USING PATH INTEGRAL SHORT TIME PROPAGATORS FOR NUMERICAL ANALYSIS OF STOCHASTIC HYBRID SYSTEMS

Gerwald Lichtenberg [*,1] **Philipp Rostalski** [*,2]

[*] *Institute of Control Systems, TU Hamburg-Harburg*

Abstract: Algorithms to approximate the evolution of probability density functions for stochastic hybrid systems rely on the knowledge of appropriate short time propagators. It is shown that a path integral propagator known for continuous stochastic systems can be adapted to the hybrid case. With this propagator, the *HybPathTree* algorithm performs well concerning precision and computational effort, e.g. in reachability analysis.

Keywords: Stochastic Hybrid Systems, Reachability Analysis, Path Integrals

1. INTRODUCTION

The prediction of probability density functions for stochastic hybrid systems is relevant to a variety of technical disciplines. State of the art methods and applications can be found e.g. in (Blom and Lygeros 2005). Even for simple systems, only numerical methods can be applied. For pure continuous stochastic systems many approaches are known which introduce a Markov chain that approximates the probability density at grid points via finite difference schemes (Kushner and Dupuis 2001). Path integrals methods can be applied to compute Markov chain approximations of continuous stochastic systems, which give better results than these classical methods (Wehner and Wolfer 1983).

The novelty of this paper can be seen from two perspectives: On the one hand, the path integral based PathTree algorithm (Ingber *et al.* 2001) is extended to a class of stochastic hybrid systems. On the other hand, existing numerical methods for analysis of stochastic hybrid systems are improved by generating the transition probabilities of the approximating Markov chain by a short time propagator resulting from path integrals.

After introducing the model class and the notion of path integrals in Sections 2 and 3, the stochastic reachability problem is given in Section 4. For its solution, extensions of path integrals for hybrid systems are derived in Section 5, which are the basis of Markov chain approximations given in Section 6 that are applied to two tank systems in Section 7.

2. MODELING

In this paper, we are using a slight adaption of the description of a generalized stochastic hybrid system given in (Bujorianu and Lygeros 2003). Let Q denote the set of discrete states and $X^q \subset \mathbb{R}^{d(q)}$ an open $d(q)$-dimension subspace assign to each of those discrete states $q \in Q$. We will refer to those as continuous state spaces. The closure of X^q is defined as $\bar{X}^q = X^q \cap \partial X^q$ with ∂X^q denoting the boundary. The hybrid state space can now be defined as $\mathcal{H} = \bigcup_{q\in Q}\{q\} \times X^q$ with $\bar{\mathcal{H}} = \bigcup_{q\in Q}\{q\} \times \bar{X}^q$ and $\partial\mathcal{H} = \bigcup_{q\in Q}\{q\} \times \partial X^q$ being its closure and boundary respectively.

[1] Current Affiliation: Automatic Control Laboratory, Swiss Federal Institute of Technology, Zürich.

The definition of a generalized stochastic hybrid system can be restated as

Definition 1. A Generalized Stochastic Hybrid System (GSHS) is a collection

$$\mathcal{M} = ((Q, d, \mathcal{X}), a, \sigma, Init, \lambda, R) \ ,$$

where

- $Q = \{1, 2, \ldots N_Q\}$ is a countable set of discrete states,
- $d : Q \to \mathbb{N}$ is a map giving the dimension of the continuous state spaces, $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ maps each $q \in Q$ into an open subset $X^q \subset \mathbb{R}^{d(q)}$ (continuous state space assigned to the discrete state $q \in Q$),
- $a : Q \times X^q \to \mathbb{R}^{d(\cdot)}$ is a vector field (describing the system dynamics in each discrete state),
- $\sigma : Q \times X^q \to \mathbb{R}^{d(\cdot)\times m}$ is a X^d-valued matrix, $m \in \mathbb{N}$ (describing the variance of the noise in each discrete state),
- $Init : \mathcal{B}(\mathcal{H}) \to [0,1]$ is a probability measure on $\mathcal{H}$ (distribution of the initial state), where $\mathcal{B}(\mathcal{H})$ is the space of σ-algebras generated by $\mathcal{H}$ (Borel σ algebra).
- $\lambda : \bar{\mathcal{H}} \to \mathbb{R}^+$ is a transition rate function giving probabilistic changes of discrete state,
- $R : \bar{\mathcal{H}} \times \mathcal{B}(\bar{\mathcal{H}}) \to [0,1]$ is a transition measure describing the distribution of the continuous state after a jump.

In the following, we will assume that there are no spontaneous jumps, i.e. a jump-rate $\lambda = 0$ and we will call models of this class stochastic hybrid systems (SHS). Thus, for the transition measure $R : \partial\mathcal{H} \times \mathcal{B}(\bar{\mathcal{H}}) \to [0,1]$ holds.

The execution of such a stochastic hybrid system can be defined as follows, compare (Bujorianu and Lygeros 2003):

Definition 2. A stochastic process $h(t) = (q(t), \boldsymbol{x}(t))$ is called a SHS execution if there exists a sequence of stopping times $T_0 = 0, \leq T_1 \leq T_2 \leq \ldots$ such that for each $k \in \mathbb{N}$:

- $h_0 = (q_0, x_0^{q_0})$ is a $Q \times X$-valued random variable extracted according to the probability measure $Init$;
- For $t \in [T_k, T_{k+1})$, $q_t = q_{T_k}$ is constant and $\boldsymbol{x}(t)$ is a (continuous) solution of the SDE $dx^{(i)} = a^{(i)}(q_{T_k}, x(t))dt + \sigma^{(i,j)}(q_{T_k, x(t)})d\Gamma^{(j)}(t)$ where $\Gamma^{(j)}(t)$ is the m-dimensional standard Wiener process
- $T_{k+1} = T_k + S^{i_k}$ where S^{i_k} is the stopping time of the process, i.e. the time where $x(t)$ first hits the boundary of ∂X^q,
- The probabilistic distribution of $x(T_{k+1})$ is governed by the law $R((q_{T_k}, \boldsymbol{x}(T_{k+1}^-)), \cdot)$.

This modelling approach can be seen as a collection of several continuous time, stochastic differential equations and its domains together with jumps. These are executed whenever the continuous state $\boldsymbol{x}$ reaches certain areas of the continuous state-space (the guards).

3. PATH INTEGRALS

The concept of path integrals will be first introduced for a continuous dynamical system, given by the following stochastic differential equation

$$dx^{(i)} = a^{(i)}(\boldsymbol{x})dt + \sigma^{(i,j)}(\boldsymbol{x})d\Gamma^{(j)}(t) \qquad (1)$$

with $i = 1 \ldots n, j = 1 \ldots m$, also known as *Langevin Equation.* The stochastic term is modelled with a m-dimensional standard Wiener noise process $\Gamma(t)$.

In the following, eqn. (1) should be interpreted in the sense of Stratonovich. For many applications, the probability $p(\boldsymbol{x}, t)$ of the state vector $\boldsymbol{x}$ at time t is of interest.

With the knowledge of some initial probability distribution $p(\boldsymbol{x}_0, t_0)$ at a certain time t_0, the conditional probability distribution $p(\boldsymbol{x}, t)$ at a later time t is given by its generator which in this simple case comes as the *Fokker-Planck Equation*

$$\frac{\partial p(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, t_0)}{\partial t} = L_{\mathrm{FP}}(\boldsymbol{x}, t) \cdot p(\boldsymbol{x}, t \,|\, \boldsymbol{x}_0, t_0) \ . \quad (2)$$

with the *Fokker-Planck Operator*

$$L_{FP}(\boldsymbol{x}, t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x^{(i)}} D_1^{(i)}(\boldsymbol{x}, t) + \\ + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} D_2^{(i,j)}(\boldsymbol{x}, t) \ . \quad (3)$$

The term $D_1^{(i)}(\boldsymbol{x}, t)$ is the i'th component of the *drift* that holds information about the underlying deterministic movement of the system, while $D_2(\boldsymbol{x}, t)$ gives the *diffusion*, i.e. the additional noise. For a multi-dimensional stochastic process, the drift operator

$$D_1^{(i)}(\boldsymbol{x}, t) = \\ a^{(i)}(\boldsymbol{x}, t) + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{m} \sigma^{(k,j)}(\boldsymbol{x}, t) \frac{\partial}{\partial x^{(k)}} \sigma^{(i,j)}(\boldsymbol{x}, t) \ ,$$

and the diffusion operator

$$D_2^{(i,j)}(\boldsymbol{x}, t) = \sum_{k=1}^{n} \sigma^{(i,k)}(\boldsymbol{x}, t) \sigma^{(j,k)}(\boldsymbol{x}, t) \ .$$

can be derived from the Langevin equation (1).

The concept of path integrals uses the fact, that the conditional probability on the left hand of (2) can be recursively applied, if the so called *short time propagator*

$$p\left(\boldsymbol{x}, t+\triangle t \,|\, \tilde{\boldsymbol{x}}, t\right) \ , \tag{4}$$

which gives the conditional transition probabilities for short times $\triangle t$ is known.

The conditional probability at a constant final time t_e is given by the so called *path integral*

$$p(\boldsymbol{x}_e, t_e \,|\, \boldsymbol{x}_0, t_0) = \lim_{N\to\infty} \underbrace{\int\cdots\int}_{N\text{ times}}$$

$$\prod_{i=0}^{N-1} d\boldsymbol{x}(t_i) \prod_{i=0}^{N-1} p\left(\boldsymbol{x}(t_{i+1}), t_{i+1} \,|\, \boldsymbol{x}(t_i), t_i\right) \tag{5}$$

by taking the limit to an infinite number N of (infinitely small) time steps $\triangle t$ constrained by the fact that the final time t_e has to be constant.

Usually, the propagator

$$p(\tilde{\boldsymbol{x}}, t+\triangle t \,|\, \boldsymbol{x}, t) \;=\; (2\,\pi)^{-\frac{n}{2}} [\det(\triangle t D_2(\boldsymbol{x}, t))]^{-\frac{1}{2}} \cdot \exp\Big(-\frac{1}{2} \sum_{l,k}^{n} \Big[(\tilde{x}^{(l)} - x^{(l)} - D_1^{(l)}(\boldsymbol{x}, t)\triangle t) \frac{1}{\triangle t} (D_2^{-1}(\boldsymbol{x}, t))^{(l,k)} (\tilde{x}^{(k)} - x^{(k)} - D_1^{(k)}(\boldsymbol{x}, t)\triangle t) \Big] \Big) \tag{6}$$

is used, but this propagator is not unique (Wehner and Wolfer 1983). With any propagator (4) satisfying eqn. (2) up to order $\triangle t^2$ eqn. (5) is a solution to the Fokker-Planck equation (Risken 1989).

For some processes, the limit of (6) can be computed analytically in closed-form if the short time propagators are appropriate, e.g. in the case of an Ornstein Uhlenbeck Process. In general, it is only possible to find numerical approximations.

4. STOCHASTIC REACHABILITY PROBLEM

Reachability analysis of hybrid systems is a topic with a lot of recent research effort (Bujorianu and Lygeros 2003, Bujorianu 2004). Usually there are two sets of states defined called target states and unsafe. The probability that a target state is reached while all unsafe states are avoided has to be computed to assess the reachability of a hybrid system (Koutsoukos and Riley 2006). In contrast to this formulation, here the only interest lies in the computation of the probability that some target states are reached at a certain time.

We assume in the sequel, that the execution of a GSHS admits a smooth probability density. This is the case e.g. for systems with bijective and deterministic reset maps $R(\cdot,\cdot)$ and eqi-dimensional continuous state-spaces X^q, i.e. $d(q) = d\,\forall\, q \in Q$ (Bect *et al.* 2006).

$$p(\boldsymbol{h}(t), t) = \text{Prob}\left(\{\tilde{\boldsymbol{x}}(t) \in [\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{x}}(t) + \boldsymbol{dx}] \tag{7}$$

$$\wedge\, \tilde{q}(t) = q(t)\}\right) , \tag{8}$$

with

$$\sum\!\!\!\!\!\!\int_{\mathcal{H}} p(\boldsymbol{h}(t), t)\boldsymbol{dh} = 1\,, \tag{9}$$

where we have introduced the symbol

$$\sum\!\!\!\!\!\!\int_{\mathcal{H}} \boldsymbol{dh} = \sum_{q(t)\in Q} \int_{X^1} \cdots \int_{X^{N_Q}} d\boldsymbol{x} \cdots d\boldsymbol{x}\,.$$

As in the case of continuous systems, the evolution of probability densities can be described by means of a partial differential operator, the so called infinitesimal generator of the stochastic process,

$$L_{\text{SHS}}(\boldsymbol{h}, t) = -\sum_{i=1}^{n} \frac{\partial}{\partial x^{(i)}} D_1^{(i)}(\boldsymbol{h}, t) + \tag{10}$$

$$+\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} D_2^{(i,j)}(\boldsymbol{h}, t)\ , \tag{11}$$

with boundary condition

$$p(\boldsymbol{h}, t) = \sum\!\!\!\!\!\!\int_{\mathcal{H}} p(\tilde{\boldsymbol{h}}, t) R(\boldsymbol{h}, \boldsymbol{d}\tilde{\boldsymbol{h}}) \boldsymbol{d}\tilde{\boldsymbol{h}}\,, \tag{12}$$

for all $\boldsymbol{h} \in \partial\mathcal{H}$.

The generator describes the behavior of the process in the interior of the state space. The interconnections of the different discrete states $q \in Q$ is given by the boundary conditions on each continuous domain ∂X^q. A mathematically rigorous introduction of infinitesimal generators can be found in (Bujorianu and Lygeros 2004) or much earlier (Feller 1952).

The conditional probability

$$p(\tilde{\boldsymbol{h}}, t \mid \boldsymbol{h}_0, t_0) = \text{Prob}\left(\boldsymbol{h}(t) \in [\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{h}} + \boldsymbol{dh}] \;\Big|\; \boldsymbol{h}(t_0) = \boldsymbol{h}_0\right) \tag{13}$$

gives the probability that a stochastic hybrid system comes from the hybrid state $\boldsymbol{h}_0 = (q_0, \boldsymbol{x}_0)^{\mathrm{T}}$ at time t_0 to a state $\boldsymbol{h}(t) \in [\boldsymbol{h}, \boldsymbol{h} + \boldsymbol{dh}]$ with $\boldsymbol{dh} = (0, \boldsymbol{dx})^{\mathrm{T}}$ at time $t > 0$.

The probability density of the hybrid state at time $\tilde{t}$ depends on the conditional probability and the initial density $p(\boldsymbol{h}(t), t)$ of the hybrid state $\boldsymbol{h} \in \mathcal{H}$ at time t. This is given by the equation

$$p(\boldsymbol{h}(\tilde{t}), \tilde{t}) = \sum\!\!\!\!\!\!\int_{\mathcal{H}} p(\boldsymbol{h}(\tilde{t}), \tilde{t} \,|\, \boldsymbol{h}(t), t)\, p(\boldsymbol{h}(t), t)\, \boldsymbol{dh(t)}$$

The stochastic reachability problem can now be posed as the determination of the probability

$$\text{Prob}\left(\boldsymbol{h}(T) \in \mathcal{H}_e \mid p(\boldsymbol{h}_0, t_0)\right) \tag{14}$$

that the hybrid system reaches an area $\mathcal{H}_e$ of the hybrid state space at time t under the knowledge of the initial hybrid state distribution $p(\boldsymbol{h}_0, t_0)$.

This probability (14) can be derived formally by calculating

$$p(\boldsymbol{h}(T),T) = \oint_{\mathcal{H}_e}\oint_{\bar{\mathcal{H}}} p(\boldsymbol{h}(T),T\,|\,\boldsymbol{h_0},t_0)\,p(\boldsymbol{h_0},t_0)\,\boldsymbol{dh_0 dh}\;.$$

5. HYBRID SYSTEMS PATH INTEGRALS

Using basic properties, an equation

$$p(\boldsymbol{h}(t_3),t_3) = \oint_{\bar{\mathcal{H}}} p(\boldsymbol{h}(t_3),t_3|\boldsymbol{h}(t_2),t_2)p(\boldsymbol{h}(t_2),t_2|\boldsymbol{h}(t_1),t_1)\;\boldsymbol{dh}(t_2) \tag{15}$$

that is similar to the Chapman-Kolmogorov equation for continuous systems can be derived.

The existence of path integrals for stochastic hybrid systems was claimed first in (Prasanth 2003). A generalization of the idea of Fokker-Planck equations for stochastic hybrid systems is described in (Bect *et al.* 2006).

If one iterates eqn. (15) while the time steps get smaller, one gets in the limit for the long time conditional probability

$$p(\boldsymbol{h}(t),t\,|\,\boldsymbol{h}(t_0),t_0) = \lim_{N\to\infty}\oint_{\bar{\mathcal{H}}}\cdots\oint_{\bar{\mathcal{H}}}\prod_{i=0}^{N-1} p(\boldsymbol{h}(t_{i+1}),t_{i+1}\,|\,\boldsymbol{h}(t_i),t_i)\,p(\boldsymbol{h}(t_0),t_0)\;\boldsymbol{dh}(t_i) \tag{16}$$

for stochastic hybrid systems, assumed that $t_{i+1} = t_i + \triangle t$ and $t = \lim_{N\to\infty} t_N$ holds.

This is similar to eqn. (5) but it will be very difficult to find propagators $p(\boldsymbol{h}(t_{i+1}),t_{i+1}\,|\,\boldsymbol{h}(t_i),t_i)$. As the limit of the integrals (5) only is computable in rare and very simple cases even for continuous systems, numerical methods have to be found to approximate the hybrid path integral (16). Known numerical methods for continuous path integrals can be extended which is discussed in the next section.

6. MARKOV CHAIN APPROXIMATIONS

By introducing finitely many mesh points $\bar{\boldsymbol{h}}_i = (\bar{q}_i, \bar{\boldsymbol{x}}_i) \in \mathcal{H}$, each representing a finite generalized volume $\triangle H_i = (\bar{q}_i, \triangle V_i)$ that partitions the hybrid state space $\mathcal{H}$ (in fact the continuous state spaces $X^{\bar{q}_i}$), we can approximate the continuous probability density $p(\boldsymbol{h}(t),t)$ by a discrete probability vector $\boldsymbol{p}(t) = (p_1(t), \dots p_{N_d}(t))^T$. The probabilities $p_i(t)$ can be calculated as

$$p_i(t) = \oint_{\triangle H_i} p(\boldsymbol{h}(t),t)\boldsymbol{dh} = \int_{\triangle V_i} p\left((\bar{q}_i,\boldsymbol{x}(t))^T,t\right)\boldsymbol{dx}\;.$$

The evolution of these approximated probability densities is given as the iteration of a Markov chain

$$\boldsymbol{p}(t+\triangle t) = \boldsymbol{T}^{\mathcal{H}}\left(\boldsymbol{p}(t) + \boldsymbol{T}^{\partial\mathcal{H}}\boldsymbol{p}(t)\right)\;. \tag{17}$$

The elements of the transition matrix $\boldsymbol{T}^{\mathcal{H}}$ are given by

$$T^{\mathcal{H}}_{i,j} = \frac{1}{\triangle V_j}\oint_{\triangle H_j}\oint_{\triangle H_i} p(\tilde{\boldsymbol{h}}(t+\triangle t),t+\triangle t|\boldsymbol{h}(t),t)\boldsymbol{dh}\,\boldsymbol{d\tilde{h}}\;, \tag{18}$$

i.e. by evaluation of the integral of the short time propagator (6) over the volume $\triangle H_j$ of all possible start points next to the mesh point j and the volume $\triangle H_i$ of end points respectively.

The term $\boldsymbol{T}^{\partial\mathcal{H}}$ represents the transitions from the boundaries due to the reset probability $R(\cdot,\cdot)$ given as

$$T^{\partial\mathcal{H}}_{i,j} = \frac{1}{\triangle V_j}\oint_{\triangle H_j}\oint_{\triangle H_i} R(\boldsymbol{h}(t),\boldsymbol{d\tilde{h}}(t))\boldsymbol{dh}(t)\;. \tag{19}$$

This is an extension of an algorithm for continuous systems called PathInt, for which local consistency and convergence have to be proven still (Wehner and Wolfer 1983).

The computational problem of this HybPathInt algorithm lies in the nonsparseness of the matrix $\boldsymbol{T} = \boldsymbol{T}^{\mathcal{H}} + \boldsymbol{T}^{\mathcal{H}}\boldsymbol{T}^{\partial\mathcal{H}}$, since the number of volumes grows exponentially with the order of the system.

This problem is overcome with the algorithm HybPathTree, which extends the PathTree algorithm for continuous systems (Ingber *et al.* 2001). This algorithm only uses transitions from $\bar{\boldsymbol{h}}_i$ to the nearest neighbors, i.e. a predefined subset of mesh points $\bar{\boldsymbol{h}}_j$ with the property that $\bar{q}_i = \bar{q}_j$, and whose volumes $\triangle V_j$ all lie next to the volume $\triangle V_i$.

This leads to a sparse transition matrix $\boldsymbol{T}$ with good approximation properties. It should be mentioned that these algorithms are similar to the Markov Chain approximation algorithms in (Kushner and Dupuis 2001), although they are mainly dealing with continuous systems and use different short-time-propagators.

7. EXAMPLES

Figure 1 shows the setup for the first example which is an extended version of a classical example in hybrid systems analysis, (Chase *et al.* 1993).

The switching rules for the outflow are

- Switch to the next tank if the current tank gets empty.
- If a tank gets full, immediately switch to that tank.

- Keep lowering the current tank if none of the above conditions are fulfilled

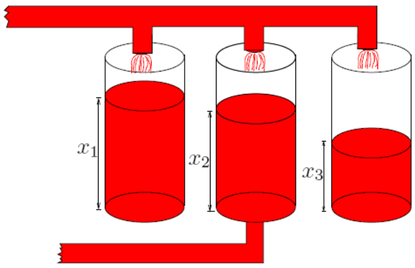

Fig. 1. Tanks System – Switched Outflow

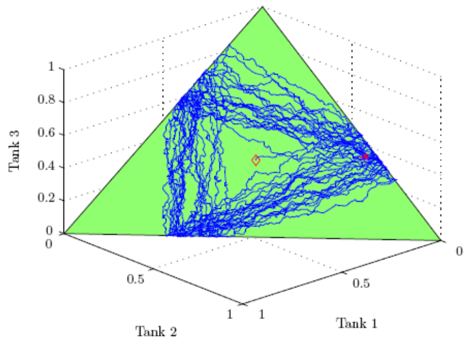

Fig. 2. Switched Outflow – Random Trajectory

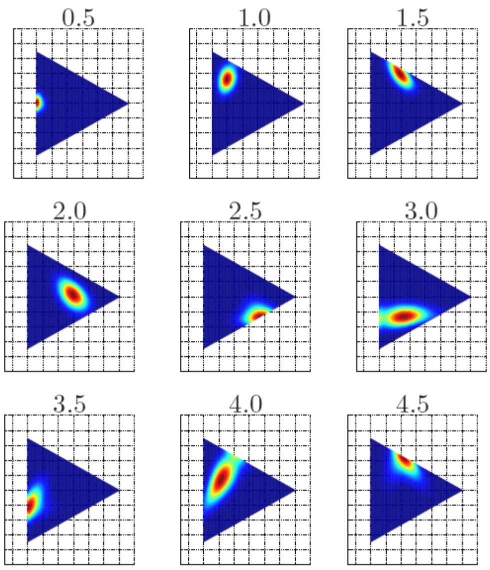

Fig. 3. Switched Outflow – HybPathTree

Changing the systems configuration as shown in Figure 4 changes the dynamics of the system completely. For the deterministic case the switched outflow system exhibits periodic behavior whereas the behavior of the switched inflow system is chaotic.

The switching conditions for the inflow are

- Switch to the next tank if the current tank gets full.
- If a tank gets empty, immediately switch to that tank.
- Keep filling the current tank if none of the above conditions are fulfilled

In order to avoid Zeno-behaviour in this case, we add the condition that if two tanks are empty at the same time, the tank with the smaller number will be filled first and the switching is locked for a small but non-zero time T_{fill}.

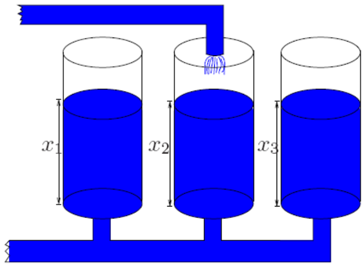

Fig. 4. Tanks System – Switched inflow

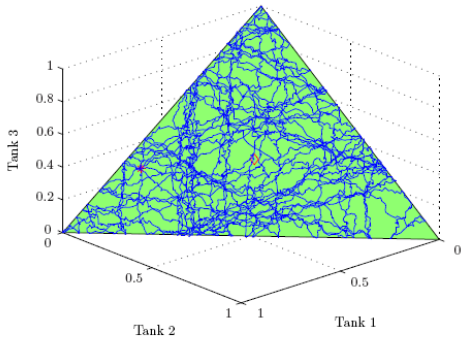

Fig. 5. Switched Inflow – Random Trajectory

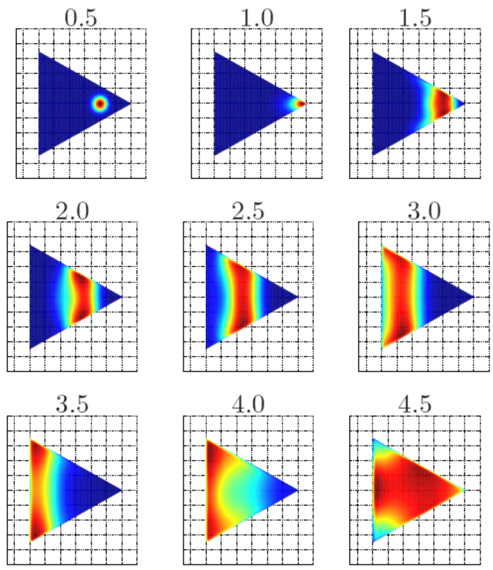

Fig. 6. Switched Inflow – HybPathTree

For both systems, the continuous state space is restricted by an invariant condition for the overall volume that can be expressed by $\sum_{i=1}^{3} x_i = 1$. The accessible state space is only two dimensional and has a triangular shape (see Figure 2). Additional Gaussian noise that doesn't contradict this conservation law is assumed. This reflects pressure fluctuations between concurring outflows or inflows respectively. Sample random trajectories are displayed in Figures 2 and 5. It is assumed further that all tanks have the same level at time 0.

The results of the HybPathTree algorithm are given in Figure 3 and Figure 6. The brightness indicates the value of the sum of probability distribution $\sum_q^3 p(\{q\} \times X^q)$ shown on the continuous state space $X = X^q \forall q \in Q$. Dark areas indicate either high or low values of the probability density whereas light areas give the areas of middle probability. The densities are given for the times $0.5, 1.0, 1.5, \ldots, 4.5$ displayed at the top of each graph. Axis labeling is omitted for the sake of simplicity - the triangles show all of the allowed 2-D state space. Computations times are some minutes on a standard PC. With the help of Monte-Carlo methods pictures can be produced with a similar resolution but with an order of magnitude higher computation times.

Figure 3 shows that the behavior of the stochastic switching outflow system is approximately periodic, similar to the deterministic case. At any time, it is very unlikely that the system is in a state where two tanks are empty, i.e. in one of the corners of the triangles. In contrast to that, Figure 6 shows that the stochastic switched inflow system after completely filling one tank (movement to the right corner until time 1) has a large ambiguity of the current state. This leads to the movement of the red area from time 1.5 to 3.5 to the left border of the triangle. The distribution shows that at time 4.5, the state can nearly be anywhere with the same probability.

8. CONCLUSIONS

It has been shown how path integral methods can be adapted to numerical prediction of probability densities for stochastic hybrid systems. The benefit for reachability analysis is ease of computation due to a sparse Markov chain structure and a good approximation quality because of precise short time propagators.

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