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CONTROL-INVARIANCE OF SAMPLED-DATA HYBRID SYSTEMS WITH PERIODICALLY CLOCKED EVENTS AND JITTER

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Abstract: Silva and Krogh formulate a sampled-data hybrid automaton to deal with time-driven events and discuss its verification. In this paper, we consider a state feedback control problem of the automaton. First, we introduce two transition systems as semantics of the automaton. Next, using these transition systems, we derive necessary and sufficient conditions for a predicate to be control-invariant. Finally, we show that there always exists the supremal control-invariant subpredicate for any predicate.

Keywords: Hybrid automaton; Jitter; Control-invariance; State feedback;

1. INTRODUCTION

A hybrid automaton is widely used as a model of hybrid systems (Henzinger, 1996). A computer-controlled system is an example of hybrid systems since it has both continuous and discrete variables associated with the physical process (the plant) and the logical dynamics (the control logic and external environment), respectively. In the computer-controlled systems, the measurements and subsequent discrete control actions are usually time-driven events and there exist the jitter variations in their occurrence times. Silva and Krogh proposed an extension of a hybrid automaton called a sampled-data hybrid automaton (SDHA) to model explicitly discrete transitions that are based on time-driven sampling of the continuous state and define a transition system called a sampled-trace transition system (STTS) as semantics to verify its dynamics (Silva and Krogh, 2000; Silva and Krogh, 2001). The SDHA is a pair of a clock structure and a hybrid automaton with clocked and unclocked events. Unclocked events are enabled when its continuous states satisfy their guards while clocked events are time-driven, that is, they are enabled only at specified sampling times in addition to constraints for their guards. A clock structure, which is given by variation interval in the initial phase, a period of clocked times, and a sampling jitter, specifies sequences of sampling times that can be generated in the system. The SDHA can be used as a model of various controlled systems with time-driven events. For example, in networked control systems, the samplings and subsequent control actions through the network can be associated with clocked events while the changes of external environments and internal model changes of plants are associated with unclocked events. Silva and Krogh (2001) propose a verification method for the SDHA using approximated quotient transition systems. But, to the best of our knowledge, a control problem of the SDHA has not been studied.

In discrete event systems, a state feedback controller is often used as a logical control problem, where a control specification is given by a predicate on their states(Ramadge and Wonham, 1987). Its control action is determined by their current states. A discrete event system is called control-invariant if there exists a state feedback controller such that all reachable states in the closed-loop system controlled by the controller satisfy the predicate. A necessary and sufficient condition for the system to be control-invariant is derived(Ramadge and Wonham, 1987). The state feedback control for the discrete event systems is extended to hybrid systems(Chen and

Hanisch, 1999) and hybrid automata with forcible events (Ushio and Takai, 2005). Forcible events are events that can be forced to occur by the control so that temporal performance can be improved (Brandin and Wonham, 1994). Ushio and Takai extend transition semantics of uncontrolled hybrid systems (Henzinger, 1996) to controlled hybrid systems with forcible events and show necessary and su cient conditions for a predicate to be control-invariant. They also show that there always exists the supremal control-invariant subpredicate for any predicate.

In this paper, we consider state feedback control of the SDHA. Since enablingness of the clocked events depends on the clock structure, a state feedback controller is time-varying in general while it is time-invariant in both hybrid systems without clocked events and discrete event systems. On the other hands, in conventional sampled-data control systems where controllers and sensors activate periodically, the sampling times are periodic but data transmission time may be uctuated so that a jitter must be taken into consideration. So, we introduce a slight modi cation of the clock structure to represent sequences of periodic sampling times with jitter so that a periodic state feedback controller is designed.

The rest of this paper is organized as follows: Section 2 reviews transition systems, several predicate transformations, and a concept of control-invariance. Section 3 introduces a controlled SDHA with forcible events, which is given by a pair of a clock structure and a hybrid automaton with clocked and unclocked events. Two labeled transition systems are introduced to de ne its semantics and necessary and su cient conditions for existence of state feedback controllers based on the transition systems are shown. Section 3 shows that there always exists the supremal control-invariant subpredicate for any predicate.

2. PRELIMINARIES

We use a labeled transition system $T{=}(Q, Act, \mathscr{T}, Q_0)$ in order to de ne semantics of controlled hybrid systems, where Q is a set of states, Act is a set of labels, $\mathscr{T}{\subseteq}Q{\times}Act{\times}Q$ is a state transition relation, $Q_0{\subseteq}Q$ is a set of initial states. $Act(T;q){\subseteq}Act$ is de ned by $Act(T;q){=}\{a{\in}Act|^{\exists}q' \in Q$ s.t $(q,a,q'){\in}\mathscr{T}\}$. Let $\mathscr{P}(Q)$ be the set of all predicates on Q. A predicate P is $true$ at state $q{\in}Q$ if $P(q){=}1$, and $false$ if $P(q){=}0$. Denoted by $\vee$, $\wedge$, and $\neg$ are disjunction, conjunction, and negation of predicates, respectively. The term "predicate" and "subset"$(=\{q{\in}Q|P(q){=}1\})$ can be used interchangeably. A partial order "$\leqslant$" for $\mathscr{P}(Q)$ is de ned as follows: for $P_1, P_2{\in}\mathscr{P}(Q)$, $P_1{\leqslant}P_2{\Leftrightarrow}P_1(q){\leq}\ P_2(q)$ for $^{\forall}q{\in}Q$. For each $a{\in}Act$, a predicate D_a is de ned by

$$D_a(q) = \begin{cases} 1 & \text{if } a \in Act(T;q), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

We de ne predicate transformations $wp_a{:}\mathscr{P}(Q){\rightarrow}\mathscr{P}(Q)$ and $wlp_a{:}\mathscr{P}(Q){\rightarrow}\mathscr{P}(Q)$ as follows:

$$wp_a(P)(q) = \begin{cases} 1 \text{ if } Post(q,a){\neq}\emptyset \text{ and} \\ \quad (\text{for } ^{\forall}q'{\in}Post(q,a))\ P(q'){=}1, \\ 0 \text{ otherwise,} \end{cases}$$

$$wlp_a(P) = wp_a(P)\ \vee\ \neg D_a,$$

where $Post(q,a){=}\{q'{\in}Q|(q,a,q'){\in}\mathscr{T}\}$. For a subset $A{\subseteq}Act$, we de ne $wp_A(P){=}\bigvee_{a\in A} wp_a(P)$. For a subset $A{\subseteq}Act$, $P{\in}\mathscr{P}(Q)$ is said to be $(T;A)$-invariant i , for $^{\forall}a{\in}A$, $P{\leqslant}wlp_a(P)$. Let $\Re_{\geq 0}$ and $\Re_{>0}$ be the sets of non-negative and positive reals, respectively. For a piecewise continuous function $h : \Re_{\geq 0} \rightarrow A$, where A is an arbitrary set, $d(h) = d_0(h)d_1(h)d_2(h)\ldots$ is the sequence of points where h is discontinuous. For $t \in \Re_{\geq 0}$, $h(t\ \ )$ and $h(t^{+})$ denote the values for the limits of h at t from the left and right, respectively.

3. SAMPLED-DATA HYBRID AUTOMATON AND STATE FEEDBACK CONTROL

Silva and Krogh proposed a sampled-data hybrid automaton (SDHA) which is modeled by a pair of a hybrid automaton with clocked and unclocked events and a clock structure (Silva and Krogh, 2001). First, we modify the SDHA to introduce a control mechanism with forcible events which are forced to occur by external control action. We de ne a controlled hybrid automaton with forcible events as follows: $H{=}(V, E,\ \ ,\ _{con},\ _{uncon},\ _{forc},\ _{cl},\ _{uncl}, X, init, Flow, jump)$.

- V, are sets of nodes, events, respectively;
- $_{con}$ and $_{uncon}$ are sets of controllable and uncontrollable events, respectively.
- $_{forc}$ is the set of forcible events and, for simplicity, we assume that $_{forc}{\subseteq}\ _{con}$;
- $_{cl}$ and $_{uncl}$ are sets of clocked and unclocked events, respectively;
- $E{\subseteq}V{\times}\ \ {\times}V$ is the set of edges with associated events, that is, $e(v,\sigma,v')$ is an edge $e \in E$ from v to v' labeled by event σ and corresponds to a discrete transition by the occurrence of σ;
- $X \subseteq \Re^n$ is the set of continuous variables;
- $init : V \rightarrow 2^X$ assigns the initial continuous states, that is, $init(v)$ is the set of all possible initial continuous states in node v;
- $Flow{=}\{f_v{:}X{\rightarrow}\Re^n|^{\forall}v{\in}V\}$ is the set of ows de ning the continuous state equation $\dot{x}{=}f_v(x)$ for each discrete state $v{\in}V$. Then let $x{=}\zeta_{v,x_0}(t)$ be a trajectory which starts from discrete state v at time $t{=}0$ and the initial continuous state $x(0){=}x_0$, on which no event occurs; and
- $jump : E{\rightarrow}2^{X\times X}$ is the jump relation, that is, $(x,x'){\in}jump(e)$ means that the continuous state x jumps to x' when σ occurs.

Note that $_{con} \cap\ _{uncon} =\ _{cl} \cap\ _{uncl} = \emptyset$ and $=\ _{con}{\cup}\ _{uncon}{=}\ _{cl}{\cup}\ _{uncl}$. In addition, $_{i,j}$ denotes $_{i} \cap\ _{j}$, where $i{\in}\{con, uncon, forc\}$ and $j{\in}\{cl, uncl\}$. The state set Q_H of hybrid automaton H is given by $Q_H{=}\{(v,x)|^{\forall}v{\in}V,^{\forall}x{\in}X\}$. Let $guard(e)$ be an occurrence condition of the discrete transition by edge $e \in E$: $guard(e){=}\{x{\in}X|^{\exists}x'{\in}X$ s.t. $(x,x'){\in}jump(e)\}$.

Assumption 1 We assume: **(1)** If, for $e(v,\sigma,v')\in E$ and $x,x'\in X$, $(x,x')\in jump(e)$, then $x'\notin guard(\tilde{e})$ for any $\tilde{e}(v',\sigma',v'')\in E$, **(2)** for any $e(v,\sigma,v')\in E$ and $\sigma\in {}_{forc}$, $guard(e)$ is a closed set, and **(3)** if $e(v,\sigma,v')\in E$, then $v\neq v'$.

The enablingness of clocked events does not depend on only the continuous states but also sampling times. To model the sequence of the sampling times, Silva and Krogh defines the clock structure such that durations between each sampling times are specified by the clock period interval and the sampling jitter interval (Silva and Krogh, 2001). In conventional computer controlled systems and networked control systems, controllers and sensors are activated periodically with a specified sampling period and computational delay in processors and/or data transmission delay in networks may cause a jitter in occurrence of clocked events. So, we will modify the clock structure. Let T_θ be a specified sampling period and J the maximum jitter. We assume for simplicity that $J<T_\theta$. Denoted by ${}_n$ is the n-th "nominal" sampling times for the sampling process. Then, we have ${}_n = {}_0+nT_\theta$. ${}_0$ is the initial nominal sampling time. Let $=\{ {}_n |\ {}_n= {}_0+nT_\theta\}$. Thus, a possible sequence of the sampling times $c=c_0c_1\ldots$ are given by the following modified clock structure:
$C(, J)=\{c_0c_1\ldots|\text{for } {}^\forall i\geq 0,\ c_i= {}_i+J_i,\ J_i\in[0,J]\}$. Thus, the controlled SDHA H_C is defined by a pair $(H, C(, J))$ of the hybrid automaton with forcible events and the modified clocked structure.

We introduce a state feedback controller $f(q,t)$ with $q\in Q_H$ and $t\in\Re_{\geq 0}$ taking control of forcible and clocked events into consideration, which is an extension of a state feedback controller with forcible events (Ushio and Takai, 2005). Let ${}_{cl}=\{\gamma|\ {}_{cl,uncon}\subseteq\gamma\subseteq {}_{cl}\}$ be the set of control patterns for clocked events. A state feedback controller f is described by 4-tuple $f = (f_{cl,1}, f_{cl,2}, f_{uncl,1}, f_{uncl,2})$, where

- $f_{cl,1}\ :\ Q_H\times\Re_{\geq 0}\to {}_{cl}$ gives a set control-enabled clocked events;
- $f_{cl,2}\ :\ Q_H\times\Re_{\geq 0}\to 2^{\Sigma_{cl,forc}}$ gives a set of forcible clocked events which are control-enabled and forced to occur; and
- $f_{uncl,1}$ and $f_{uncl,2}$ are given in similar ways to above two definitions, respectively.

Note that a state feedback controller f does not only depend on state, but also time since the sampling times have an effect on the behavior of the controlled SDHA and the control action depends on when clocked events are enabled. In addition, the nominal sampling times ${}_i$ are periodic, the fluctuation of the actual sampling times c_i are in a time interval given by the jitter, the continuous flows are determined by a time-invariant system in each node, and a control specification is given by a predicate on Q_H independently of time. So, it is sufficient to consider a periodic state feedback controller f as follows: for ${}^\forall q\in Q_H$ and ${}^\forall t\geq {}_0$, $f_{i,j}(q,t)=f_{i,j}(q,t+T_\theta)$, where $i\in\{cl,uncl\}$ and $j\in\{1,2\}$. Denoted by H_C^f is the SDHA controlled by the state feedback controller f.

Henzinger (1996) introduces two transition systems, called timed and timed-abstract transition systems, in order to represent semantics of the hybrid automaton and Ushio and Takai (2005) extended them to a controlled hybrid automaton with a control specification given by a predicate. Next, we extend them to H_C^f. This is also an extension of STTS (Silva and Krogh, 2001). To define semantics of the controlled SDHA by a transition system, the transition system must have a state variable which indicates duration between the current time and sampling times since both the sampling times and the current state in Q_H determine its behavior. In the STTS, the state is composed of the state variable $q\in Q_H$ and two variables ρ and ω which indicate time and elapsed time from the latest sampling time, respectively. Since we modify the clock structure, we introduce a new variable as a state variable, which indicates if a clocked event may occur at the current time. Thus, we can define two semantics for the controlled SDHA H_C^f given by a controlled timed/time-abstract transition system.

A controlled timed transition system is defined by

$$S_C^t(H_C^f,P)=(Q_t, Act_t, \mathscr{T}_t^f, Q_{t0}). \tag{2}$$

$Q_t\subseteq Q_H\times\{-1,0,1\}\times[0,\max({}_0,T_\theta)]\times\{0,1\}$ is the state set of the transition system. Each element of a state $(q,\rho,\omega,\ )\in Q_t$ is defined as follows: q indicates a state of H_C. ρ indicates that the current time is before the first nominal sampling time ${}_0$ ($\rho=-1$), at ${}_0$ ($\rho=0$), after ${}_0$ ($\rho=1$). ω indicates absolute time until the current time becomes ${}_0$, where the system starts at $\omega=0$. When the current time is equal to or greater than ${}_0$, ω is an elapsed time from the latest nominal sampling time and is reset to zero at each nominal sampling time. Note that ω is zero if ρ is zero. is reset to one at each nominal sampling time and to zero if $\omega>J$ or after a clocked event occurs. $Q_{t0}=\{(v,x,-1,0,0)\in Q_t|x\in init(v),\ {}^\forall v\in V\}$ is the set of initial states in Q_t. $Act_t=\ \cup\Re_{>0}$. To define the transition relation $\mathscr{T}_t^f$ of $S_C^t(H_C^f,P)$, we introduce a set of clock sequence $STS_C(q_t,\delta)$ as follows: for $q_t=(q,\rho,\omega,\ )\in Q_t$ and $\delta\in\Re_{\geq 0}$, $STS_C(q_t,\delta)=$

$$\begin{cases}\{t_0t_1\ldots t_N\mid t_i\in[\ {}_i-\omega,\ {}_i-\omega+J],\\ \qquad i=0,1,\ldots,\ t_N<\delta\leq t_{N+1}\}\\ \hfill \text{if } \rho=-1,\\ \{t_0t_1\ldots t_N\mid t_{i\ 1}\in[iT_\theta-\omega, iT_\theta-\omega+J],\\ \qquad i=1,2,\ldots,\ t_N<\delta\leq t_{N+1}\}\\ \hfill \text{if } \rho\neq -1,\ \ =0,\\ \{t_0t_1\ldots t_N\mid t_i\in[iT_\theta,\ iT_\theta+J],\ i=0,1,2,\ldots,\\ \qquad t_N<\delta\leq t_{N+1}\}\\ \hfill \text{if } \rho\neq -1,\omega=0,\ \ =1,\\ \{t_0t_1\ldots t_N\mid t_{i\ 1}\in[iT_\theta-\omega, iT_\theta-\omega+J],\\ \qquad i=1,2,\ldots,\ t_N<\delta\leq t_{N+1}\}\cup\{t_0t_1\ldots t_N\mid\\ \qquad t_i\in\ [iT_\theta-\omega,\ iT_\theta-\omega+J],\ i=1,2,\ldots,\\ \qquad t_0\in[0,J-\omega],\ t_N<\delta\leq t_{N+1}\}\\ \hfill \text{if } \rho=1,\omega\neq 0,\ \ =1.\end{cases}$$

$\mathscr{T}_t^f \subseteq Q_t \times Act_t \times Q_t$ is de ned as follows: Consider states $q_t = (q, \rho, \omega, \ )$ and $q'_t = (q', \rho', \omega', \ ') \in Q_t$.

(A) For each $\sigma \in \ $, $(q_t, \sigma, q'_t) \in \mathscr{T}_t^f$ i the following conditions are satis ed:

(1) $\omega = \omega'$, $\rho = \rho'$, and $e(v, \sigma, v') \in E$.
(2) $(x, x') \in jump(e)$.
(3) if $\rho = \rho' = -1$, then
 (i) $\sigma \in f_{uncl,1}(q, \ \omega)$ and (ii) $\ = \ ' = 0$.
(4) if $\rho = \rho' = 0$, then
 - if $\sigma \in \ _{cl}$, then
 (i) $\sigma \in f_{cl,1}(q, \ _0)$ and (ii) $\ = 1, \ \ ' = 0$,
 - otherwise
 (i) $\sigma \in f_{uncl,1}(q, \ _0)$ and (ii) $\ = \ ' = 1$.
(5) if $\rho = \rho' = 1$, then
 - if $\sigma \in \ _{cl}$, then (i) $\omega \in [0, J]$, (ii) $\sigma \in f_{cl,1}(q, \ _0 + \omega)$, and (iii) $\ = 1, \ \ = 0$,
 - otherwise
 (i) $\sigma \in f_{uncl,1}(q, \ _0 + \omega)$ and (ii) $\ = \ '$.

(B) For $\delta \in \Re_{>0}$, $(q_t, \ \delta, \ q'_t) \in \mathscr{T}_t^f$ i the following conditions are satis ed:

(1) $v = v'$, $x' = \zeta_q(\delta)$.
(2) For $^\forall \ , \ \in (0, \delta)$, $P(v, \zeta_q(\ )) = P(v, \zeta_q(\ ))$.
(3) One of the following conditions is satis ed:
 - if $\rho = \rho' = -1$, then (i) $\delta = \omega' - \omega$, (ii) $\ = 0$, $\ ' = 0$, and (iii) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E, ^\forall t \in [0, \delta)$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{uncl,2}(v, \zeta_q(t), \omega + t)$.
 - if $\rho = -1, \rho' = 0$, then(i) $\delta = \ _0 - \omega$, (ii) $\ = 0$, $\ ' = 1$, and (iii) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E, ^\forall t \in [0, \delta)$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{uncl,2}(v, \zeta_q(t), \omega + t)$.
 - if $\rho = -1$, $\rho' = 1$, then (i) $^\exists K \geq 0$ s.t. $\delta = \omega' + \ _K - \omega$, (ii) if $K = 0$, $\omega' \neq 0$, (iii) if $\omega' \in [0, J]$, $\ ' = 1$, (iv) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, $^\forall t \in [0, \delta)$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{uncl,2}(v, \zeta_q(t), \omega + t)$, and (v) $^\exists \{t_k\} \in STS_C(q_J, \delta)$ s.t. for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, $^\forall t \in \{t_k\}$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{cl,2}(v, \zeta_q(t), \omega + t)$.
 - if $\rho \neq -1$, $\rho' = 1$, then (i) $^\exists n \in Z_{\geq 0}$ s.t. $\delta = \omega' - \omega + nT_\theta$, (ii) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, $^\forall t \in [0, \delta)$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{uncl,2}(v, \zeta_q(t), t + \omega + \ _0)$, (iii) $^\exists \{t_k\} \in STS_C(q_t, \delta)$ s.t for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, $^\forall t \in \{t_k\}$, $\zeta_q(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{cl,2}(v, \zeta_q(t), t + \omega + \ _0)$, (iv) if $\ = 1$, $\omega' \in [0, J] \Rightarrow \ ' = 1$, and (v) if $\ = 0$, for any n satisfying (i), the following equations holds. if $n \geq 1$ and $\omega' \in [0, J]$, $\ ' = 1$. if $n = 0$, $\ ' = 0$.

Let $SDS_{C,f}(q_t, \delta, q'_t)$ be the set of sampled state sequences on a transition relation $(q_t, \delta, q'_t) \in \mathscr{T}_t^f$ de ned as follows: $SDS_{C,f}(q_t, \delta, q'_t) = \{(v, \zeta_q(t_0)), (v, \zeta_q(t_1)) \ldots (v, \zeta_q(t_N)) | \{t_k\}_{k=0}^N \in STS_C(q_t, \delta), {}^\exists q_{t,k} \in Q_t, \ k = 0, 1, \ldots, N-1$ s.t. $(q_t, t_0, q_{t,0}) \in \mathscr{T}_t^f$, $(q_{t,k}, t_{k+1} - t_k, q_{t,k+1}), (q_{t,N}, \delta - t_N, q'_t) \in \mathscr{T}_t^f, q_k = (v, \zeta_q(t_k))\}$, where $q_{t,k} = (q_k, \rho_k, \omega_k, \ _k)$. In the controlled timed transition system, $\Re_{>0} \subseteq Act_t$ is the set of elapsed times from the latest occurrence of events. So, by aggregating such events into two events indicating that time elapses, a controlled time-abstract transition system is de ned by

$$S_C^a(H_C^f, P) = (Q_a, Act_a, \mathscr{T}_a^f, Q_{a0}). \quad (3)$$

The sets Q_a and Q_{a0} are the same as those of S_C^t, that is, $Q_a = Q_t$ and $Q_{a0} = Q_{t0}$. $Act_a = \ \cup \{\ _{con}, \ _{uncon}\}$, where $\ _{con}$ and $\ _{uncon} \notin \ $ are events. $\mathscr{T}_a^f \subseteq Q_a \times Act_a \times Q_a$ is de ned as follows: Transitions related to events in $\ $ are the same as those in $\mathscr{T}_t^f$ in $S_C^t(H_C^f, P)$. Consider two states $q_a = (q, \rho, \omega, \ )$, $q'_a = (q', \rho', \omega', \ ') \in Q_a$, where $q = (v, x)$, $q' = (v', x') \in Q_H$. Let $\ (q_a, q'_a) = \{\delta \in \Re_{>0} \mid (q_a, \delta, q'_a) \in \mathscr{T}_t^f\}$.

(A) $\ (q_a, \ _{con}, q'_a) \in \mathscr{T}_a^f$ i $\ (q_a, q'_a) \neq \emptyset$ and for any $\delta \in \ (q_a, q'_a)$,

$$wp_{\Sigma_{cl,forc}}(P)(q_a) \vee \left[\bigwedge_{\{q_k\} \in SDS_{C,f}} \left\{ \bigvee_{\hat{q} \in \{q_k\}} wp_{\Sigma_{cl,forc}}(P)(\hat{q}) \right\} \right] \vee \left\{ \bigvee_{\epsilon \in [0, \delta)} wp_{\Sigma_{uncl,forc}}(P)(v, \zeta_q(\epsilon)) \right\} = 1, \quad (4)$$

where $SDS_{C,f} = SDS_{C,f}(q_a, \delta, q'_a)$, $P(q_a) = P(q)$.

(B) $\ (q_a, \ _{uncon}, q'_a) \in \mathscr{T}_a^f$ i $\ (q_a, q'_a) \neq \emptyset$ and there exists $\delta \in \ (q_a, q'_a)$ such that Eq. 4 does not hold.

Let $open = (open_{cl,1}, open_{cl,2}, open_{uncl,1}, open_{uncl,2})$ be a state feedback controller de ned as follows: for any $q \in Q_H$ and any $t \in \Re_{\geq 0}$, $open_{cl,1}(q, t) = \ _{cl}$, $open_{cl,2}(q, t) = \emptyset$, $open_{uncl,1}(q, t) = \ _{uncl}$, $open_{uncl,2}(q, t) = \emptyset$. Two transition systems controlled by the controller $open$ are denoted as follows:

$$S_C^t(P) = (Q_t, Act_t, \mathscr{T}_t, Q_{t0}) = S_C^t(H_C^{open}, P),$$
$$S_C^a(P) = (Q_a, Act_a, \mathscr{T}_a, Q_{a0}) = S_C^a(H_C^{open}, P).$$

Note that $S_C^t(P)$ and $S_C^a(P)$ correspond to semantics of the uncontrolled system. From the above de nitions, the following lemma is easily shown.

Lemma 1 Let $f = (f_{cl,1}, f_{cl,2}, f_{uncl,1}, f_{uncl,2})$ be a state feedback controller for a controlled SDHA $H_C(H, C(\ , J))$. Then, for two states $q_a = (q, \ \rho, \ \omega, \ \ )$, $q'_a \in Q_a = Q_t$, and event $\sigma \in \ $,

- $(q_a, \ \sigma, \ q'_a) \in \mathscr{T}_t^f \Rightarrow (q_a, \ \sigma, \ q'_a) \in \mathscr{T}_t$.
- $(q_a, \ \sigma, \ q'_a) \in \mathscr{T}_t$ and
$$\begin{cases} \sigma \in f_{uncl,1}(q, \ \omega) & \text{if } \rho = -1 \\ \sigma \in f_{cl,1}(q, \ \ _0 + \omega) & \text{if } \sigma \in \ _{cl} \\ \sigma \in f_{uncl,1}(q, \ \ _0 + \omega) & \text{otherwise} \end{cases}$$
$\Rightarrow (q_a, \ \sigma, \ q'_a) \in \mathscr{T}_t^f$, where $q = (v, x) \in Q_H$.

Let r be a run for H_C^f, and r is de ned as follows:

$$r = (v, x, c), \quad (5)$$

where $v(t) \in V$ is a trajectory for the discrete variable, $x(t) \in X$ is a trajectory for the continuous variable, and $c \in C$ is the sampling times synchronizing with H_C^f. It is said that r is a run for H_C^f if the following conditions hold:

(1) $v(0) \in V, \ x(0) \in init(v(0))$.
(2) if $t \in d(v)$, then there exists $\sigma \in \ $ such that the following three conditions hold: (i) $e(v(t \ ), \sigma, v(t^+)) \in E$, (ii) if $\sigma \in \ _{cl}$, $\sigma \in f_{cl,1}((v(t \ ), x(t \ )), t)$ and $t \in c$, (iii) if $\sigma \in \ _{uncl}$, $\sigma \in f_{uncl,1}((v(t \ ), x(t \ )), t)$.
(3) if $t \notin d(v)$, then (i) $\dot{x}(t) = f_{v(t)}(x(t))$, (ii) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, if $x(t) \in guard(e)$, $\tilde{\sigma} \notin f_{uncl,2}(v(t), x(t), t)$, and (iii) (if $t \in c$) for $^\forall e(v, \tilde{\sigma}, \tilde{v}) \in E$, $x(t) \in guard(e) \Rightarrow \tilde{\sigma} \notin f_{cl,2}(v(t), x(t)), t)$.

The following propositions indicate that the transition systems are the semantics of the controlled SDAH H_C^f.

Proposition 1 Consider a controlled SDHA H_C^f and its associated timed transition system $S_C^t(H_C^f, P)$. If r is a run for H_C^f, then there exists a corresponding sequence of states $q_t^r = q_{t,0}^r q_{t,1}^r \ldots$ that is a state trajectory for $S_C^t(H_C^f, P)$.

Proposition 2 Consider a controlled SDHA H_C^f and its associated timed transition system $S_C^t(H_C^f, P)$. If $q_t = q_{t,0}q_{t,1}q_{t,2}\ldots$ is a state trajectory for $S_C^t(H_C^f, P)$, there exists a corresponding run r^t for H_C^f.

The above propositions are also true for the time-abstract transition system $S_C^a(H_C^f, P)$.

We extend a predicate P on Q_H to $Q_t = Q_a$ as follows: for each state $q_t = (q, \rho, \omega, \ ) \in Q_t$, $P : Q_t \to \{0,\ 1\}$ is defined by $P(q_t) = P(q)$.

4. CONTROL-INVARIANCE

A concept of control-invariance plays an important role in state feedback control of discrete event systems (Ramadge and Wonham, 1987). A predicate $P \in \mathscr{P}(Q_a)$ is said to be control-invariant if there exists a state feedback controller f such that P is $(S_C^t(H_C^f, P); Act_t)$-invariant. Such a controller f is called a permissive feedback controller. We show necessary and sufficient conditions for P to be control-invariant in the controlled SDHA. We define predicates for trajectories of the continuous variables as follows: for a predicate $P \in \mathscr{P}(Q_a)$, states $q_a = (q, \rho, \omega, \ ) \in Q_a$, $q'_a = (v', \zeta_q(\delta), \rho', \omega', \ ') \in Q_a$, and time $\delta \in \Re_{>0}$,

$$pc_{C,\delta}(P)(q_a) = \begin{cases} 1 & \text{if } P(v, \zeta_q(\epsilon)) = 1, \forall\, \epsilon \in (0, \delta), \\ 0 & \text{if } P(v, \zeta_q(\epsilon)) = 0, \forall\, \epsilon \in (0, \delta), \end{cases}$$

$$pwp_{\Sigma_{forc},\delta}(P)(q_a) = \left\{ \bigvee_{\epsilon \in [0,\ \delta)} wp_{\Sigma_{uncl,forc}}(P)(v,\ \zeta_q(\epsilon)) \right\} \vee \left[\bigwedge_{\{q_k\} \in SDS_C} \left\{ \bigvee_{\hat{q} \in \{q_k\}} wp_{\Sigma_{cl,forc}}(P)(\hat{q}) \right\} \right],$$

$$twp_{C,\delta}(P)(q_a) = P(q') \vee pwp_{\Sigma_{forc},\delta}(P)(q_a),$$

where $SDS_C = SDS_{C,open}(q_a, \delta, q'_a)$. If a predicate P is a closed set, twp can be rewritten as follows: $twp_{C,\delta}(P)(q_a) = pwp_{\Sigma_{forc},\delta}(P)(q_a)$. A predicate P is said to be $(S_C^t(P);\ _{uncon}, \Re_{>0},\ _{forc})$-invariant if the following conditions hold:

(1) P is $(S_C^t(P);\ _{uncon})$-invariant, and

(2) in $S_C^t(P)$, for any $\delta \in \Re_{>0}$,

$$P \leqslant \neg D_\delta \vee wp_{\Sigma_{forc}}(P) \vee (twp_{C,\delta}(P) \wedge pc_{C,\delta}(P))\,. \quad (6)$$

We show necessary and sufficient conditions for the control-invariance.

Theorem 1 Consider the controlled SDHA H_C and a predicate $P \in \mathscr{P}(Q_a)$. Then, the following three statements are equivalent:

(i) P is control-invariant.

(ii) P is $(S_C^t(P)\ ;\ _{uncon}, \Re_{>0},\ _{forc})$-invariant.

(iii) P is $(S_C^a(P)\ ;\ _{uncon} \cup \{\ _{uncon}\})$-invariant.

Proof:(i)⇒(ii). Suppose that P is control-invariant. Let f be a permissive feedback controller. Suppose that P is not $(S_C^t(P);\ _{uncont}, \Re_{>0},\ _{forc})$-invariant. Then, we have the following cases:

- Consider the case that there exist $q_t, q'_t \in Q_t$, and $\sigma \in\ _{uncon}$ such that $(q_t, \sigma, q'_t) \in \mathscr{T}_t$, $P(q_t) = 1$, and $P(q'_t) = 0$. Since $_{uncon,cl} \subseteq f_{cl,1}$ and $_{uncon,uncl} \subseteq f_{uncl,1}$, we have $(q_t, \sigma, q'_t) \in \mathscr{T}_t^f$ by Lemma 1. Since f is a permissive feedback controller, we have $P(q_t) = 0$, which is a contradiction.
- Consider the case that there exist $\delta \in \Re_{>0}$ and $q_t = (q, \rho, \omega, \ ) \in Q_t$ which do not satisfy Eq. (6). Then, $P(q_t) = 1$, there exists $q'_t = (q', \rho', \omega', \ ') \in Q_t$ such that $(q_t, \delta, q'_t) \in \mathscr{T}_t$, one of the following conditions is satisfied:
  (A) $P(v, \zeta_q(\epsilon)) = 0$ for any $\epsilon \in (0, \delta)$ and $wp_{\Sigma_{forc}}(P)(q_t) = 0$. Since $wp_{\Sigma_{forc}}(P)(q_t) = 0$, there exist $0 < \tilde{\epsilon} < \delta$ and $\tilde{q}_a = (v, \zeta_q(\tilde{\epsilon}), \tilde{\rho}, \tilde{\omega}, \tilde{\ })$ $\in Q_t$ such that $(q_t, \epsilon, \tilde{q}_a) \in \mathscr{T}_t^f$. Then we have $P(\tilde{q}_a) = 0$, which is a contradiction since f is a permissive feedback controller.
  (B) $P(v, \zeta_q(\epsilon)) = 1$ for any $\epsilon \in (0, \delta)$, $P(q'_t) = 0$, and $pwp_{\Sigma_{forc},\delta}\ (P)(q_t) = 0$. Then, we have $(q_t, \delta, q'_t) \in \mathscr{T}_t^f$, which is also a contradiction.

From the above contradictions, P is shown to be $(S_C^a(P);\ _{uncon}, \Re_{>0},\ _{forc})$-invariant.

(ii)⇒(iii) For $q_a, q'_a \in Q_a = Q_t$, and $\sigma \in\ _{uncon}$, the following implication is easily shown: $(q_a, \sigma, q'_a) \in \mathscr{T}_t \Rightarrow (q_a, \sigma, q'_a) \in \mathscr{T}_a$. Thus, we have P is $(S_C^a(P)\ ;\ _{uncon})$-invariant if P is $(S_C^t(P);\ _{uncon})$-invariant. Suppose that P is not $(S_C^a(P);\ _{uncon})$-invariant. Then, there exist q_a and q'_a such that $(q_a,\ _{uncon}, q'_a) \in \mathscr{T}_a$, $P(q_a) = 1$, and $P(q'_a) = 0$. From the definition of $_{uncon}$, there exits $\delta \in \Re_{>0}$ such that $(q_a, \delta, q'_a) \in \mathscr{T}_t$, which is a contradiction since P is $(S_C^t(P);\ _{uncon}, \Re_{>0},\ _{forc})$-invariant. Thus, P is $(S_C^a(P);\ _{uncon} \cup \{\ _{uncon}\})$-invariant.

(iii)⇒(i) Suppose that P is $(S_C^a(P);\ _{uncon} \cup \{\ _{uncon}\})$-invariant. Then, we consider the following state feedback controller $f = (f_{cl,1}, f_{cl,2}, f_{uncl,1}, f_{uncl,2})$: for each $q \in Q_H$, $t \in \Re_{\geq 0}$, $f_{cl,1}(q,t) =\ _{cl,uncon} \cup \{\sigma \in\ _{cl,con} \mid wp_\sigma(P)(q, \rho, \omega, 1) = 1$ in $S_C^t(P)\}$, $f_{uncl,1}(q,t) =\ _{uncl,uncon} \cup \{\sigma \in\ _{uncl,con} \mid wp_\sigma(P)(q, \rho, \omega, 0) = 1$ in $S_C^t(P)\}$, where ρ and ω satisfy **(a)** if $t <\ _0$, then $\rho = -1$, and $\omega = t$. **(b)** if $t =\ _0$, then $\rho = 0$, and $\omega = 0$. **(c)** if $t >\ _0$, then $\rho = 1$, and $\omega = t -\ _0 - \lfloor (t -\ _0)/T_\theta \rfloor T_\theta$. $f_{cl,2}(q,t) =\ _{cl,forc} \cap f_{cl,1}$. $f_{uncl,2}(q,t) =\ _{uncl,forc} \cap f_{uncl,1}$. Thus, it is easy to prove that P is $(S_C^t(H_C^f, P); Act_t)$-invariant. ■

5. SUPREMAL CONTROL-INVARIANT SUBPREDICATE

In general, a given predicate $P \in \mathscr{P}(Q_a)$ is not necessarily control-invariant. In this section, we propose a procedure for computation of the supremal control-invariant subpredicate. We introduce some definitions for the predicate P as follows:

- $\mathscr{C}I(P)$ is the set of all control-invariant subpredicates of $P \in \mathscr{P}(Q_a)$.

- $\mathbf{0}\in\mathscr{P}(Q_a)$ is the predicate such that for any $q_a\in Q_a$, $\mathbf{0}(q_a)=0$. Since $\mathbf{0}\in\mathscr{C}I(P)$, $\mathscr{C}I(P)\neq\emptyset$.
- A predicate $P^{\uparrow}\in\mathscr{C}I(P)$ called the supremal control-invariant subpredicate of P is de ned as follows: for any $P'\in\mathscr{C}I(Q_a)$, $P'\leqslant P^{\uparrow}$.

Ushio and Takai (2005) showed that there always exists $P^{\uparrow}$ for the hybrid systems with forcible event. In this section, we show the same property also holds for the controlled SDHA.

Since $P'\leqslant P \Rightarrow \text{for}^{\forall}\sigma\in\ , wp_{\sigma}(P')\leqslant wp_{\sigma}(P)$, the following lemma is easily shown.

Lemma 2 Let P and $P'\in\mathscr{P}(Q_a)$ be predicates such that $P'\leqslant P$ and P' is $(S^a_C(P');\ _{uncon}\cup\{\ _{uncon}\})$-invariant. For any $q_a\in\{q_a=(q,\rho,\omega,\ )\in Q_a|P'(q_a)=1\}$, if there exists $q'_a\in Q_a$ such that $(q_a,\ _{uncon},\ q'_a)\in\mathscr{T}_a$ in $S^a_C(P)$, then $(q_a,\ _{uncon},q'_a)\in\mathscr{T}_a$ also holds in $S^a_C(P')$

Using Lemma 2, we prove the following theorem.

Theorem 2 Let I be any index set. if $P_i\in\mathscr{P}(Q_a)$ is $(S^a_C(P_i)\mid\ _{uncon}\cup\{\ _{uncon}\})$ -invariant for each $i\in I$, then, $P_I=\bigvee_{i\in I}P_i$ is $(S^a_C(P_I)\ ;\ \ _{uncon}\cup\{\ _{uncon}\})$ -invariant.

By Theorem 2, there exists its supremal control-invariant subpredicate $P^{\uparrow}$ for any predicate $P\in\mathscr{P}(Q_a)$.

The following theorem gives an iterative scheme for computing the supremal control-invariant subpredicate.

Theorem 3 For any $P\in\mathscr{P}(Q_a)$, consider the following iterative computation: $P_{j+1}=P_j\wedge\ (P_j)\ (^{\forall}j\geq 0)$, where $P_0:=P$.

Then the following implication holds:

$$^{\exists}k\geq 0 \text{ s.t. } P_{k+1}=P_k\Rightarrow P^{\uparrow}=P_k, \quad (7)$$

where $\ :\mathscr{P}(q_a)\rightarrow\mathscr{P}(q_a)$ is de ned as follows:

$$(P)(q_a)=$$
$$\begin{cases}1 & \text{if} \displaystyle\bigwedge_{\sigma\in\Sigma_{uncon}\cup\{\tau_{uncon}\}} wlp_{\sigma}(P)(q_a)=1 \text{ in } S^a_C(P)\\ 0 & \text{otherwise.}\end{cases}$$

Proof: Assume that there exists k such that $P_{k+1}=P_k$. For the above iterative scheme, we have $P_k{=}P_{k+1}{=}P_k\wedge\ (P_k){\leqslant}wlp_a(P_k)$ for any $a\in\ _{uncon}\cup\{\ _{uncon}\}$ and $P_{j+1}{=}P_j\wedge\ (P_j){\leqslant}P_0{=}P$ for any $j{\geq}0$. So P_k is a control-invariant subpredicate of P i.e. $P_k\in\mathscr{C}I(P)$, which implies $P_k{\leqslant}P^{\uparrow}$.

Next, we prove that $P^{\uparrow}\leqslant P_l$ for $l=0,1,\ldots,k$ by induction. **(1)** $l=0$. Since $P^{\uparrow}$ is a subpredicate of P, we have $P^{\uparrow}\leqslant P_0$. **(2)** Suppose that $P^{\uparrow}\leqslant P_l$ holds. Then, we show by a contradiction that $P^{\uparrow}\leqslant P_{l+1}$ holds. If $P^{\uparrow}\leqslant P_{l+1}$ does not holds, then there exists $q_a\in Q_a$ such that $P^{\uparrow}(q_a)=1$ and $\ (P_l)(q_a)=0$ since $P^{\uparrow}(q_a)=1\Rightarrow P_l(q_a)=1$ for any $q_a\in Q_a$. If $\ (P_l)(q_a)=0$ holds, One of the following cases always holds: **(a)** In $S^a_C(P_l)$, there exists $\sigma\in\ _{uncon}$ such that $wlp_{\sigma}(P_l)(q_a)=0$. Since D_{σ} in $S^a_C(P^{\uparrow})$ is equivalent to that in $S^a_C(P_l)$, there exists $q'_a\in Q_a$ such that $(q_a,\sigma,q'_a)\in\mathscr{T}_a$ in $S^a_C(P^{\uparrow})$ and $P_l(q'_a)=0$, which implies that $P^{\uparrow}(q'_a)=0$. This contradicts the assumption that $P^{\uparrow}$ is control-invariant. **(b)** $wlp_{\tau_{uncon}}(P_l)(q_a)=0$ holds in $S^a_C(P_l)$. Then there exists $q'_a\in Q_a$ such that $(q_a,\ _{uncon},q'_a)\in\mathscr{T}_a$ in $S^a_C(P_l)$ and $P_l(q'_a)=0$. By Lemma 2, we have $(q_a,\ _{uncon},q'_a)\in\mathscr{T}_a$ in $S^a_C(P^{\uparrow})$. Since $P^{\uparrow}(q'_a)=0$, this contradicts the assumption that $P^{\uparrow}$ is control-invariant.
For the above cases, we have $P^{\uparrow}\leqslant P_{l+1}$ and $P^{\uparrow}\leqslant P_k$. Therefore, we have $P_k=P^{\uparrow}$. ■

Note that $P^{\uparrow}$ computed by the above scheme depends on time in general while the control speci cation $P\in\mathscr{P}(Q_H)$ is independent of time.

6. CONCLUSION

This paper considered state feedback control of a sampled-data hybrid automaton as a model of computer-controlled systems where control speci- cations are given by predicates.

We introduced two transition systems as semantics for the controlled sampled-data hybrid automata and proved necessary and su cient conditions for the control-invariance, and showed that there always exists the supremal control-invariant subpredicate for any predicate.

In general, the procedure for computation of the supremal control-invariant subpredicate is not decidable. So it is future work to obtain an approximation method for the computation.

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