified Twisting} \text{ (M-TW)}: \\ u = -U_2 sign(\dot{s}) - (U_1 - U_2) sign(\ddot{s})\end{array} \quad (10)$$

where U_0, U_1 and U_2 are constant tuning parameters.

Because of both the AU and M-TW structures use only the sign of the phase variables s, $\dot{s}$, $\ddot{s}$,then by Assumption 5 we can disregard the measurement nonlinearities in the stability analysis.

We shall demonstrate that it is possible to evaluate a-priori a constant F and set, correspondingly, the controller parameters U_0, U_1 and U_2 such that the following inequality holds

$$|f(\mathbf{x}, \xi)| \leq F \qquad t \geq t_0 \quad (11)$$

and variables s, $\dot{s}$ and $\ddot{s}$ tend to zero in finite time.

Let parameters U_0, U_1, U_2 satisfy the following inequalities.

$$\begin{array}{c} G_m U_i - F > 0, \quad i = 0,1,2 \\ G_m U_1 - F > G_M U_2 + F \end{array} \tag{12}$$

Under the tuning condition (12) for its parameter U_0, the AU control (9), (12) enforces a sequence of time instants $t_{z,i}$ $(i = 1, 2, \ldots)$ such that $s(t_{z,i}) = 0$. We denote as "*zero crossing of* s" the occurrence of such a condition. Unfortunately, the sequences $(t_{z,i} - t_{z,i-1})$, $|\dot{s}(t_{zi})|$ and $|\ddot{s}(t_{zi})|$ are all diverging (Anosov, 1959).

The M-TW control (10), (12) gives the closed-loop trajectories a finite-time converging "Twisting" behaviour in the $\dot{s} - \ddot{s}$ plane (see Fig. 2). Then vector $[s, \dot{s}, \ddot{s}]$ converges in finite time to a stable "*equilibrium point*" $P_e \equiv (s_e, 0, 0)$, with, in general, $s_e \neq 0$. (Levant, 1993).

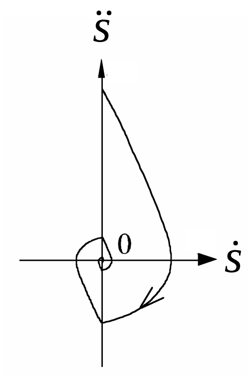

Fig. 2. Trajectories of the Modified Twisting (M-TW) in the $\dot{s} - \ddot{s}$ plane

Thus, when used alone, the two control laws (9) and (10) are not effective. On the contrary, the hereafter described switching policy between the AU and M-TW structures causes the occurrence of a sequence of equilibrium points

$$P_{e,i} \equiv (s_{e,i}, 0, 0) \qquad i = 1, 2, \ldots \tag{13}$$

such that the strict contraction property

$$|s_{e,i+1}| \leq \varepsilon |s_{e,i}|, \quad 0 < \varepsilon < 1, \quad i = 1, 2, \ldots \tag{14}$$

is fulfilled and the convergence to zero of s, $\dot{s}$ and $\ddot{s}$ is guaranteed. Fig. 4 shows a typical closed-loop time evolution of the s variable which fulfills the contraction condition (14).

The proposed controller can be schematized as in Fig. 3.

Remark 1: On-line detection of the transition-enabling conditions, zero-crossings and equilibrium points, is now discussed. Zero-crossing detection implies detection of both positive-to-negative and negative-to-positive sign changings of s. Detection of the equilibrium points is less immediate. In fact, checking condition $\dot{s} = \ddot{s} = 0$ is unpractical in real systems. Following the idea proposed in (Bartolini *et al.*, 2002), the attainment of the equilibrium points $P_{e,i}$ can be easily and efficiently detected by monitoring on-line the switching frequency of the discontinuous control variable in a receding-horizon time interval. The switching frequency of the control signal tends to infinity while approaching the sliding condition which establishes at the equilibrium points. This criterion is also suitable for digital implementation: the sequence $\{u[k], u[k-1], \ldots, u[k-N]\}$ $(u[k] = u(kT_s)$, T_s is the sampling period, N is an integer number) can be stored and processed, and updated, at each sampling time instant. A minimum number of sign variations between the adjacent elements of the sequence can reveal the approaching of an equilibrium point.

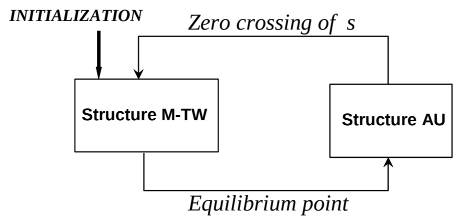

Fig. 3. The proposed switching mechanism

The above treatment is summarized in the following Theorem:

Theorem 1 *Consider system* (1)-(2) *satisfying assumptions A1-A5. Perform the following sequence of steps*

A. Set the desired contraction rate $\varepsilon \in (0,1)$ *appearing in (14) and the arbitrary coefficients* $\eta > 1$, $\gamma \in (0,1)$, $\theta_1 \in (0,1)$, $\theta_2 \in (0,1)$.

B. Compute the unique positive root F^* *of equation (46)-(49) and set* $F \geq F^*$.

C. Compute the unique positive root x^* *of equation* $x^*(3 + 2x^*) + \gamma\varepsilon(1 + x^*)^{3/2} = \varepsilon$.

D. Set the controller parameters U_0, U_1, U_2 *as*

$$U_0 = \eta \frac{F}{G_m}, \quad U_1 = \frac{1}{G_m}\left(\frac{\eta \frac{G_M}{G_m} + 1}{\rho} + 1\right) F$$
$$U_2 = \frac{1}{G_m}\left(\frac{\eta \frac{G_M}{G_m} + 1}{\frac{1}{4}\theta_1 \gamma^2 \varepsilon^2} - 1\right) F \tag{15}$$

$$\rho = min\left\{\frac{1}{4}\theta_1 \gamma^2 \varepsilon^2, \theta_2 x^*\right\} \tag{16}$$

and apply the switched controller described in Fig. 3, where the Structures AU and M-TW are defined *in (9), (10). Then, s, $\dot{s}$ and $\ddot{s}$ are steered to zero in finite time.*

Proof. See the Appendix.

Remark 2: By (15), tuning of constant parameters U_0, U_1, U_2 depends on the unknown F. Theorem 1 guarantees that a constant F^* exists such that for any $F \geq F^*$ the desired performance is achieved, and gives a procedure to compute a conservative overestimation of F^*. For practical purposes, the calibration of the control parameters can be performed by simply increasing the single parameter F until satisfactory behaviour is observed in the closed-loop system. This method of experimental tuning of a single gain parameter is not unusual in the VSC context (Utkin, 1992).

4. SIMULATION RESULTS

To validate the present analysis consider the following fifth-order nonlinear system

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\xi_1 + \xi_2) \qquad \mathbf{x} = [x_1, x_2]^T \\ &\dot{\xi}_i = \xi_{i+1}, \quad i = 1, 2 \\ &\dot{\xi}_3 = \frac{\xi_2}{1+\xi_2^2} + \xi_1 + \xi_3 + \|\mathbf{x}\| + (2 + cos(x_1 + \xi_2)u \end{aligned} \tag{17}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{18}$$

The considered dynamics meets Assumptions A1-A5. Constants appearing in (3)-(5) can be evaluated as: $F_0 = 0.5$, $F_1 = F_2 = 1$, $G_m = 1$, $G_M = 3$, $K_1 = K_2 = 1$, $K_3 = 0$. Initial conditions are taken as follows: $[\xi_1(0), \xi_2(0), \xi_3(0)] = [1, 1, 1]$, $[x_1(0), x_2(0)] = [1, 1]$. The free design parameters are set as $\eta = 1.1$, $\theta_1 = \theta_2 = \gamma = 0.9$, and constant F is taken as 15.

Let us choose a "contraction factor" $\varepsilon = 0.5$. The resulting control parameters are $U_0 = 16.5$, $U_1 = 3598$, $U_2 = 451$. Euler integration algorithm with step $T_s = 0.0005ms$ has been used in the Matlab-Simulink environment.

Occurrence of at least 5 sign changings among the most recent 40 samples of u (see Remark 1) was chosen as the criterion to enable the transition from the M-TW to the AU structure. Fig. 4 reports the time profile of s. It is apparent that the local maxima of s feature the imposed contraction rate of 0.5 in perfect accordance with (14).

Choosing $\varepsilon = 0.7$, and leaving unchanged the free parameters, we obtained the following control magnitudes from the tuning procedure: $U_0 = 16.5$, $U_1 = 2576$, $U_2 = 184$. The plot of the s variable vs. time is shown in Fig. 5. As expected, increasing the value of ε, i.e. reducing the prescribed contraction rate, the required control magnitudes decrease. Again the observed contraction rate of 0.7 is according to (14), and, at the same time, the transient length obviously increases.

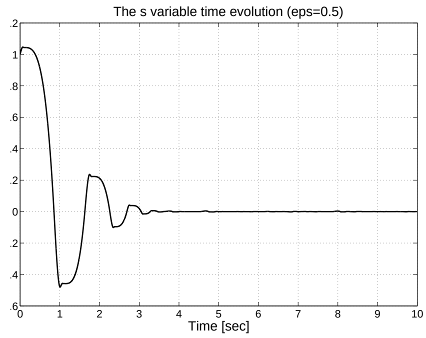

Fig. 4. The output quantity $s(t)$ when $\epsilon = 0.5$

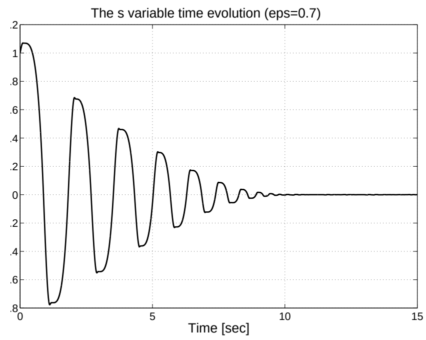

Fig. 5. The output quantity $s(t)$ when $\epsilon = 0.7$.

The dependence of the accuracy on the discretisation step has been analyzed. The sampling step was changed in order to investigate the accuracy order of the proposed method with respect to the sampling period. By comparing the two plots in Fig. 6, it follows that by using a discretization step which is ten times smaller (reducing it to $T_s = 0.05ms$) the accuracy is improved by a factor of almost 1000, which means that the accuracy is $O(T_s^3)$.

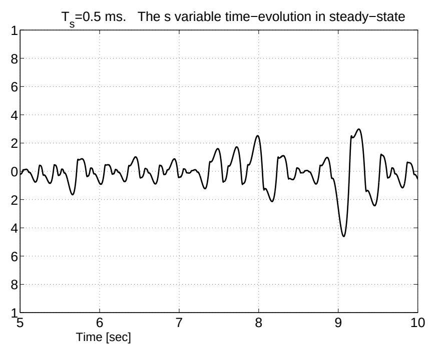

Fig. 6. The sliding accuracy with $\tau = 0.5ms$

The robustness of the proposed technique against the additive measurement noise corrupting the phase variables has been checked as well. We considered a uniformly-distributed random noise with a maximal magnitude of 0.1 superimposed

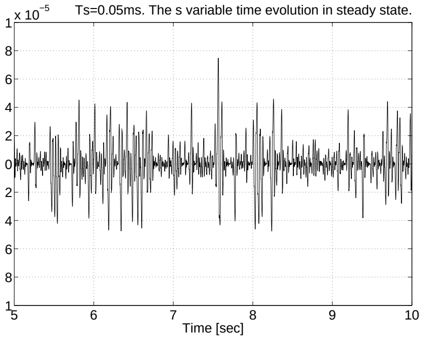

Fig. 7. The sliding accuracy with $\tau = 0.25ms$

to the three feedback variables. The noise causes a detriment of sliding accuracy in the steady state. Fig. 8 shows the time evolution of s is the steady state (the control parameters and sampling period are the same as those used in Fig. 6).

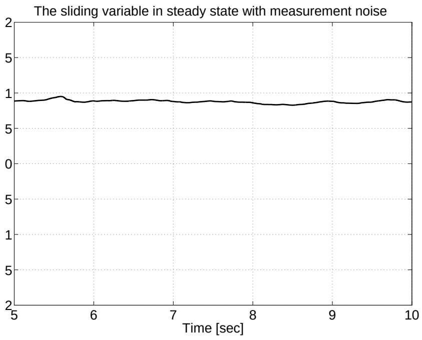

Fig. 8. The sliding accuracy in steady-state in the presence of noise

5. CONCLUSIONS

In this paper a novel control algorithm is presented which stabilizes a class of nonlinear uncertain systems in finite time. We assume that uncertain nonlinearities corrupt the measurement of the phase variables such that their sign of the unique reliable information. The control law consists of two VSC and a supervisor that implements a proper switching logic. The accuracy-order under discretization and sampling, and the robustness against measurement noise, are investigated by simulation.

REFERENCES

Anosov, D.V. (1959). On stability of equilibrium points of relay systems. *Automatica i telemechanica (Automation and Remote Control)* **2**, 135–149.

Atassi, N.A. and H.K. Khalil (1999). A separation principle for the stabilization of a class of nonlinear systems. *IEEE Trans. on Aut. Control* **44**, 1672–1687.

Bartolini, G., A. Levant, A. Pisano and E. Usai (2002). A real-sliding criterion for control adaptation. *Proc. 2002 Workshop on Variable Structure Systems (VSS 2002).*

Bartolini, G., Pisano, A. Punta E. and Usai E. (2003). A survey of applications of second order sliding mode control to mechanical systems. *Int. J. Control* **76**, 875–892.

Cao, Y.-Y., Lin, Z. and Chen, B.M. (2003). An output feedback H_∞ controller design for linear systems subjected to sensor nonlinearities. *IEEE Trans. Circ. Syst.* **7**, 914–921.

Krishnamurthy, P., Khorrami, F. and Chandra, R.S. (2001). Global high-gain-based observer and backstepping controller for generalized output-feedback canonical form. *IEEE Trans. Aut. Contr.* **48**, 2277–2284.

Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *Int. J. Control* **58**, 1247–1263.

Oh, S. and H.K. Khalil (1997). Nonlinear output feedback tracking using high-gain observers and variable structure control. *Automatica* **33**, 1845–1856.

Orlov Y. (2004). Finite Time Stability and Robust Control Synthesis of Uncertain Switched Systems. *SIAM Journal on Control and Optimization* **43**,4, 1253–127.

Sontag, E. (1998). *Mathematical Control Theory.* Springer-Verlag. New York.

Tao, G. and P. Kokotovic (1996). *Adaptive Control of Systems with Actuator and Sensor Nonlinearities.* Wiley. New York.

Teel, A. and L. Praly (1995). Tools for semi-global stabilization via partial state and output feedback. *SIAM J. Contr. Opt.* **33**, 1443–1488.

Utkin, V.I. (1992). *Sliding modes in control and optimization.* Springer Verlag. Berlin.

Zuo, Z. (2005). Output feedback H_∞ controller design for linear discrete-time systems with sensor nonlinearities. *Proc. IEE* **152**, 19–26.

Proof of Theorem 1

Closed-loop stability

We assume, temporarily, that condition (11) holds for some known value of F and we demonstrate the semi-global finite-time convergence of $s, \dot{s}, \ddot{s}$ to zero.

The initial structure is the M-TW, so that the system trajectory is similar to that reported in Fig. 2. An equilibrium point $P_{e,1} \equiv (s_{e,1}, 0, 0)$ is reached in finite time $t_{e,1}$. Assume with no loss of generality that $s_{e,1} < 0$. Fig. 9, shows a typical time evolution of the s variable starting from point P_{e1} subject to the switched controller in Fig. 3 stopped at the first iteration.

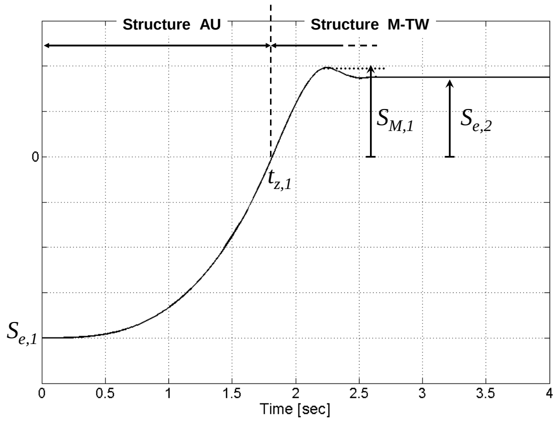

Fig. 9. Time evolution of s leaving from an equilibrium point $P_{e,1}$.

The AU control law is active until a zero crossing of s is attained at the time instant $t_{z1} \geq t_{e1}$. During this time interval the system is governed by the following differential inclusion

$$s^{(3)} \in [-F, F] - [G_m, G_M]U_0 sign(s_{e1}), \quad (19)$$

Since $\dot{s}(t_{e1}) = \ddot{s}(t_{e1}) = 0$, by simple worst-case analysis it follows that the first zero crossing of s is reached after a finite time $t_{z,1}$ such that

$$t_{z,1} - t_{e,1} \in \left[\left(\frac{6|s_{e,1}|}{G_M U_0 + F} \right)^{1/3}, \left(\frac{6|s_{e,1}|}{G_m U_0 - F} \right)^{1/3} \right] \quad (20)$$

By computing the limit solutions of (19), and taking into account (11) and (20), it results that $\ddot{s}(t_{z,1})$ and $\dot{s}(t_{z,1})$ are both positive and such that

$$\begin{aligned} \ddot{s}(t_{z,1}) &\leq \bar{\ddot{s}}_{z,1} \equiv \sqrt[3]{6}(G_M U_0 + F)^{2/3} |s_{e,1}|^{1/3} \\ \dot{s}(t_{z,1}) &\leq \bar{\dot{s}}_{z,1} \equiv \frac{\sqrt[3]{6}}{2}(G_M U_0 + F)^{1/3} |s_{e,1}|^{2/3} \end{aligned} \quad (21)$$

At $t = t_{z,1}$ the controller switches to the M-TW structure, and the origin of the $\dot{s} - \ddot{s}$ phase plane is reached at $t = t_{e,2}$. Fig. 10 depicts the corresponding system trajectory in the $\dot{s} - \ddot{s}$ plane.

Define

$$s_{M,1} = \sup_{t \in [t_{e,1}, t_{e,2}]} |s| \quad (22)$$

Let $t = t_{f,1}$ the time instant at which point $P_{f,1}$ in Fig. 10 is reached. It is easy to show that

$$t_{f,1} \leq \bar{t}_{f,1} \equiv t_{z,1} + \frac{\bar{\ddot{s}}_{z,1}}{G_m U_1 - F} \quad (23)$$

Hence, $s(t_{f,1})$ and $\dot{s}(t_{f,1})$ are bounded as follows

$$\begin{aligned} s(t_{f,1}) &\leq \bar{s}(t_{f,1}) \equiv \bar{\dot{s}}_{z,1}(\bar{t}_{f,1} - t_{z,1}) + \frac{1}{3}\frac{\bar{\ddot{s}}_{z,1}^3}{(G_m U_1 - F)^2} = \\ &= \frac{G_M U_0 + F}{G_m U_1 - F}\left[3 + 2\frac{G_M U_0 + F}{G_m U_1 - F}\right]|s_{e,1}| \end{aligned} \quad (24)$$

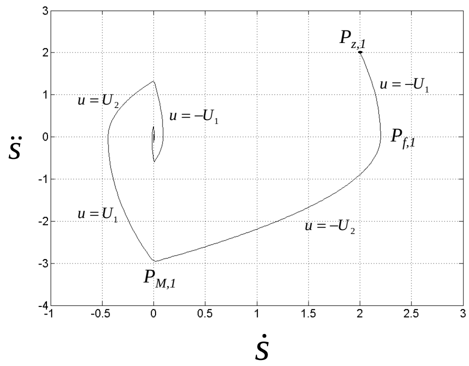

Fig. 10. Phase trajectory of $\dot{s}$ and $\ddot{s}$ leaving from the zero crossing $P_{z,1}$.

$$\begin{aligned} \dot{s}(t_{f,1}) &\leq \bar{\dot{s}}_{f,1} = \bar{\dot{s}}_{z,1} + \frac{1}{2}\frac{\bar{\ddot{s}}_{z,1}^2}{G_m U_1 - F} = \\ &= \frac{1}{2}(G_M U_0 + F)^{1/3}\left[1 + \frac{G_M U + F}{G_m U_1 - F}\right]|6 s_{e,1}|^{2/3} \end{aligned} \quad (25)$$

$|s|$ features the maximum overshoot at the point $P_{M,1} \equiv (s_{M_1}, 0, \ddot{s}_{M_1})$ (see Figs. 9 and 10). Then we get

$$|\ddot{s}_{M_1}|^2 \leq 2(G_M U_2 + F)\bar{\dot{s}}_{f,1} \quad (26)$$

The increase of s along the trajectory between points $P_{f,1}$ and $P_{M,1}$ can be expressed as a function of $\bar{\ddot{s}}_{M_1}$ (or, equivalently, of $\bar{\dot{s}}_{f,1}$) according to

$$s_{M_1} \leq s(t_{f,1}) + \frac{1}{3}\frac{\bar{\ddot{s}}_{M_1}^3}{(G_M U_2 + F)^2} \quad (27)$$

Therefore, by combining (24), (26) and (27) we obtain

$$\begin{aligned} s_{M_1} \leq \bar{s}_{M_1} &= \frac{G_M U_0 + F}{G_m U_1 - F}\left[3 + 2\frac{G_M U_0 + F}{G_m U_1 - F}\right]|s_{e,1}| + \\ &+ 2\left[\frac{G_M U_0 + F}{G_M U_2 + F}\right]^{1/2}\left[1 + \frac{G_M U_0 + F}{G_m U - F}\right]|s_{e,1}| \end{aligned} \quad (28)$$

which can be rewritten as

$$s_{M,1} \leq \Sigma(\Delta_1, \Delta_2)|s_{e,1}| \quad (29)$$

$$\begin{aligned} \Sigma(\Delta_1, \Delta_2) &= \Delta_1(3 + 2\Delta_1) + 2\sqrt{\Delta_2}(1 + \Delta_1)^{3/2} \\ \Delta_1 &= \frac{G_M U_0 + F}{G_m U_1 - F}, \quad \Delta_2 = \frac{G_M U_0 + F}{G_M U_2 + F} \end{aligned} \quad (30)$$

Since, by definition, $|s_{M,1}| \geq |s_{e,2}|$, a sufficient condition guaranteeing the contraction condition $|s_{e,2}| \leq \varepsilon |s_{e,1}|$ is to choose U_0, U_1 and U_2 according to the following inequalities

$$G_m U_i - F > 0, \quad i = 0, 1, 2 \quad (31)$$

$$G_m U_1 - F > G_M U_2 + F \quad (32)$$

$$\Sigma(\Delta_1, \Delta_2) \leq \varepsilon < 1 \quad (33)$$

By iteration, sequence $|s_{e,i}|$ will meet the strict contraction condition (14).

To conclude the design process, a triple U_0, U_1, U_2 which satisfies inequalities (31)-(33) have to be computed . By (31), U_0 can be simply set as in (15) with $\eta > 1$.

Set

$$\Delta_2 = \Delta_2^* = \frac{1}{4}\gamma^2\varepsilon^2 \qquad 0 < \gamma < 1 \qquad (34)$$

Condition (34) derives from imposing $\Sigma(0, \Delta_2^*) = \gamma\varepsilon$, with $\gamma \in (0, 1)$. This makes the inequality (33) satisfied when $\Delta_1 = 0$. Since function $\Sigma(\Delta_1, \cdot)$ is continuous and not decreasing, condition (34) guarantees the existence of a solution interval to (33) in the form $\Delta_1 \in (0, \Delta_1^*)$ where Δ_1^* is the smallest root of equation

$$\Sigma(\Delta_1^*, \Delta_2^*) \equiv \Delta_1^*(3 + 2\Delta_1^*) + \gamma\varepsilon(1 + \Delta_1^*)^{3/2} = \varepsilon \quad (35)$$

Now to guarantee that Δ_1 belongs to the solution set $(0, \Delta_1^*)$ and, at the same time, that $0 \leq \Delta_1 < \Delta_2$ (as required by (32)) we can set

$$\Delta_1 = min(\theta_1\Delta_2^*, \theta_2\ \Delta_1^*),\ 0 < \theta_1 < 1,\ 0 < \theta_2 < 1, \quad (36)$$

and compute U_1 and U_2 on the basis of the given values of U_0, Δ_1 and Δ_2. Manipulation of the given formulas (34)-(36) taking into account (30) yields the tuning rules (15)-(16).

Finite transient length

To assess the finite time convergence let T_{ei} ($i = 1, 2, \ldots$) be such that $t_{e,i+1} - t_{ei} \leq T_{ei}$. It was shown in (Levant, 1993; Bartolini *et al.*, 2003) that the Twisting algorithm features a finite convergence time that can be overestimated by a bounded function of the initial conditions. By applying such a formula to the actual case it yields

$$T_{e,i} = \lambda|s_{e,i}|^{1/3} \qquad (37)$$

where λ is a positive constant. Then, the following condition holds

$$\frac{T_{e,i}}{T_{e,i-1}} = \left(\frac{s_{e,i}}{s_{e,i-1}}\right)^{1/3} \leq \varepsilon^{1/3} < 1 \qquad (38)$$

which implies that the total convergence time can be upper bounded by the finite sum of a convergent geometric series.

$$\overline{T} = t_{z,1} + \sum_{i=1}^{\infty} T_{ei} \leq \infty \qquad (39)$$

Existence and computation of F

Let

$$X_0 = \sup_{\mathbf{x}\in X_0} \|\mathbf{x}\|, \qquad S_0 = \sup_{\xi\in\Omega_0} |s| \qquad (40)$$

$$\dot{S}_0 = \sup_{\xi\in\Omega_0} |\dot{s}| \qquad \ddot{S}_0 = \sup_{\xi\in\Omega_0} |\ddot{s}| \qquad (41)$$

and define

$$\overline{s} = \sup_{t\geq t_0} |s|, \qquad \overline{\dot{s}} = \sup_{t\geq t_0} |\dot{s}|, \qquad \overline{\ddot{s}} = \sup_{t\geq t_0} |\ddot{s}|, \quad (42)$$

By combining (3) and (5) it follows that

$$|f(\mathbf{x}, \xi)| \leq H_0 + H_1\overline{s} + H_2\overline{\dot{s}} + H_3\overline{\ddot{s}} \qquad (43)$$

with the constants H_0, H_1 and H_2 given as follows:

$$\begin{array}{ll} H_0 = F_0 + F_1X_0 & H_1 = F_1K_1 + F_2 \\ H_2 = F_1K_2 + F_2 & H_3 = F_1K_3 + F_2 \end{array} \qquad (44)$$

Let condition (11) be satisfied for some unknown constant F. Then we can compute the quantities $\overline{s}$, $\overline{\dot{s}}$ and $\overline{\ddot{s}}$, which depend on the unknown F.

If the inequality

$$H_0 + H_1\overline{s}(F) + H_2\overline{\dot{s}}(F) + H_3\overline{\ddot{s}}(F) \leq F \qquad (45)$$

admits a semi-infinite solution interval of the type $F \geq F^*$ then the existence of F is proven. Let F^* be the unique positive root of equation

$$H_0 + H_1\overline{s}(F^*) + H_2\overline{\dot{s}}(F^*) + H_3\overline{\ddot{s}}(F^*) = F^* \quad (46)$$

By solving (46) not only the existence of F such that (11) holds is proven, but also one achieves an overestimate of it. The initial conditions of s, $\dot{s}$ and $\ddot{s}$ are taken on the neighbor of the compact domain Ω_0: $s(t_0) = S_0$, $\dot{s}(t_0) = \dot{S}_0$, $\ddot{s}(t_0) = \ddot{S}_0$.

By worst case analysis it can be computed the following overestimates:

$$\begin{aligned} \overline{s}(F) = S_0 &+ \frac{\dot{S}_0\ddot{S}_0}{G_mU_1 - F} + \frac{1}{3}\frac{\ddot{S}_0^3}{(G_mU_1 - F)^2} + \\ &+ \frac{1}{3}\frac{2\sqrt{2}[G_MU_2 + F]^{3/2}[\dot{S}_0 + \frac{1}{2}\frac{\ddot{S}_0^2}{G_mU_1 - F}]^{3/2}}{(G_mU_2 + F)^2} + \\ &+ \frac{\dot{S}_{z1}\ddot{S}_{z1}}{G_mU_1 - F} + \frac{1}{3}\frac{\ddot{S}_{z1}^3}{(G_mU_1 - F)^2} + \\ &+ \frac{1}{3}\frac{2\sqrt{2}[G_MU_2 + F]^{3/2}[\dot{S}_{z1} + \frac{1}{2}\frac{\ddot{S}_{z1}^2}{G_mU_1 - F}]^{3/2}}{(G_mU_2 + F)^2} \end{aligned} \qquad (47)$$

$$\overline{\dot{s}}(F) = \dot{S}_0 + \frac{1}{2}\frac{\ddot{S}_0^2}{G_mU_1 - F} + \dot{S}_{z1} + \frac{1}{2}\frac{\ddot{S}_{z1}^2}{G_mU_1 - F} \qquad (48)$$

$$\begin{aligned} \overline{\ddot{s}}(F) = \ddot{S}_0 &+ \sqrt{2(G_MU_2 + F)}\sqrt{\dot{S}_0 + \frac{1}{2}\frac{\ddot{S}_0^2}{G_mU_1 - F}} + \\ &+ \ddot{S}_{z1} + \sqrt{2(G_MU_2 + F)}\sqrt{\dot{S}_{z1} + \frac{1}{2}\frac{\ddot{S}_{z1}^2}{G_mU_1 - F}} \end{aligned} \qquad (49)$$

By considering (47)-(49) into (45) it is easy to observe that the left-hand side of the latter is growing with order $O(F^{2/3})$. This guarantees the existence of F^* such that any $F \geq F^*$ fulfills the inequality (11). This concludes the proof. □