STOCHASTIC HYBRID NETCAD SYSTEMS FOR MODELING CALL ADMISSION AND ROUTING CONTROL IN NETWORKS

Zhongjing Ma * **Peter E. Caines** **
Roland Malhame ***

* *mazhj@cim.mcgill.ca*
** *peterc@cim.mcgill.ca*
*** *roland.malhame@polymtl.ca*

Abstract: In this paper a stochastic hybrid systems framework is established for the formulation of call admission control (CAC) and routing control (RC) problems in networks. The hybrid state process of the underlying system is a piecewise deterministic Markovian process (PDMP) evolving deterministically between random event instants at which times the state jumps to another state value. The random events in the system correspond to the arrival of call requests or the departure of connections. The resulting NETCAD stochastic state space systems framework permits the formulation and analysis of centralized optimal stochastic control with respect to specified utility functions [3,8,10,11].

Keywords: Call Admission Control, Routing Control, Stochastic Hybrid Systems, Piecewise Deterministic Markovian Processes, Point Processes

1. INTRODUCTION

Call admission control (CAC) and routing control (RC) in telecommunication networks have been topics of active research for decades (see e.g. [1,5,6,12]). In the 1960s, Benes pioneered routing control in telephone networks, providing a general mathematical formulation and the extensive associated analysis for telephone systems presented in [1].

In this paper, CAC and RC in networks are modeled as stochastic control problems for the so-called NETwork Connection Assignment and Departure (NETCAD) systems. The state process of the underlying NETCAD systems has some particular characteristics: (1) the state process has two parts: the first is a piecewise constant integer-valued point process [2], while the second is a variable dimension real-valued piecewise deterministic process; (2) it is a piecewise deterministic Markov process [4] with respect to a Markovian control law, where the state value evolves deterministically between the random event instants and jumps to some other state value at random event instants subject to controlled transition probabilities. NETCAD systems may be viewed as stochastic hybrid systems generalizing the class of deterministic hybrid systems which was defined in [13] and the references therein; as indicated above, the state process is composed of two components: a discrete component, denoting the connections along the set of (origin-destination) routes in the NETCAD network, and a continuous component constituting the vector of ages of the call requests and active connections in the NETCAD system.

The distinction between the work in this paper and that found in standard telecommunication texts and papers (see e.g. [1,5,6,12]) is that here a network system is represented within a formal stochastic systems framework with a specified class of input stochastic processes and a stochastic hybrid state space process with a controlled evolution equation while such a fine low level analysis is not formulated in [1,5,6,12]. This permits the formulation of an optimal stochastic control theory for NETCAD systems in the current work [8].

The paper is organized as follows. In Section 2, we present the formal definition of the networks upon which NETCAD systems are based; in Section 3, we formulate the network connection assignment and departure (NETCAD) systems and the Markov property of the state process is proved. Section 4 contains the conclusions and outlines future work.

2. THE NETWORK OF A NETCAD SYSTEM

2.1 NETCAD Networks

A NETCAD network is a capacitated network $Net(\mathbb{V},\mathbb{L},\mathbb{C})$ as defined below. Based upon this notion, a NETCAD system is defined in Definition 3.7 at the end of Section 3.

Definition 2.1. A *network*, or *graph*, $Net(\mathbb{V},\mathbb{L})$ consists of a set of vertices $\mathbb{V} = \{v_1, \cdots, v_V\}$, $V \in \mathbb{Z}_1$, and a set of lines $\mathbb{L} = \{l_1, \cdots, l_L\}, L \in \mathbb{Z}_1$, where each line $l \in \mathbb{L}$ is an ordered pair $(v', v'') \in \mathbb{V} \times \mathbb{V}$ of distinct vertices.
A *network* $Net(\mathbb{V},\mathbb{L})$ *with (line) capacities* $\mathbb{C} = \{c_s \equiv c(l_s) : 1 \le s \le L, c_s \in \mathbb{Z}_1\}$, shall be denoted by $Net(\mathbb{V},\mathbb{L},\mathbb{C})$. □

Definition 2.2. A *route*, r in the network $Net(\mathbb{V},\mathbb{L})$, connecting a vertex $o \in \mathbb{V}$ to a vertex $d \in \mathbb{V}$, $d \neq o$, is a finite sequence of vertices $r = (v'_1, \cdots, v'_k)$, such that

$$v'_1 = o,\ v'_k = d,$$
$$v'_i \neq v'_j,\ \text{for } i \neq j,$$
$$(v'_i, v'_{i+1}) \in \mathbb{L},\ \text{for } i = 1, \cdots, k-1.$$

The *set of routes* in the network $Net(\mathbb{V},\mathbb{L})$ is denoted by $\mathcal{R}$, and we denote R as the cardinality of $\mathcal{R}$, i.e. $R = |\mathcal{R}|$. □

Fig.1 is an illustration of 3 distinct routes between node v_1 and v_8, which are $(v_1, v_2, v_5, v_4, v_8)$, (v_1, v_4, v_8) and (v_1, v_3, v_7, v_8) respectively in a network.

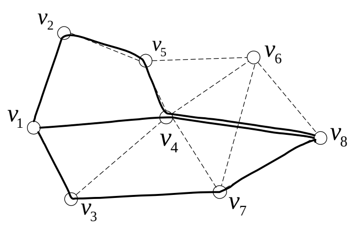

Fig. 1. Distinct routes in a network

Definition 2.3. The *feasible set* of *origin destination vertex pairs*, denoted by $\mathbb{V}^{\triangle}$, is defined as

$$\mathbb{V}^{\triangle} = \Big\{\langle o, d\rangle \in \mathbb{V} \times \mathbb{V};\ \exists\, r \in \mathcal{R}, \text{ s.t. } r = (v'_1, \cdots, v'_j), v'_1 = o, v'_j = d,\ o \neq d\Big\} \quad (1)$$

□

Remark: Call requests from any node to itself are excluded in this paper.

Definition 2.4. The *admissible set* of *connections*, denoted by $\mathcal{N}$, in $\mathcal{R}$ in the network with capacities $Net(\mathbb{V},\mathbb{L},\mathbb{C})$, is defined as

$$\mathcal{N} = \Big\{\mathbf{n} = (\mathbf{n}_r) \in \mathbb{Z}_+^R : \sum_{r \in \mathcal{R};\, l_s \in r} \mathbf{n}_r \le c_s,\ \forall s, 1 \le s \le L\Big\} \quad (2)$$

□

We observe that in the definition of $\mathcal{N}$, for each fixed l_s, the set of $r \in \mathcal{R}$ appearing in the sum is the set of routes each of which contains l_s as a line.
Since the routes in $\mathcal{R}$ are in one-to-one correspondence with the index of the components of a vector in $\mathbb{Z}_+^R \subset \mathbb{R}^R$, we shall by abuse of notation let $r \in \mathcal{R}$ also denote the integer indexing the corresponding vector component in $\mathbb{R}^R$.

2.2 A Simple Example of a Capacitated Network

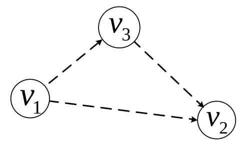

Fig. 2. A three node capacitated network

We consider a simple network $Net(\mathbb{V},\mathbb{L},\mathbb{C})$, see Fig.2, with

$$\mathbb{V} = \{v_1, v_2, v_3\}$$
$$\mathbb{L} = \{l_1 = (v_1, v_2), l_2 = (v_1, v_3), l_3 = (v_3, v_2)\}$$
$$\mathbb{C} = \{c_l = 2;\ l \in \mathbb{L}\}$$

Hence the set of routes, $\mathcal{R}$, is defined as

$$\mathcal{R} = \{r_1 = (v_1, v_3, v_2), r_2 = (v_1, v_2), r_3 = (v_1, v_3), r_4 = (v_3, v_2)\},$$

and the admissible connections set, $\mathcal{N}$, is defined as

$$\mathcal{N} = \{\mathbf{n} = (\mathbf{n}_{r_1}, \mathbf{n}_{r_2}, \mathbf{n}_{r_3}, \mathbf{n}_{r_4}) \in \mathbb{Z}_+^4;\ \sum_{r_i \in \mathcal{R};\, l \in r_i} \mathbf{n}_{r_i} \le 2,\ \forall l \in \mathbb{L}\}$$

3. THE NETWORK CONNECTION ASSIGNMENT AND DEPARTURE (NETCAD) SYSTEMS

3.1 The NETCAD System Framework

We consider the following:

(1) The probability space $(\Omega, \mathcal{F}, P)$ carries the family of independent $\mathbb{R}_+$ valued random variables

$$\{\tau_k^{\langle o,d\rangle}, \tau_j^c;\ k,j \in \mathbb{Z}_1, \langle o,d\rangle \in \mathbb{V}^{\triangle}\}, \quad (3)$$

where $\tau_k^{\langle o,d\rangle}$ denotes the length of the interval between the $(k-1)$th and the kth $\langle o,d\rangle$ call request; τ_j^c denotes the lifetime of the jth allocated connection in the network.

(2) For each $\langle o,d\rangle \in \mathbb{V}^{\triangle}$, the random variables $\{\tau_k^{\langle o,d\rangle}, k \in \mathbb{Z}_1\}$ are assumed to have a common arbitrary distribution $A^{\langle o,d\rangle}(t)$ with density function $a^{\langle o,d\rangle}(t)$, i.e.

$$\mathbb{P}(\tau_k^{\langle o,d\rangle} \le t) = A^{\langle o,d\rangle}(t) = \int_{-\infty}^{t} a^{\langle o,d\rangle}(s)ds. \quad (4)$$

(3) The random variables $\{\tau^{c_j}, j \in \mathbb{Z}_1\}$ are assumed to have a common arbitrary distribution $B(t)$ with density function $b(t)$, i.e.

$$\mathbb{P}(\tau^{c_j} \le t) = B(t) = \int_{-\infty}^{t} b(s)ds. \quad (5)$$

Remarks: For each $\langle o,d\rangle \in \mathbb{V}^{\triangle}$, we shall denote the call (connection) request process of $\langle o,d\rangle$ by $N^+_{\langle o,d\rangle}$. Then from the above assumption, we observe that, for each $\langle o,d\rangle \in \mathbb{V}^{\triangle}$, the $N^+_{\langle o,d\rangle}$ process is an autonomous, i.e. control independent, point process in $\mathbb{R}_+$ with independent inter-event times $\{\tau_k^{\langle o,d\rangle}, k \in \mathbb{Z}_1\}$. In other words, for each $\langle o,d\rangle \in \mathbb{V}^{\triangle}$, the call request process $N^+_{\langle o,d\rangle}$ is a point process with independent interarrival times distributed $A^{\langle o,d\rangle}(.)$.

Definition 3.1. The *sub-state space*, $Z_{\mathbf{n}}$, with respect to $\mathbf{n} = (\mathbf{n}_{r_1}, \cdots, \mathbf{n}_{r_R}) \in \mathcal{N}$, is defined as the following collection of index and age pairs:

$$Z_{\mathbf{n}} = \{z \equiv (\mathbf{n};\ \zeta);\ \zeta \in \mathbb{R}_+^{m(\mathbf{n})}\}, \quad \text{where} \quad (6)$$

$$m(\mathbf{n}) \triangleq |\mathbb{V}^{\triangle}| + \sum_{r \in R} \mathbf{n}_r,$$

$$\zeta \triangleq \Big(\{\overbrace{\zeta^{\langle o,d\rangle_1}, \zeta^{\langle o,d\rangle_2}, \cdots, \zeta^{\langle o,d\rangle_{|\mathbb{V}^{\triangle}|}}}^{|\mathbb{V}^{\triangle}|}\}_{od},$$

$$\{\overbrace{\zeta^{c_{r_1,1}}, \cdots, \zeta^{c_{r_1,\mathbf{n}_{r_1}}}}^{\mathbf{n}_{r_1}}\}_{r_1}, \cdots,$$

$$\{\overbrace{\zeta^{c_{r_R,1}}, \cdots, \zeta^{c_{r_R,\mathbf{n}_{r_R}}}}^{\mathbf{n}_{r_R}}\}_{r_R} \Big),$$

where the following constraints necessarily hold:

$$\zeta^{c_{r_i,1}} > \cdots > \zeta^{c_{r_i,\mathbf{n}_{r_i}}} \ge 0,\ \forall\, i \in \{1, \cdots, R\},$$

and where $\zeta^{\langle o,d\rangle_i}$ denotes the elapsed time since an $\langle o,d\rangle_i$ call request and $\zeta^{c_{r_i,j}}$ denotes the age of connection $c_{r_i,j}$.

The *state space*, denoted by Z, is defined as

$$Z \triangleq \dot{\bigcup}_{\mathbf{n} \in \mathcal{N}} Z_{\mathbf{n}}. \quad (7)$$

□

Remarks:

(1) *We set, for each $\langle o,d\rangle \in \mathbb{V}^{\triangle}$, a unique index number i, $i \in \{1, 2, \cdots, |\mathbb{V}^{\triangle}|\}$, and denote this $\langle o,d\rangle$ pair by $\langle o,d\rangle_i$, i.e. for each i, $i \in \{1, 2, \cdots, |\mathbb{V}^{\triangle}|\}$, there is a unique $\langle o,d\rangle \in \mathbb{V}^{\triangle}$ and $\langle o,d\rangle_i \equiv \langle o,d\rangle$.*

(2) *Since there are $\mathbf{n}_{r_i}$ connections along route r_i, then each of these connections can be uniquely denoted by $c_{r_i,j}$, $j(i) \in \{1, \cdots, \mathbf{n}_{r_i}\}$ and its age is denoted by $\zeta^{c_{r_i,j}}$.*

(3) *Specifically, the sequence of $\mathbf{n}_{r_i}$ connections along route r_i can be indexed by their age, or time since birth time, such that $\zeta^{c_{r_i,1}} > \zeta^{c_{r_i,2}} > \cdots > \zeta^{c_{r_i,\mathbf{n}_{r_i}}}$, corresponding to the fact that the earlier a connection was established, the smaller is its index number.*

Here we give an example to illustrate the definition of the state z.

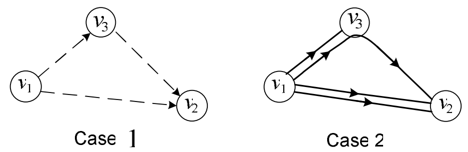

Fig. 3. Network state values with respect to different connection allocations

In Case 1 in Fig.3 there is no connection in the network, and the corresponding state value z takes the following form:

$$z = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta^{\langle v_1,v_2\rangle} \\ \zeta^{\langle v_1,v_3\rangle} \\ \zeta^{\langle v_2,v_3\rangle} \end{bmatrix} \right)$$

In Case 2 in Fig.3 there are respectively one, two and one connections along the route $r_1 \equiv (v_1, v_3, v_2)$, $r_2 \equiv (v_1, v_2)$ and $r_3 \equiv (v_1, v_3)$. The corresponding state value z takes the following form:

$$z = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta^{\langle v_1,v_2\rangle} \\ \zeta^{\langle v_1,v_3\rangle} \\ \zeta^{\langle v_2,v_3\rangle} \end{bmatrix}, \begin{bmatrix} \zeta^{c_{r_1,1}} \\ \zeta^{c_{r_2,1}} \\ \zeta^{c_{r_2,2}} \\ \zeta^{c_{r_3,1}} \end{bmatrix} \right)$$

Definition 3.2. The *(call request and connection departure) event set*, $E_{\mathbf{n}}$, with respect to a connection vector value $\mathbf{n} \in \mathcal{N}$, s.t. $\mathbf{n} = (\mathbf{n}_{r_1}, \cdots, \mathbf{n}_{r_R})$, is defined as:

$$E_{\mathbf{n}} = e^0_{\mathbf{n}} \dot{\bigcup} E^+_{\mathbf{n}} \dot{\bigcup} E^-_{\mathbf{n}}, \quad \text{where} \quad (8)$$

$$E^+_{\mathbf{n}} = \dot{\bigcup}_{\langle o,d\rangle_q \in \mathbb{V}^\triangle} e^+_{\langle o,d\rangle_q}$$

$$E^-_{\mathbf{n}} = \dot{\bigcup}_{r_i\in\mathcal{R}} \left\{ \dot{\bigcup}_{j\in\{1,\cdots,\mathbf{n}_{r_i}\}} e^-_{c_{r_i,j}} \right\},$$

where $\dot{\bigcup}$ denotes the disjoint union of the indicated entities and

(1) $e^0_{\mathbf{n}} \equiv \mathbf{0} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix} \in \mathbb{R}^M$ denotes absence of call request or connection departure event and $\mathbf{0}$ is the zero vector in $\mathbb{R}^M$, and

$$M \equiv M(\mathbf{n}) = |\mathbb{V}^\triangle| + \sum_{r\in R} \mathbf{n}_r;$$

(2) $e^+_{\langle o,d\rangle_q} \equiv \mathbf{1}_m = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \end{bmatrix} \in \mathbb{R}^M$ denotes the call request (event) from vertex o to vertex d, with $\langle o,d\rangle = \langle o,d\rangle_q$ and $\mathbf{1}_m$ is the m-th unit vector in $\mathbb{R}^M$, where $m = q$;

(3) $e^-_{c_{r_i,j}} \equiv \mathbf{1}_m = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \end{bmatrix} \in \mathbb{R}^M$ denotes the connection $c_{r_i,j}$ departure (event) and $\mathbf{1}_m$ is the m-th unit vector in $\mathbb{R}^M$, and

$$m = |\mathbb{V}^\triangle| + \sum_{l=1}^{i-1} \mathbf{n}_{r_l} + j.$$

The *(total) event set*, E, is defined as

$$E \triangleq \dot{\bigcup}_{\mathbf{n}\in\mathcal{N}} E_{\mathbf{n}}. \quad (9)$$

□

Definition 3.3. The *(extended) sub-state space*, X_n, with respect to a *connection vector*, $\mathbf{n} = (\mathbf{n}_{r_1}, \cdots, \mathbf{n}_{r_R}) \in \mathcal{N}$, is defined as the following: $X_{\mathbf{n}} \triangleq Z_{\mathbf{n}} \times E_{\mathbf{n}}$.

The *(total extended) state space*, X, is defined as: $X \triangleq \dot{\bigcup}_{\mathbf{n}\in\mathcal{N}} X_{\mathbf{n}}$. □

Definition 3.4. The *feasible control (value) set*, $U(x)$, with respect to an extended state value $x = (z, e) \equiv (\mathbf{n}, \zeta, e) \in X$, s.t. $\mathbf{n} = (\mathbf{n}_{r_1}, \cdots, \mathbf{n}_{r_R})$ and an event $e \in E^+_{\mathbf{n}}$, is defined as:

$$U(x) = \mathbf{0}^{\langle o,d\rangle}(x) \dot{\bigcup} \mathbf{1}^{\langle o,d\rangle}_r(x) \dot{\bigcup} - \mathbf{1}_c(x), \quad (10)$$

where

$$\mathbf{0}^{\langle o,d\rangle}(x) = \dot{\bigcup}_{\langle o,d\rangle_q\in\mathbb{V}^\triangle} 0^{\langle o,d\rangle_q}(x), \quad \text{with } 0^{\langle o,d\rangle_q}(x) \equiv 0^{\langle o,d\rangle_q(x)}(x),$$

$$\mathbf{1}^{\langle o,d\rangle}_r(x) = \dot{\bigcup}_{\langle o,d\rangle_q\in\mathbb{V}^\triangle} \left\{ \dot{\bigcup}_{\substack{r\in\mathcal{R}_{\langle o,d\rangle_q} \\ \mathbf{n}+\mathbf{1}_r\in\mathcal{N}}} 1^{\langle o,d\rangle_q}_r(x) \right\}, \quad \text{with } 1^{\langle o,d\rangle_q}_r(x) \equiv 1^{\langle o,d\rangle_q(x)}_r(x),$$

$$-\mathbf{1}_c(x) = \dot{\bigcup}_{r_i\in\mathcal{R}} \left\{ \dot{\bigcup}_{j\in\{1,\cdots,\mathbf{n}_{r_i}\}} - 1_{c_{r_i,j}}(x) \right\}, \quad \text{with } -1_{c_{r_i,j}}(x) \equiv -1_{c_{r_i,j}(x)}(x),$$

where

(1) $0^{\langle o,d\rangle_q}(x) \equiv \mathbf{0} \in \mathbb{R}^R$ denotes the fact that the call request $e^+_{\langle o,d\rangle_q}$ is rejected and $\mathbf{0}$ is the zero vector in $\mathbb{R}^R$;

(2) $1^{\langle o,d\rangle_q}_r(x) \equiv \mathbf{1}_i = \begin{bmatrix} \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \end{bmatrix} \in \mathbb{R}^R$ denotes that the call request $e^+_{\langle o,d\rangle_q}$ is accepted and is allocated on the route $r_i = r$, s.t. $r = \{v_{q_1}, \cdots, v_{q_m}\} \in \mathcal{R}$, $\langle v_{q_1}, v_{q_m}\rangle = \langle o,d\rangle_q$ and $\mathbf{n} + \mathbf{1}_r \in \mathcal{N}$;

(3) $-1_{c_{r_i,j}}(x) \equiv -\mathbf{1}_i = -\begin{bmatrix} \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \end{bmatrix} \in \mathbb{R}^R$ denotes that a connection in r_i departs.

The *control (value) set* U is defined as

$$U = \dot{\bigcup}_{x\in X} U(x). \quad (11)$$

Here we give an example to display a feasible control with respect to an index $n \in \mathcal{N}$. See Fig.4.

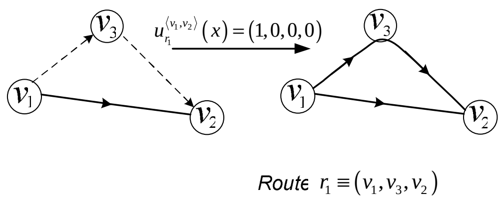

Fig. 4. A feasible control with respect to a state value $\big((0,1,0,0),\zeta\big)$

Suppose that $x = \big((0,1,0,0),\zeta\big) \in X$ and $e = (1,0,0,0,0)$, i.e. a call request $\langle v_1, v_2\rangle$ occurs, a feasible control can be $u^{\langle v_1,v_2\rangle}_{r_1}(x) = (1,0,0,0)$, i.e. the call from v_1 to v_2 is allocated to the route $r_1 \equiv (v_1, v_3, v_2)$.

Definition 3.5. When z and e depend in a progressively measurable way on $(\Omega, \mathcal{F}, \mathbb{P})$, we refer to $z = \{z(t,\omega);\ t \in [0,T], \omega \in \Omega\}$ and $e = \{e(t,\omega);\ t \in [0,T], \omega \in \Omega\}$ as *state* and *event processes*; and x as the *(extended) state process*

$$x = \Big\{ x(t,\omega) \triangleq \big(z(t^-,\omega), e(t,\omega)\big), t \in [0,T], \omega \in \Omega \Big\}, \text{ with } z(t^-,\omega) \equiv \lim_{s\uparrow t} z(s,\omega). \quad (12)$$

We define $\mathcal{F}_t \triangleq \bigvee_{r\in[0,t]} \sigma(x_r) \in \mathcal{F}$, i.e. $\mathcal{F}_{.}$ is the *natural filtration* extended by the process x, where $\sigma(x_r)$ denotes the σ-field generated by the random variable x_r. □

Definition 3.6. The set of *state dependent*, or *Markovian (measurable), control laws* is denoted by $\mathcal{U}[0,T]$, $0 \le T < \infty$, and is given by,

$$\mathcal{U}[0,T] = \big\{ u : [0,T] \times \Omega \to U; \text{ s.t. } u_t \text{ is } \sigma(x_t) \text{ measurable}, t \in [0,T] \big\} \quad (13)$$

$$\mathcal{U}[0,\infty) = \cup_{T\ge 0}\, \mathcal{U}[0,T] \quad (14)$$

□

Definition 3.7. A family of state processes $\{x_t \triangleq (z_{t^-}, e_t)\}$ taking values in X in a capacitated network $Net(\mathbb{V}, \mathbb{L}, \mathbb{C})$, together with a family of feasible controls u, is called a *network connection assignment and departure (stochastic) system*, or a *NETCAD system*, for short. □

Definition 3.8. We term a *sequence* of *event instants* $\{t_j(\omega)\}$ in $\mathbb{R}_+$

$$0 \le t_1(\omega) < \cdots < t_j(\omega) < t_{j+1}(\omega) < \cdots, \quad (\Omega, \mathcal{F}, \mathbb{P}), \ \omega \in \Omega, \tag{15}$$

at which random call request or connection departure event occurs as a sequence of *(random) event instants* $t_. : \mathbb{Z}_+ \times \Omega \to \mathbb{R}_+$. The sequence $\tau_. : \mathbb{Z}_1 \times \Omega \to \mathbb{R}_+$, with $\tau_{k+1} \triangleq t_{k+1}(\omega) - t_k(\omega)$, where $t_0(\omega) \equiv 0$ is defined as the sequence of *event intervals* (associated to $t_.(\omega)$). □

Definition 3.9. State response or *transition equation*, with a measurable Markov control law $u \in \mathcal{U}[0,\infty)$, i.e. u_t is $\sigma(x_t^u)$ measurable, for the evolution of the state process

$$x^u : [0,\infty) \times \Omega \to X, \tag{16}$$

with initial state value $x_{t_0}^u \equiv (z_0^u, e_0^u) = \big((\mathbf{n}, \zeta), \mathbf{0}\big)$, is given by

$$x_t^u = (z_t^u, e_t^u), \quad t_{i-1} \le t < t_i, \ i \in \mathbb{Z}_1 \tag{17}$$

$$\begin{aligned} z_t^u &= z_{t_{i-1}}^u + \int_{t_{i-1}}^t (\mathbf{0_n}, \mathbf{I}_\zeta) dr \\ &= \big(\mathbf{n}_{t_{i-1}}, \zeta_{t_{i-1}} + [t - t_{i-1}]\, \mathbf{I}_\zeta\big), \end{aligned} \tag{18}$$

where $t_{i-1} \le t < t_i$, $i \in \mathbb{Z}_1$.

$$e_t^u = \begin{cases} e \in E_{\mathbf{n}}, & t = t_i, \ i \in \mathbb{Z}_1, \quad \mathbf{n} \equiv \mathbf{n}(x_{t_{i-1}}^u) \\ \mathbf{0}, & \text{otherwise} \end{cases} \tag{19}$$

$$\begin{aligned} x_{t_i^+}^u &\equiv \lim_{s \downarrow t_i} x_s = \lim_{s \downarrow t_i} \big(z_{s^-}^u, e_s^u\big) = \big(\lim_{s \downarrow t_i} z_{s^-}^u, \lim_{s \downarrow t_i} e_s^u\big) \\ &= \big(\lim_{s \downarrow t_i} \lim_{r \uparrow s} z_r^u, \mathbf{0}\big) = \big(z_{t_i^+}^u, \mathbf{0}\big) = \big(z_{t_i}^u, \mathbf{0}\big), \end{aligned} \tag{20}$$

where $\mathbf{0}$ denotes the zero vector with some proper dimension, $\mathbf{I}_\zeta$ denotes $(1, \cdots, 1)$ with a dimension depending on the value of ζ.

The state transition equation at the event instant t_i is the following:

$$z_{t_i}^u = \Big(\mathbf{n}_{t_i^-} + u_{t_i}(x_{t_i}^u), \ \mathbf{A}[\mathbf{I}_{M\times M} - e_{t_i} e_{t_i}^T]\, \zeta_{t_i^-}\Big), \tag{21}$$

where M is the dimension of the vector $\zeta_{t_i^-}$ and

$$\begin{aligned} \mathbf{A} &\equiv \mathbf{A}(x_{t_i}^u, u_{t_i}(x_{t_i}^u)) \\ &= \begin{cases} \mathbf{A}^+_{(M+1)\times M}, & \text{if } u_{t_i}(x_{t_i}^u) > 0 \\ \mathbf{A}^-_{(M-1)\times M}, & \text{if } u_{t_i}(x_{t_i}^u) < 0 \\ \mathbf{I}_{M\times M}, & \text{otherwise} \end{cases} \end{aligned}$$

$$\mathbf{A}^+_{(M+1)\times M} = \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{0}_{(m+1)\times(M-m)} \\ \mathbf{0}_{(M-m+1)\times m} & \mathbf{I}_{(M-m)\times(M-m)} \end{bmatrix},$$

$$\text{where } m = |\mathbb{V}^\triangle| + \sum_{j=1}^{l} \mathbf{n}_{r_j}(t_i^-), u_{t_i}(x_{t_i}^u) = 1^{r_l}_{\langle o,d\rangle_q}$$

$$\begin{aligned} &\mathbf{A}^-_{(M-1)\times M} \\ &= \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{0}_{m\times(M-m)} \\ \mathbf{0}_{(M-m-1)\times(m+1)} & \mathbf{I}_{(M-m-1)\times(M-m-1)} \end{bmatrix}, \end{aligned}$$

$$\text{where } m = |\mathbb{V}^\triangle| + \sum_{j=1}^{l-1} \mathbf{n}_{r_j}(t_i^-) + [k-1] \quad \text{and} \quad u_{t_i}(x_{t_i}^u) = -1_{c_{r_l,k}}$$

Remarks: $\mathbf{I}_{j\times j}$, *$j \in \mathbb{Z}_1$, denotes the j-dimension identity matrix.* □

3.2 An Example of a NETCAD System

Considering the capacitated network defined in Section 2.2, we specify a realization of the controlled state process z^u during $[0, t_2)$ to help the audiences to understand the state transition procedure.

Suppose that $z_0^u = (\mathbf{n}_0^u, \zeta_0^u) = \left(\begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} a\\b\\c \end{bmatrix}\right)$, $a, b, c \in \mathbb{R}_+$, then for $0 < t \le t_1$,

$$z_{t^-}^u = \left(\begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} t+a\\t+b\\t+c \end{bmatrix}\right), \quad z_{t_1^-}^u = \left(\begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} t_1+a\\t_1+b\\t_1+c \end{bmatrix}\right),$$

Remarks: during $[0, t_1)$, the dimension of the vector ζ_t is 3, since there is no connection in the network during this interval.

Suppose at t_1, $e_{t_1} = \begin{bmatrix} 1\\0\\0 \end{bmatrix}$ and $u_{t_1}(x_{t_1}^u) = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}$, i.e. an $\langle o,d\rangle_1 \equiv \langle v_1, v_2\rangle$ call request occurs at t_1 and this call request is allocated to the route r_2, then

$$\begin{aligned} z_{t_1}^u &= \Big(\mathbf{n}_{t_1^-} + u_{t_1}(x_{t_1}^u), \ \mathbf{A}[\mathbf{I}_{3\times 3} - e_{t_1} e_{t_1}^T]\zeta_{t_1^-}\Big) \\ &= \left(\begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} + \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \mathbf{AB}\begin{bmatrix} t_1+a\\t_1+b\\t_1+c \end{bmatrix}\right), \quad \text{where} \\ &\qquad \mathbf{A} = \mathbf{A}^+_{4\times 3} = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0 \end{bmatrix} \\ &\qquad \mathbf{B} = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} - \begin{bmatrix} 1\\0\\0 \end{bmatrix}\begin{bmatrix} 1&0&0 \end{bmatrix} \\ &= \left(\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\t_1+b\\t_1+c\\--- \\0 \end{bmatrix}\right). \end{aligned}$$

Then we obtain that, for any $t_1 < t \le t_2$,

$$z_{t^-}^u = z_{t_1}^u + \int_0^{t-t_1} (\mathbf{0_n}, \mathbf{1}_\zeta) dr = \left(\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} t-t_1\\t+b\\t+c\\---\\t-t_1 \end{bmatrix}\right),$$

and $z_{t_2^-}^u = \left(\begin{bmatrix} 0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} t_2-t_1\\t_2+b\\t_2+c\\---\\t_2-t_1 \end{bmatrix}\right)$.

Remarks: A connection was established in the network at t_1. So $Dim(\zeta_{t_1}) = Dim(\zeta_{t_1^-}) + 1 = 3 + 1 = 4$, i.e during $[t_1, t_2)$ the dimension of the vector ζ_t is 4. □

3.3 Hybrid Nature of the NETCAD System

The state process z^u of the NETCAD system is composed of the two parts, such that

$$z_t^u = \begin{pmatrix} \mathbf{n}_t \\ \zeta_t \end{pmatrix}, \quad \text{at any instant } t,\ t \in [0, T], \quad (22)$$

where $\mathbf{n}$, a discrete process, keeps unchanged between the random event instants and is transferred to a state value $\mathbf{n}' \in \mathcal{N} \subset \mathbb{Z}_+^R$, with some controlled transition probability; while ζ, a continuous process, evolves deterministically between the random event instants and is transferred to a state value $\zeta' \in \mathbb{R}^{M(\mathbf{n}')}$, where the dimension of ζ' is dependent on the value of $\mathbf{n}'$.

3.4 Markov Property of the State Process

Lemma 3.1. [9] For each event instant, t_j, $j \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$, $\{t_j \leq t\} \in \mathcal{F}_t$, i.e. t_j is a stopping time of the filtration $\mathcal{F}_\cdot$. □

Theorem 3.1. [9] For all $t, s \geq 0$ and any $\Gamma \in \sigma(X)$, where $\sigma(X)$ denotes the σ-field generated by X,

$$\mathbb{P}\big(x_{t+s}^u \in \Gamma \,|\mathcal{F}_t\big) = \mathbb{P}\big(x_{t+s}^u \in \Gamma \,|\sigma(x_t^u)\big), \quad (23)$$

i.e. with extended state feedback the overall closed loop NETCAD system generates a Markov state process x^u. □

4. CONCLUSION

The stochastic state space dynamical systems framework for call request and routing in what are termed NETCAD networks has been introduced in this paper. A feature of the resulting stochastic NETCAD systems is that they are hybrid stochastic systems with variable dimension state processes; for these processes certain properties, such as the Markovian property and piecewise continuity, have been established.

The NETCAD framework permits the formulation and analysis of centralized optimal stochastic control with respect to specified utility functions; in particular, this entails the derivation of the Hamilton-Jacobi-Bellman equation for optimally controlled NETCAD systems [3,8,10,11]. In addition this framework provides the foundation for the current work [7] on decentralized suboptimal control based upon state aggregation and estimation.

5. ACKNOWLEDGEMENTS

The authors gratefully acknowledge many discussions of this work with Lorne G. Mason.

REFERENCES

[1] V.E. Benes, "Mathematical theory of connecting networks and telephone traffic", Academic Press, New York, 1965.

[2] R. Boel and P. Varaiya, "Optimal Control of Jump Processes", *SIAM J. Control and Optimization*, vol. 15, No.1, January 1977.

[3] P.E. Caines, Z. Ma and L.G. Mason, "Stochastic Control of Communication Networks". *Mediterranean Conference on Complex Systems*, Aiya Napa, Cyprus, July, 2005.

[4] M.H.A. Davis, "Markov Models and Optimization", Chapman & Hall, London, 1993.

[5] Z. Dziong and L.G. Mason, "Call Admission and Routing in Multi-Service Loss Networks", *IEEE Trans. Commun.*, vol. 42, pp. 2011-2022, Feb./Mar./Apr. 1994.

[6] F.P. Kelly and R.J. Williams, "Dynamic routing in stochastic networks", *IMA Volumes in Mathematics and its Applications*, 71. Springer-Verlag, New York, 1995. 169-186.

[7] Z. Ma, "Call Admission Control and Routing Control in Integrated Communication Networks via Dynamic Programming", Masters Thesis, McGill U., Feb. 2005.

[8] Z. Ma, P.E. Caines and R.P. Malhamé, "Optimal Stochastic Control of Networks: Call Admission and Routing for Simple Networks" . Submitted to *International Conference on Intelligent Systems And Computing: Theory And Applications (ISYC'06)*.

[9] Z. Ma, P.E. Caines and R.P. Malhamé, "Stochastic Control of Network Systems I: NETCAD State Space Structure & Dynamics", Submitted to *45th IEEE Int. Conf. Decision and Control*.

[10] Z. Ma, P.E. Caines and R.P. Malhamé, "Stochastic Control of Network Systems II: NETCAD Optimal Control & the HJB Equation", Submitted to *45th IEEE Int. Conf. Decision and Control*.

[11] Z. Ma, P.E. Caines and R.P. Malhamé, "Viscosity Solutions of HJB Equations for CAC and RC Problems in NETCAD Systems", Research Report Dept. ECE, McGill U., Feb, 2006.

[12] P. Marbach, O. Mihatsch, and J.N. Tsitsiklis, "Call Admission Control and Routing in Integrated Services Networks Using Neuro-Dynamic Programming", *IEEE Trans. Commun.*, vol. 18, no.2, pp. 197-207, Feb. 2000.

[13] M. S. Shaikh and P. E. Caines, "On the optimal control of hybrid systems: Analysis and algorithms for trajectory and schedule optimization". In Proc. *42nd IEEE Int. Conf. Decision and Control*, pp 2144-2149, Maui, Hawaii, 2003.