NON-DETERMINISTIC REACTIVE SYSTEMS, FROM HYBRID SYSTEMS AND BEHAVIOURAL SYSTEMS PERSPECTIVES

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Abstract: A reactive system is an entity which takes as inputs signals from a certain set, and transforms them to produce as outputs signals in some further set, where a signal is modeled as a function from a time domain to a value space, and time domains are linearly ordered sets. Building on our previous work which generalizes Aubin's Evolutionary Systems model, we develop a general formulation of non-deterministic input-output reactive systems that is uniform for discrete, continuous and hybrid time, and which extends Willems' Behavioural Systems input-output model by allowing hybrid time, and by allowing different time domains for input and output signals. However, our approach differs from these existing frameworks in that instead of taking as primitive signals over infinite-length time domains, we work with finite-length paths and utilize their algebraic and partial-order structure. We illustrate our framework with a generic model of event-driven A/D conversion, transforming a continuous-time input signal into a discrete-time output signal via an intermediate hybrid-time signal.

Keywords: hybrid systems, input-output systems, behavioural systems.

1. INTRODUCTION

In previous work (Davoren *et al.*, 2004), we identify and investigate a class of non-deterministic dynamical systems we call *general flow systems* which include discrete-time transition systems, continuous-time differential inclusions, and hybrid-time systems such as hybrid automata and impulse differential inclusions. Over any value space X and a (non-negative) time line L, a general flow system Φ maps each point $x \in X$ to the set $\Phi(x)$ of all signals or paths $\gamma : T \to X$ of the system with $\gamma(0) = x$ and time domain $T \subseteq L$; in the non-deterministic setting, there may be none, exactly one, or many possible Φ-paths starting from x. In (Davoren *et al.*, 2004), we adapt Aubin's model of *Evolutionary Systems* (Aubin and Dordan, 2002; Aubin *et al.*, 2002) to provide semantics for a temporal logic that is uniform for discrete-time, continuous-time and hybrid-time systems. Over the hybrid time line $L = \mathbb{N} \times \mathbb{R}_0^+$ (ordered lexicographically), hybrid paths $\gamma : T \to X$ have time domains $T \subset L$ of the form $T = \bigcup_{i<N} [\,(i,0),\,(i,\Delta_i)\,]$, with $\Delta_i \in \mathbb{R}_0^+$ the duration of the i-th interval. Within T, time $(i+1,0)$ is the immediate *discrete successor* of time (i,Δ_i), but in the underlying line L, there is a continuum-length "gap" in between. To deal with hybrid signals, two moves were crucial. The

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first was to work with finite-length paths having a start-point and an end-point, instead of infinite-length paths over the whole time line, as used for the usual discrete and continuous time lines in the work of Aubin, and also within the *Behavioural Systems* framework (Willems, 1989; Willems, 1991; Moor and Raisch, 1999). The second was to develop a theory of finite-length paths $\gamma : T \to X$ with time-domains $T \subset L$ that are not simply intervals $T = [0, b]$ but rather finite disjoint unions of intervals, with "gaps" in between. We then build up a theory of *maximal extensions* of finite paths to compare with signal models over unbounded time domains. By working with finite-length paths, we also get to use monoidal and partial-order structure that is not accessible within frameworks based on infinite-length signals. In this respect, our approach is close to that of (Tabuada *et al.*, 2004).

In the present paper, we develop a general formulation of non-deterministic input-output reactive systems that is uniform for discrete, continuous and hybrid time. In broad terms, a *reactive system* Γ takes as input a signal η from a certain set, and transforms it to produce as output one or more signals $\gamma \in \Gamma(\eta)$ in some further set. While such input-output systems have been well-studied within Behavioural Systems theory (Willems, 1989; Willems, 1991) for the usual discrete or continuous time lines, that framework does not extend to hybrid time or to systems where the time-line L_{out} for output paths may differ from the time-line L_{in} for input paths. Basic examples are A/D converters transforming continuous-time signals into discrete-time signals (particularly event-driven rather than fixed periodic sampling), and D/A converters transforming discrete-time signals into continuous-time signals. Our approach is also commensurate with the *Tagged Signal* "Models of Computation" metamodel (Lee and Sangiovanni-Vincentelli, 1998), restricted to the timed case of linearly ordered signal domains, in that it provides a general framework which includes within it diverse model classes, and within which various system properties can be formulated and compared.

In a follow-on paper, we will extend the present work with a study of *compositions* of input-output reactive systems by sequential, parallel and feedback constructions, with the aim of giving an explicit set-theoretic semantics for block-diagram based modeling and description frameworks, such as CHARON (Alur *et al.*, 2000), that allow non-determinism, hybrid signals and time-translations. Additionally, we will give a representation within our framework of Lynch and Vandraager's *hybrid I/O automata* model (Lynch *et al.*, 1996), as well as of Aubin's *impulse differential inclusion* hybrid model (Aubin *et al.*, 2002), extended with inputs and outputs, which are not possible within this short paper.

The body of the paper is as follows. Section 2 covers preliminaries on set-valued maps and linear orders. In Section 3, we develop some basic theory of paths with "gappy" time domains, and tersely review the theory of general flows and their infinitary extensions, enabling a first comparison in Section 4 with Aubin's Evolutionary Systems and with state behaviours in Willems' Behavioural Systems Theory. In Section 5, we develop input-output reactive systems and some basic system properties and their inter-relationships. The main result concerns the relationship between Willems' concept of *non-anticipation*, which is a non-deterministic analogue of *causality*, and the system property of being *extension-preserving* with respect to the natural partial order of extension of finite paths (that is not available in frameworks based on signals with infinite time domains). We also including several illustrative examples, including event-driven A/D conversion. In Section 6, we compare our work with other input-output system models. In particular, we show that for the usual discrete and continuous time lines, our finite-path formulations of system properties such as *non-anticipation*, *input freedom*, and *time-invariance* (suffix-closure), all have an exact correspondence with their infinite signal formulations in Willems' input-output Behavioural Systems theory.

2. PRELIMINARIES: RELATIONS/SET-VALUED MAPS, MONOIDS, AND LINEAR ORDERS

We write $\mathbb{R}_0^+ := [0, \infty)$ and $\mathbb{N}$ for the non-negative reals and natural numbers, respectively.

We write $r : X \rightsquigarrow Y$ to mean $r : X \to 2^Y$ is a *set-valued map*, with $r(x) \subseteq Y$ for every $x \in X$ (possibly $r(x) = \varnothing$); equivalently, $r \subseteq X \times Y$ is a *relation* – we don't distinguish between a set-valued map and its graph. Let $[\, X \rightsquigarrow Y \,] := 2^{X \times Y}$ denote the set of all maps $r : X \rightsquigarrow Y$ from X to Y; it is partially ordered by the subset relation, with least element ϵ (empty map), and closed under *relational converse*: for a map $r : X \rightsquigarrow Y$, the converse $r^{-1} : Y \rightsquigarrow X$ is given by: $x \in r^{-1}(y)$ iff $y \in r(x)$. The *domain* is $\text{dom}(r) := \{x \in X \mid r(x) \neq \varnothing\}$, and the *range* is $\text{ran}(r) := \text{dom}(r^{-1}) \subseteq Y$. A map $r : X \rightsquigarrow Y$ is *total on* X if $\text{dom}(r) = X$. Given $r_1 : X \rightsquigarrow Y$ and $r_2 : Y \rightsquigarrow Z$, their *sequential composition* is $(r_1 \circ r_2) : X \rightsquigarrow Z$, defined by $(r_1 \circ r_2)(x) := \{z \in Z \mid (\exists y \in Y)\, y \in r_1(x) \wedge z \in r_2(y)\,\}$.

We distinguish several sub-classes of *deterministic maps*. We write $r : X \to Y$ to mean r is a single-valued *function* that is *total* on domain X, with values in Y, written $r(x) = y$ (rather than

$r(x) = \{y\}$). Let $[\,X \to Y\,]$ denote the set of all $r : X \to Y$. We also need *partial functions*, and write $r : X \dashrightarrow Y$ to mean that on $\mathrm{dom}(r) \subseteq X$, r is a single-valued function; let $[\,X \dashrightarrow Y\,]$ denote the set of all such maps. For partial functions, we also write $r(x) = y$ when $x \in \mathrm{dom}(r)$, and may write $r(x) = \text{UNDEF}$ when $x \notin \mathrm{dom}(r)$. As sets of maps, $[\,X \to Y\,] \subseteq [\,X \dashrightarrow Y\,] \subseteq [\,X \rightsquigarrow Y\,]$.

On any set X, we define a *monoidal family over* X to be a structure $(X, M, \cdot, \iota)$ where the functions $\cdot : X \to [\,(M \times M) \dashrightarrow M\,]$ and $\iota : X \to M$, are such that, if $\cdot_x := \cdot(x) : (M \times M) \dashrightarrow M$ and $\iota_x := \iota(x) \in M$, for each $x \in X$, the following conditions are satisfied, for all $m, m', m'' \in M$:
(*a*) *family associativity*: if $(m, m') \in \mathrm{dom}(\cdot_x)$ and $(m', m'') \in \mathrm{dom}(\cdot_y)$, then $(m \cdot_x m', m'') \in \mathrm{dom}(\cdot_y)$ and $(m, m' \cdot_y m'') \in \mathrm{dom}(\cdot_x)$ and $(m \cdot_x m') \cdot_y m'' = m \cdot_x (m' \cdot_y m'')$;
(*b*) *right identity*: if $(m, m') \in \mathrm{dom}(\cdot_x)$ then $(m, \iota_x) \in \mathrm{dom}(\cdot_x)$ and $m \cdot_x \iota_x = m$;
(*c*) *left identity*: if $(m', m) \in \mathrm{dom}(\cdot_x)$ then $(\iota_x, m) \in \mathrm{dom}(\cdot_x)$ and $\iota_x \cdot_x m = m$.

Let $(L, <, 0)$ be a *linear order* with least element 0 and no largest element, and let $\leqslant$ be the reflexive closure of $<$. We use usual interval notation: for $a, b \in L$, the *bounded intervals* in L include $[a, b] := \{l \in L \mid a \leqslant l \leqslant b\}$ and $(a, b) := \{l \in L \mid a < l < b\}$. The right *unbounded intervals* are: $[a, \infty) := \{l \in L \mid a \leqslant l\}$ and $(a, \infty) := \{l \in L \mid a < l\}$. Given $(L_1, <_1, 0_1)$ and $(L_2, <_2, 0_2)$, a map $g : L_1 \dashrightarrow L_2$ is:

- [*strictly*] *order-preserving* if $(\forall l, k \in \mathrm{dom}(g))$, if $l \leqslant_1 k$ [if $l <_1 k$] then $g(l) \leqslant_2 g(k)$ [then $g(l) <_2 g(k)$];
- an *order-isomorphism* if g is a total function and a bijection, and both g and g^{-1} are strictly order-preserving.

We take a (future) *time line* to be any linear order $(L, <, 0)$ that is *shift-invariant* in the sense that it is equipped with a family of maps $\{\sigma^{-a}\}_{a \in L}$ such that $\sigma^{-0} : L \to L$ is the identity map, and for each $a \in L$, $\sigma^{-a} : [a, \infty) \to L$ (the *left a-shift*) is an order-isomorphism, with inverse $\sigma^{+a} := (\sigma^{-a})^{-1} : L \to [a, \infty)$ (the *right a-shift*).

The basic examples are the discrete time line $\mathbb{N}$, and the dense continuum $\mathbb{R}_0^+$, the non-negative cones of linearly ordered abelian groups under addition, $\mathbb{Z}$ and $\mathbb{R}$ respectively. The group operation trivially gives shift-invariance: take $\sigma^{-a}(l) = l - a$ for $l \in [a, \infty)$, and inverse $\sigma^{+a}(l) = l + a$.

The hybrid time line $\mathbb{H} := \mathbb{N} \times \mathbb{R}_0^+$ is linearly ordered *lexicographically*: $(i, t) <_{\mathrm{lex}} (j, s)$ iff $i < j$ or $i = j$ and $t < s$. The least element is $\mathbf{0} := (0, 0)$. This ordering does *not* admit any natural addition operation to make it a linearly ordered semigroup, but it is shift-invariant: for $a = (k, r) \in \mathbb{H}$, define $\sigma^{-a} : [a, \infty) \to \mathbb{H}$ by $\sigma^{-a}(i, t) := (0, t - r)$ if $i = k$ and $\sigma^{-a}(i, t) := (i - k, t)$ if $i > k$, for all $l = (i, t) \in [a, \infty)$.

We will also require some additional properties of time lines. A time line $(L, <, 0)$ will be called *ω-compact* if there exists an ω-length sequence $\{b_n\}_{n<\omega}$ such that for all $n < \omega$, $b_n \in L$ and $0 < b_n < b_{n+1}$, and $L = \bigcup_{n<\omega}[0, b_n]$. A time line $(L, <, 0)$ is called *Dedekind-complete* if for every subset $A \subseteq L$ such that $A \neq \varnothing$, if A has an upper bound $b \in L$ such that $l \leqslant b$ for all $l \in A$, then $\sup(A)$ exists in L, and likewise for lower bounds and inf's. The time lines $\mathbb{N}$, $\mathbb{R}_0^+$ and $\mathbb{H}$ each have both these properties.

3. PATHS, GENERAL FLOW SYSTEMS, AND THEIR EXTENSIONS

Let $(L, <, 0)$ be a time line. We define a *bounded time domain* in L to be a subset $T \subset L$ such that $T = \bigcup_{n<N}[a_n, b_n]$ with $N \in \mathbb{N}^+$, $a_0 = 0$, $b_{N-1} = b_T$ the maximum element, and $a_n \leqslant b_n < a_{n+1} \leqslant b_{n+1}$ for all $n < N - 1$. Let $\mathsf{BT}(L) \subset 2^L$ denote the set of all bounded time domains in L, and let $\mathsf{BT}_\varnothing(L) = \mathsf{BT}(L) \cup \{\varnothing\}$. Define $\mathsf{BI}(L) := \{T \in \mathsf{BT}(L) \mid (\exists b \in L)\, \mathrm{dom}(\gamma) = [0, b]\,\}$ and $\mathsf{BI}_\varnothing(L) = \mathsf{BI}(L) \cup \{\varnothing\}$.

Over any set $X \neq \varnothing$ (an arbitrary *value space* or *signal space*), define the set of *L-paths in X*, and the set of *interval L-paths in X*:

$$\mathsf{Path}(L, X) := \{\gamma : L \dashrightarrow X \mid \mathrm{dom}(\gamma) \in \mathsf{BT}(L)\}$$
$$\mathrm{IPath}(L, X) := \{\gamma : L \dashrightarrow X \mid \mathrm{dom}(\gamma) \in \mathsf{BI}(L)\}$$

For $\gamma \in \mathsf{Path}(L, X)$, define $b_\gamma := b_{\mathrm{dom}(\gamma)}$ to be the maximum of $\mathrm{dom}(\gamma)$, so $\gamma(b_\gamma) \in X$ is the end-point and $\gamma(0) \in X$ the start-point of γ. Where ϵ is the empty path, let $\mathsf{Path}_\epsilon(L, X) := \mathsf{Path}(L, X) \cup \{\epsilon\}$ and $\mathcal{P}_\epsilon := \mathcal{P} \cup \{\epsilon\}$ for path sets $\mathcal{P} \subseteq \mathsf{Path}(L, X)$.

We utilise three operations on $\mathsf{Path}_\epsilon(L, X)$; for $\gamma, \gamma' \in \mathsf{Path}_\epsilon(L, X)$, $t \in L$ and $x \in X$, define:

- *restriction* or *prefix* ending at $t \in \mathrm{dom}(\gamma)$: $\gamma|_t \in \mathsf{Path}_\epsilon(L, X)$ where $\gamma|_t := \gamma\restriction_{[0,t] \cap \mathrm{dom}(\gamma)}$.
- *translation* or *suffix* starting at $t \in \mathrm{dom}(\gamma)$: ${}_t|\gamma \in \mathsf{Path}_\epsilon(L, X)$ where $({}_t|\gamma)(l) := \gamma(\sigma^{+t}(l))$ for all $l \in \mathrm{dom}({}_t|\gamma) := \sigma^{-t}([t, b_\gamma] \cap \mathrm{dom}(\gamma))$.
- *point-concatenation* at $x \in X$: $\gamma *_x \gamma' \in \mathsf{Path}_\epsilon(L, X)$ where, for all $l \in L$: $(\gamma *_x \gamma')(l) := \gamma(l)$ if $l \in \mathrm{dom}(\gamma)$ and $\gamma'(0) = x = \gamma(b_\gamma)$; $(\gamma *_x \gamma')(l) := \gamma'(\sigma^{-b_\gamma}(l))$ if $l \in \sigma^{+b_\gamma}(\mathrm{dom}(\gamma'))$ and $\gamma'(0) = x = \gamma(b_\gamma)$; and $(\gamma *_x \gamma')(l) := \text{UNDEF}$ for all other l and x.

For each $x \in X$, the *trivial path* $\theta_x : [0, 0] \to X$ is given by $\theta_x(0) := x$. Then the map $\theta : X \to \mathsf{Path}(L, X)$ given by $\theta(x) := \theta_x$ is an embedding (injective function) of X into $\mathsf{Path}(L, X)$.

For any subset of paths $\mathcal{P} \subseteq \mathrm{Path}_\epsilon(L, X)$, we say $\mathcal{P}$ is *closed under suffixes* (*closed under prefixes*) if for all $\gamma \in \mathcal{P}$ and all $t \in \mathrm{dom}(\gamma)$, the path ${}_t|\gamma \in \mathcal{P}$ (respectively, $\gamma|_t \in \mathcal{P}$). Define $X_\mathcal{P} := \{x \in X \mid (\exists \gamma \in \mathcal{P})\ \gamma(0) = x \vee \gamma(b_\gamma) = x\}$. The structure $(X_\mathcal{P}, \mathcal{P}, *, \theta)$ constitutes a *monoidal family* over the set $X_\mathcal{P}$ if for all $\gamma, \gamma' \in \mathcal{P}$ and all $x \in X_\mathcal{P}$, if $(\gamma, \gamma') \in \mathrm{dom}(*_x)$, then $\gamma *_x \gamma' \in \mathcal{P}$ and $\theta_x \in \mathcal{P}$.

For discrete time $L = \mathbb{N}$, the interval path set $\mathrm{IPath}_\epsilon(\mathbb{N}, X) = X^*$ is the set of all finite *words* or sequences over X. The usual operation of *word-concatenation* from automata and DES theory equips the set X^* as a (total) monoid with identity ϵ. Denoting it by $\cdot$, that operation is definable in terms of point-concatenation as follows: for all $\gamma, \gamma' \in \mathrm{IPath}(\mathbb{N}, X) = X^+$, we have $\gamma \cdot \gamma' = \gamma *_x (\nu_{xy}) *_y \gamma'$, where $\mathrm{dom}(\nu_{xy}) = \{0, 1\}$, $\nu_{xy}(0) = x = \gamma(b_\gamma)$, and $\nu_{xy}(1) = y = \gamma'(0)$.

Note that for all non-empty $\gamma \in \mathrm{Path}(L, X)$ and all $x, y, z \in X$ such that $\gamma(0) = z$ and $\gamma(b_\gamma) = y$, we have: $\epsilon *_x \epsilon = \epsilon$, $\gamma *_y \epsilon = \gamma$, and $\epsilon *_z \gamma = \gamma$. Hence ϵ as well as θ_x functions as a monoid identity for the $*_x$ operation. The following structures are monoidal families over X: $(X, \mathrm{Path}(L, X), *, \theta)$, $(X, \mathrm{Path}_\epsilon(L, X), *, \varepsilon)$, $(X, \mathrm{IPath}(L, X), *, \theta)$ and $(X, \mathrm{IPath}_\epsilon(L, X), *, \varepsilon)$.

The point-concatenation operation is usefully related to the notion of *extensions* of paths which continue from the end value of a given path. Define a partial order on $\mathsf{Path}_\epsilon(L, X)$ from the subset relation and the underlying linear order on L; (re-using notation) we define: $\gamma < \gamma'$ iff $\gamma \subset \gamma'$ and $t < t'$ for all $t \in \mathrm{dom}(\gamma)$ and for all $t' \in \mathrm{dom}(\gamma') - \mathrm{dom}(\gamma)$, in which case we say γ' is a *proper extension* of γ. As usual, $\gamma \leqslant \gamma'$ iff either $\gamma < \gamma'$ or $\gamma = \gamma'$.

Proposition 3.1. Let L be any time line, and X any value space. Then for all $\gamma, \gamma' \in \mathsf{Path}(L, X)$, the following are equivalent:

(i) $\gamma < \gamma'$;

(ii) for all $t \in \mathrm{dom}(\gamma)$, $\gamma|_t = \gamma'|_t$ and for all $t' \in \mathrm{dom}(\gamma') - \mathrm{dom}(\gamma)$, $t < t'$;

(iii) $\gamma' = \gamma *_x \gamma''$ for some $\gamma'' \in \mathsf{Path}(L, X)$ and $x \in X$ with $\gamma''(0) = x$ and $\gamma(b_\gamma) = \gamma'(b_\gamma) = x$ and $\gamma'' \neq \theta_x$.

Now consider the hybrid time line $\mathbb{H} = \mathbb{N} \times \mathbb{R}_0^+$. Define $DS := \mathrm{IPath}_\epsilon(\mathbb{N}, \mathbb{R}_0^+)$ to be the set of all (finite) *duration sequences*. For $\Delta \in DS$, define $HT(\Delta)$ to be the *hybrid time domain* determined by Δ, and over any $X \neq \varnothing$, define $\mathsf{HPath}_\epsilon(X) \subset \mathsf{Path}_\epsilon(\mathbb{H}, X)$ to be the set of all *hybrid paths* over X, as follows:

$$HT(\Delta) := \bigcup_{i < \mathrm{length}(\Delta)} [\,(i, 0),\, (i, \Delta_i)\,]$$
$$\mathsf{HT} := \{\, HT(\Delta) \in \mathsf{BT}(\mathbb{H}) \mid \Delta \in DS \,\}$$
$$\mathsf{HPath}(X) := \{\xi \in \mathsf{Path}(\mathbb{H}, X) \mid \mathrm{dom}(\xi) \in \mathsf{HT}\}$$

General flow systems are dynamical systems over a state space or signal value space. As expected from Behavioural Systems theory (Willems, 1989; Willems, 1991), and as we shall see in examples in Section 5, we can for certain input-output systems associate a general flow dynamical system over an input-state-output product space.

Definition 3.2. Let $(L, <, 0)$ be a time line and let $X \neq \varnothing$ be any space. A *general flow system* over X with time line L is a map $\Phi\colon X \rightsquigarrow \mathsf{Path}(L, X)$ satisfying, for all $x \in \mathrm{dom}(\Phi)$, for all $\gamma \in \Phi(x)$, and for all $t \in \mathrm{dom}(\gamma)$:

(GF0) the *initialization property*: $\gamma(0) = x$;

(GF1) the *time-invariance* or *suffix-closure property*: ${}_t|\gamma \in \Phi(\gamma(t))$;

(GF2) the *point-concatenation property*: $\gamma|_t *_y \gamma' \in \Phi(x)$ for all $\gamma' \in \Phi(y)$ with $y = \gamma(t)$.

- Φ is *non-blocking* if $\Phi(x) \neq \{\theta_x\}$ for all $x \in \mathrm{dom}(\Phi)$;
- Φ is *prefix-closed* if $\gamma|_t \in \Phi(x)$ for all $x \in \mathrm{dom}(\Phi)$, $\gamma \in \Phi(x)$ and $t \in \mathrm{dom}(\gamma)$;
- Φ is *deterministic* if for every $x \in \mathrm{dom}(\Phi)$, the path set $\Phi(x)$ is linearly-ordered by $<$.
- Φ is *$<$-unbounded* if for all $x \in \mathrm{dom}(\Phi)$ and $\gamma \in \Phi(x)$, there exists $\gamma' \in \Phi(x)$ such that $\gamma < \gamma'$.

Among other results (Davoren *et al.*, 2004), it is readily established that Φ is non-blocking iff Φ is $<$-unbounded. Examples of general flow systems include state transition systems over $L = \mathbb{N}$, differential inclusions over $L = \mathbb{R}_0^+$, and hybrid automata and impulse differential inclusions over $L = \mathbb{H}$. Any path set $\mathcal{P} \subseteq \mathsf{Path}(L, X)$ that is monoidal and suffix-closed determines a general flow system.

It is clear that if a flow Φ is non-blocking, then for each $x \in dom(\Phi)$ and $\gamma \in \Phi(x)$, there exists an infinite sequence of paths $\{\gamma_n\}$ with $\gamma_0 = \gamma$ and $\gamma_n \in \Phi(x)$ and $\gamma_n < \gamma_{n+1}$ for all n. Motivated by this fact, we view "maximal extensions" or "completions" of paths as infinitary objects, arising as limits of infinite ordered sequences of finite bounded paths.

Definition 3.3. Let L be an ω-compact time line. For any path set $\mathcal{P} \subseteq \mathsf{Path}_\epsilon(L, X)$, define the *$\omega$-extension* of $\mathcal{P}$, and the *maximized ω-extension* of $\mathcal{P}$, as follows:

$$\begin{aligned} &\mathsf{Ext}^\omega(\mathcal{P}) \\ &:= \{\, \beta \in [L \dashrightarrow X] \mid (\exists\, \overline{\gamma} \in [\omega \to \mathsf{Path}(L, X)]\,) \\ &\qquad (\forall k < \omega)\ \gamma_k := \overline{\gamma}(k) \ \wedge\ \gamma_k \in \mathcal{P} \ \wedge \\ &\qquad\qquad \gamma_k < \gamma_{k+1} \ \wedge\ \beta = \bigcup_{k < \omega} \gamma_k \quad \} \end{aligned}$$

$$\begin{aligned} &\mathsf{M}^\omega(\mathcal{P}) \\ &:= \{\, \beta \in \mathsf{Ext}^\omega(\mathcal{P}) \mid (\forall \gamma \in \mathcal{P}\,)\ \beta \not< \gamma \,\} \end{aligned}$$

Define $\mathsf{EPath}^\omega(L, X) := \mathsf{Ext}^\omega(\,\mathsf{Path}_\epsilon(L, X)\,)$, and $\mathsf{EIPath}^\omega(L, X) := \mathsf{Ext}^\omega(\,\mathsf{IPath}_\epsilon(L, X)\,)$.

The ω-extension $\mathsf{Ext}^\omega(\mathcal{P})$ contains all the partial functions $\beta : L \dashrightarrow X$ that arise as the limit of an

ω-length strictly extending sequence of paths in $\mathcal{P}$. The *maximised* ω-extension $\mathsf{M}^\omega(\mathcal{P})$ throws out from $\mathsf{Ext}^\omega(\mathcal{P})$ all limit paths β that are properly extended by some finite-length path γ in $\mathcal{P}$. (The path extension partial ordering on bounded paths straightforwardly lifts to limit paths.)

Definition 3.4. Let L be an ω-compact time line. Given a general flow system $\Phi\colon X \rightsquigarrow \mathsf{Path}(L, X)$, define the maximised ω-extension $\mathsf{M}^\omega\Phi\colon X \rightsquigarrow \mathsf{EPath}^\omega(L, X)$ by $(\mathsf{M}^\omega\Phi)(x) := \mathsf{M}^\omega(\Phi(x))$ for all $x \in dom(\Phi)$. A flow Φ will be called:
• *ω-extendible* if for all $x \in dom(\Phi)$ and $\gamma \in \Phi(x)$, there exists $\alpha \in (\mathsf{M}^\omega\Phi)(x)$ such that $\gamma < \alpha$.
• *ω-full* if for all $x \in dom(\Phi)$ and $\beta \in \mathsf{Ext}^\omega(\Phi(x))$, there exists $\alpha \in (\mathsf{M}^\omega\Phi)(x)$ such that $\beta \leqslant \alpha$.

In general, $dom(\mathsf{M}^\omega\Phi) \subseteq dom(\Phi)$, and we have $dom(\mathsf{M}^\omega\Phi) = dom(\Phi)$ iff Φ is ω-extendible.

Proposition 3.5. Given a general flow system $\Phi\colon X \rightsquigarrow \mathsf{Path}(L, X)$ over any time line L,

Φ is ω-extendible
iff Φ is $<$-unbounded and ω-full.

For $L = \mathbb{N}$, all limit paths $\beta \in \mathsf{EIPath}^\omega(\mathbb{N}, X)$ have infinite time domain, so $(\mathsf{M}^\omega\Phi)(x) = \mathsf{Ext}^\omega(\Phi(x))$ for any non-blocking Φ. For $L = \mathbb{H}$ and ω-extendible Φ, limit paths $\alpha \in \mathrm{ran}(\mathsf{M}^\omega\Phi)$ include those with time domains $\mathrm{dom}(\alpha) = \bigcup_{n<\omega}\{n\} \times [0, \Delta_n]$, as well as those with $\mathrm{dom}(\alpha) = T_0 \cup (\{i\} \times [0, d))$ for some $T_0 \in \mathsf{BT}(\mathbb{H})$ and $(i, 0) >_{\mathrm{lex}} b_{T_0}$ and $d \in \mathbb{R}_0^+ \cup \{\infty\}$.

4. COMPARISON WITH EVOLUTIONARY AND BEHAVIOURAL SYSTEM MODELS

Aubin's evolutionary system's (Aubin and Dordan, 2002) are over the usual time lines $L = \mathbb{R}_0^+$ or $L = \mathbb{N}$, and consist of a map $\Psi : X \rightsquigarrow [L \to X]$, with extended whole line paths $\beta : L \to X$, where Ψ is closed under the operations of translation/suffix and point-concatenation, analogous to clauses **(GF1)** and **(GF2)** of Definition 3.2.

In Willems' Behavioural Systems theory (Willems, 1989; Willems, 1991), a *dynamical system* is a structure $\Sigma = (L, X, \mathfrak{B})$ where $L \subseteq \mathbb{R}$ is the time line, X is the signal space, and $\mathfrak{B} \subseteq [L \to X]$ is the *behaviour.* A behaviour $\mathfrak{B}$ is called *time invariant* if for all $\beta : L \to X$ and all $t \in L$, if $\beta \in \mathfrak{B}$ then the t-translation/suffix ${}_t|\beta \in \mathfrak{B}$; and is called a *state behaviour* if for all $t \in L$ and for all $\beta, \beta' \in \mathfrak{B}$, if $x = \beta(t) = \beta'(t)$ then the point-concatenation $(\beta|_t) *_x ({}_t|\beta') \in \mathfrak{B}$. A behaviour $\mathfrak{B}$ is called *complete* if for all $\beta : L \to X$, if $\big[(\forall a, b \in L)\, \beta\!\upharpoonright_{[a,b]} \in \mathfrak{B}\!\upharpoonright_{[a,b]}\big]$ then $\beta \in \mathfrak{B}$ (− and note that the reverse implication is trivially true).

Theorem 4.1. Let the time line be either $L = \mathbb{N}$ or $L = \mathbb{R}_0^+$, and $X \neq \varnothing$. Let $\Psi\colon X \rightsquigarrow [L \to X]$ and let $\mathfrak{B} \subseteq [L \to X]$. Then:

Ψ is an Aubin evolutionary system;
iff $\mathfrak{B}$ is a time-invariant and complete state behaviour in the sense of Willems;
iff there exists an ω-extendible interval path general flow system $\Phi\colon X \rightsquigarrow \mathsf{IPath}(L, X)$ such that $\Psi = \mathrm{M}^\omega\Phi$ and $\mathfrak{B} = \mathrm{ran}(\mathrm{M}^\omega\Phi)$.

5. INPUT-OUTPUT REACTIVE SYSTEMS

To allow that the time-line L_{out} for output paths may differ from the time-line L_{in} for input paths, we need to use at least *order-preserving* partial functions $\tau : L_{\mathrm{out}} \dashrightarrow L_{\mathrm{in}}$ which do a *time translation* by *looking in reverse*, starting from a "now" instant $t \in \mathrm{dom}(\gamma) \subset L_{\mathrm{out}}$ in an output path γ, and asking at what time $\tau(t) \in \mathrm{dom}(\eta) \subset L_{\mathrm{in}}$ is the corresponding "now" time point in the input path η which generates γ among its output.

Definition 5.1. Let L_{in} and L_{out} be time-lines, and let U, Y be any non-empty sets. Define

$$\mathsf{TT}(L_{\mathrm{out}}, L_{\mathrm{in}}) := \{\, \tau \in \mathsf{Path}(L_{\mathrm{out}}, L_{\mathrm{in}}) \mid \tau \textit{ is order-preserving} \wedge \tau(0_{\mathrm{out}}) = 0_{\mathrm{in}} \wedge \mathrm{ran}(\tau) \in \mathsf{BT}(L_{\mathrm{in}}) \,\}$$

A *reactive system* is any map Γ of type

$$\Gamma\colon \mathsf{Path}_\epsilon(L_{\mathrm{in}}, U) \rightsquigarrow (\mathsf{Path}_\epsilon(L_{\mathrm{out}}, Y) \times \mathsf{TT}_\epsilon(L_{\mathrm{out}}, L_{\mathrm{in}}))$$

such that for all inputs $\eta \in \mathrm{dom}(\Gamma)$, and for all outputs $(\gamma, \tau) \in \Gamma(\eta)$, we have $\mathrm{ran}(\tau) \subseteq \mathrm{dom}(\eta)$ and $\mathrm{dom}(\tau) = \mathrm{dom}(\gamma)$. A reactive system Γ will be called:
• *time-homogeneous* if: $L_{\mathrm{out}} = L_{\mathrm{in}} = L$, and $(\forall \eta \in \mathrm{dom}(\Gamma))(\forall (\gamma, \tau) \in \Gamma(\eta))$, $\mathrm{dom}(\gamma) = \mathrm{dom}(\eta)$ and τ is the identity function restricted to $\mathrm{dom}(\gamma)$, in which case we treat Γ as a map of type $\Gamma : \mathsf{Path}(L, U) \rightsquigarrow \mathsf{Path}(L, Y)$;
• [*strictly*] *extension-preserving* if:
$(\forall \eta \in \mathrm{dom}(\Gamma))(\forall \eta' \in \mathrm{dom}(\Gamma))$,
if $\eta \leqslant \eta'$ [*if* $\eta < \eta'$]
then $(\forall (\gamma, \tau) \in \Gamma(\eta)(\exists (\gamma', \tau') \in \Gamma(\eta'))$ such that $\tau \leqslant \tau'$ and $\gamma \leqslant \gamma'$ [$\tau < \tau'$ and $\gamma < \gamma'$];
• *time-invariant/suffix-closed* if: the path set $\mathrm{dom}(\Gamma)$ is suffix-closed, and
$(\forall \eta \in \mathrm{dom}(\Gamma))(\forall (\gamma, \tau) \in \Gamma(\eta))(\forall t \in \mathrm{dom}(\gamma))$,
$(\exists \tau' \in \mathsf{TT}(L_{\mathrm{out}}, L_{\mathrm{in}}))$, such that
$\tau' = ({}_t|\tau) \circ \sigma^{-t}$ and $({}_t|\gamma,\, \tau') \in \Gamma\big({}_{\tau(t)}|\eta\big)$;
• *prefix-closed* if: the path set $\mathrm{dom}(\Gamma)$ is prefix-closed, and $(\forall \eta \in \mathrm{dom}(\Gamma))(\forall (\gamma, \tau) \in \Gamma(\eta))$
$(\forall t \in \mathrm{dom}(\gamma))$, $(\gamma|_t,\, \tau|_t) \in \Gamma\big(\eta|_{\tau(t)}\big)$;
• *non-anticipating* (the output has a non-anticipatory dependence on the input) if:
$(\forall \eta, \eta' \in \mathrm{dom}(\Gamma))(\forall s \in \mathrm{dom}(\eta) \cap \mathrm{dom}(\eta'))$
if $\eta|_s = \eta'|_s$
then $(\forall (\gamma, \tau) \in \Gamma(\eta))(\exists (\gamma', \tau') \in \Gamma(\eta'))$
$(\forall t \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\gamma'))$ *if* $\tau(t) \leqslant s$
then $\tau|_t = \tau'|_t$ and $\gamma|_t = \gamma'|_t$;
• *strictly non-anticipating* if:

$(\forall \eta, \eta' \in \mathrm{dom}(\Gamma))(\forall s \in \mathrm{dom}(\eta) \cap \mathrm{dom}(\eta'))$
if $D_s = [0, s) \cap \mathrm{dom}(\eta) = [0, s) \cap \mathrm{dom}(\eta')$
and $\eta \restriction_{D_s} = \eta' \restriction_{D_s}$
then $(\forall (\gamma, \tau) \in \Gamma(\eta))(\exists (\gamma', \tau') \in \Gamma(\eta'))$
$(\forall t \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\gamma'))$ *if* $\tau(t) \leqslant s$
then $\tau|_t = \tau'|_t$ and $\gamma|_t = \gamma'|_t$;

• *input-time-unbounded* if: for every input path $\eta \in \mathrm{dom}(\Gamma)$, there exists an $\alpha \in \mathsf{M}^\omega(\mathrm{dom}(\Gamma))$ such that $\eta < \alpha$ and the time domain $\mathrm{dom}(\alpha)$ is an unbounded set in L_{in};

• *totally free in input* if: $\mathrm{dom}(\Gamma) = \mathsf{Path}_\epsilon(L_{\mathrm{in}}, U)$; i.e. Γ is total as a map;

• *monoidal* if: **(a)** $(U_{\mathcal{D}}, \mathcal{D}, *, \theta)$ and $(Y_{\mathcal{R}}, \mathcal{R}, *, \theta)$ are both monoidal families, where $\mathcal{D} := \mathrm{dom}(\Gamma)$ and $\mathcal{R} := \pi_Y(\mathrm{ran}(\Gamma))$; and **(b)** $(\forall \eta, \eta' \in \mathrm{dom}(\Gamma))$ $(\forall u \in U_{\mathcal{D}})$ *if* $\eta(b_\eta) = u = \eta'(0)$ *then* $(\forall (\gamma, \tau) \in \Gamma(\eta))$, $(\forall (\gamma', \tau') \in \Gamma(\eta'))$, if $\gamma(b_\gamma) = \gamma'(0)$, then $(\gamma'', \tau'') \in \Gamma(\eta *_u \eta')$, where $\gamma'' = \gamma *_y \gamma'$, $y = \gamma(b_\gamma)$, $\tau'' = \tau *_t (\tau' \circ \sigma^{+t})$ and $t = b_\gamma$.

Given any non-empty set X, a map Ψ will be called a *parameterized reactive system* if it is of type $\Psi : (X \times \mathsf{Path}_\epsilon(L_{\mathrm{in}}, U)) \rightsquigarrow (\mathsf{Path}_\epsilon(L_{\mathrm{out}}, Y) \times \mathsf{TT}_\epsilon(L_{\mathrm{out}}, L_{\mathrm{in}}))$, and for all $(x, \eta) \in \mathrm{dom}(\Psi)$, and for all $(\gamma, \tau) \in \Psi(x, \eta)$, $\mathrm{ran}(\tau) \subseteq \mathrm{dom}(\eta)$ and $\mathrm{dom}(\tau) = \mathrm{dom}(\gamma)$. Most of these properties of reactive systems can be simply extended to parameterized reactive systems by substituting quantification over $\eta \in \mathrm{dom}(\Gamma)$ with $(x, \eta) \in \mathrm{dom}(\Psi)$, and substituting $(\gamma, \tau) \in \Gamma(\eta)$ with $(\gamma, \tau) \in \Psi(x, \eta)$.

In (Willems, 1989; Willems, 1991), over the discrete and the continuous time lines, the *non-anticipating* property for input-output behaviours is proposed as the non-deterministic analogue of the *causality* property in deterministic systems. Our first major result is that, under the assumption of prefix-closure, the properties of being *non-anticipating* and of being *extension-preserving* are equivalent.

Theorem 5.2. Let $\Gamma : \mathsf{Path}_\epsilon(L_{\mathrm{in}}, U) \rightsquigarrow (\mathsf{Path}_\epsilon(L_{\mathrm{out}}, Y) \times \mathsf{TT}_\epsilon(L_{\mathrm{out}}, L_{\mathrm{in}}))$ be a reactive system, where the time lines L_{in} and L_{out} are ω-compact and Dedekind-complete.
(1.) If Γ is strictly non-anticipating, then Γ is non-anticipating.
(2.) If Γ is strictly extension-preserving, then Γ is extension-preserving
(3.) If Γ is extension-preserving and prefix-closed, then Γ is non-anticipating.
(4.) If Γ is non-anticipating, then Γ is extension-preserving.
From (3.) and (4.), if Γ is prefix-closed, then Γ is extension-preserving iff Γ is non-anticipating.
(5.) If Γ is monoidal and suffix-closed, then Γ is strictly extension-preserving.
(6.) If Γ is totally free in input, then Γ is input-time-unbounded.

Before discussing some examples of reactive systems, we first apply ω-extension constructions to the input-output setting.

Definition 5.3. Given a reactive system Γ with time lines L_{in} and L_{out} both ω-compact, define the ω-*extension* of Γ to be the map $\mathsf{Ext}^\omega\Gamma \colon \mathsf{EPath}^\omega(L_{\mathrm{in}}, U) \rightsquigarrow \mathsf{EPath}^\omega(L_{\mathrm{out}}, Y \times L_{\mathrm{in}})$ such that:

$$\begin{aligned}
&(\mathsf{Ext}^\omega\Gamma)(\alpha)\\
:= \{\, &(\beta, \upsilon) \in \mathsf{Ext}^\omega(\mathrm{ran}(\Gamma)) \mid\\
&(\exists\, \overline{\boldsymbol{\eta}} \in [\omega \to \mathsf{Path}(L_{\mathrm{in}}, U)]\,)\\
&(\exists\, (\overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}}) \in [\omega \to \mathsf{Path}(L_{\mathrm{out}}, Y \times L_{\mathrm{in}})]\,)\\
&(\forall k < \omega)\ \eta_k := \overline{\boldsymbol{\eta}}(k) \ \wedge\ \gamma_k := \overline{\boldsymbol{\gamma}}(k) \ \wedge\\
&\tau_k := \overline{\boldsymbol{\tau}}(k) \ \wedge\ \eta_k \in \mathrm{dom}(\Gamma) \ \wedge\ (\gamma_k, \tau_k) \in \Gamma(\eta_k)\\
&\wedge\ \eta_k < \eta_{k+1} \ \wedge\ (\gamma_k, \tau_k) < (\gamma_{k+1}, \tau_{k+1}) \ \wedge\\
&\alpha = \bigcup_{k<\omega} \eta_k \ \wedge\ \beta = \bigcup_{k<\omega} \gamma_k \ \wedge\ \upsilon = \bigcup_{k<\omega} \tau_k \,\}
\end{aligned}$$

for all $\alpha \in \mathrm{dom}(\mathsf{Ext}^\omega\Gamma) := \mathsf{Ext}^\omega(\mathrm{dom}(\Gamma))$.

In defining the *maximized ω-extension* $\mathsf{M}^\omega\Gamma$ of a reactive system Γ, we want to restrict to limit input paths $\alpha \in \mathrm{dom}(\mathsf{Ext}^\omega\Gamma) \subseteq \mathsf{EPath}^\omega(L_{\mathrm{in}}, U)$ that are not only maximized ω-extensions of finite input paths, so $\alpha \in \mathsf{M}^\omega(\mathrm{dom}(\Gamma))$, but that are also *unbounded* in the length of their time domain. This rules out limit input paths with finite escape time. When a system Γ is input-time-unbounded, then there are no limit input paths with bounded time duration, hence for such systems, there will be no loss in the move from Γ to the maximized ω-extension $\mathsf{M}^\omega\Gamma$.

Note, however, that on the output side, we will *not* be assuming that all resulting limit output paths must be time-unbounded; rather, we will identify that condition into a system property of *output-time-unboundedness*.

Definition 5.4. Let L be an ω-compact time line, and let $X \neq \varnothing$ be any value space. For any path set $\mathcal{P} \subseteq \mathsf{Path}_\epsilon(L, X)$, define the *time-unbounded maximized ω-extension* of $\mathcal{P}$ to be the limit path set $\mathsf{TM}^\omega(\mathcal{P})$, with $\mathsf{TM}^\omega(\mathcal{P}) \subseteq \mathsf{M}^\omega(\mathcal{P}) \subseteq \mathsf{Ext}^\omega(\mathcal{P}) \subseteq \mathsf{EPath}^\omega(L, X)$, defined by:

$$\mathsf{TM}^\omega(\mathcal{P}) := \{\alpha \in \mathsf{M}^\omega(\mathcal{P}) \mid (\forall t \in L)\ (\exists s \in \mathrm{dom}(\alpha))\ s \geqslant t\,\}$$

So the limit path set $\mathsf{TM}^\omega(\mathcal{P})$ is that subset of $\mathsf{M}^\omega(\mathcal{P})$ obtained by throwing away all limit paths α where the time domain $\mathrm{dom}(\alpha)$ is a bounded subset of L. We shall require time-unbounded maximized extensions on the input side.

Definition 5.5. Given a reactive system Γ with ω-compact time lines L_{in} and L_{out}, define the *maximized ω-extension* of Γ to be the map $\mathsf{M}^\omega\Gamma$ of type:

$$\mathsf{M}^\omega\Gamma \colon \mathsf{EPath}^\omega(L_{\mathrm{in}}, U) \rightsquigarrow \mathsf{EPath}^\omega(L_{\mathrm{out}}, Y \times L_{\mathrm{in}})$$

such that $\mathrm{dom}(\mathsf{M}^\omega\Gamma) := \mathsf{TM}^\omega(\mathrm{dom}(\Gamma))$, and $(\mathsf{M}^\omega\Gamma)(\alpha) := (\mathsf{Ext}^\omega\Gamma)(\alpha)$ for all $\alpha \in \mathrm{dom}(\mathsf{M}^\omega\Gamma)$.

A reactive system Γ will be called:
• *ω-responsive* if it is input-time-unbounded and $(\forall \eta \in \mathrm{dom}(\Gamma))(\forall \alpha \in \mathrm{dom}(\mathsf{M}^\omega\Gamma))$, *if* $\eta < \alpha$ *then* $(\forall(\gamma,\tau) \in \Gamma(\eta))(\exists(\beta,\upsilon) \in (\mathsf{M}^\omega\Gamma)(\alpha))$ such that $(\gamma,\tau) < (\beta,\upsilon)$
• *output-time-unbounded* if every limit output path has an unbounded time domain, which means $\mathrm{ran}(\mathsf{M}^\omega\Gamma) = \mathsf{TM}^\omega(\mathrm{ran}(\Gamma))$.

The following theorem gives a first characterisation of the ω-responsiveness property.

Theorem 5.6. For any reactive system Γ,
(7.) Γ is ω-responsive iff Γ is strictly extension-preserving and input-time-unbounded.
(8.) If Γ is monoidal, suffix-closed and input-time-unbounded, then Γ is ω-responsive.

Example 5.7. A *state machine* is a structure $\mathcal{S} = (X, U, Y, \mathit{UpDt})$ with state set X, input set U, output set Y, and $\mathit{UpDt} : (X \times U) \rightsquigarrow (X \times Y)$ the state/output update map; equivalently, $\mathcal{S}$ is a non-deterministic Mealy machine. Associate with $\mathcal{S}$ three maps on interval paths over $L = \mathbb{N}$.

The first map is the *full input-state-output map* $\Phi_\mathcal{S} : (U \times X \times Y) \rightsquigarrow \mathsf{IPath}(\mathbb{N}, U \times X \times Y)$ with:

$$\begin{aligned} &\Phi_\mathcal{S}(u,x,y) \\ := \; &\{ (\eta,\xi,\gamma) \in \mathsf{IPath}(\mathbb{N}, U \times X \times Y) \mid \\ &\quad u = \eta(0) \;\wedge\; x = \xi(0) \;\wedge\; y = \gamma(0) \;\wedge \\ &\quad (\forall i < b_\eta)\; (\xi(i+1), \gamma(i)) \in \mathit{UpDt}(\xi(i), \eta(i)) \wedge \\ &\quad (\exists x' \in X)\; (x', \gamma(b_\eta)) \in \mathit{UpDt}(\xi(b_\eta), \eta(b_\eta)) \quad \} \end{aligned}$$

It is readily seen that $\Phi_\mathcal{S}$ is a prefix-closed general flow system and that $\Phi_\mathcal{S}$ is non-blocking (and ω-extendible) iff the map UpDt is total.

We can then define two reactive systems from the flow $\Phi_\mathcal{S}$ via projections. First, the *input-state* system: an parameterized time-homogeneous reactive system

$$\Psi_\mathcal{S} : (X \times \mathsf{IPath}(\mathbb{N}, U)) \rightsquigarrow \mathsf{IPath}(\mathbb{N}, X)$$

and second, the (external behaviour) *input-output* system: a time-homogeneous reactive system

$$\Gamma_\mathcal{S} : \mathsf{IPath}(\mathbb{N}, U) \rightsquigarrow \mathsf{IPath}(\mathbb{N}, Y)$$

transforming input-paths into output-paths. These two maps are defined as follows:

$$\begin{aligned} \Psi_\mathcal{S}(x,\eta) &:= \{\xi \in \mathsf{IPath}(\mathbb{N}, X) \mid (\exists \gamma \in \mathsf{IPath}(\mathbb{N}, Y)) \\ &\qquad (\eta,\xi,\gamma) \in \Phi_\mathcal{S}(\eta(0), x, \gamma(0)) \} \\ \Gamma_\mathcal{S}(\eta) &:= \{\gamma \in \mathsf{IPath}(\mathbb{N}, Y) \mid (\exists \xi \in \mathsf{IPath}(\mathbb{N}, X)) \\ &\qquad (\eta,\xi,\gamma) \in \Phi_\mathcal{S}(\eta(0), \xi(0), \gamma(0)) \} \end{aligned}$$

In particular, $(x,\eta) \in \mathrm{dom}(\Psi_\mathcal{S})$ iff $(x, \eta(0)) \in \mathrm{dom}(\mathit{UpDt})$, and $\mathrm{dom}(\Gamma_\mathcal{S}) = \pi_U(\mathrm{dom}(\Psi_\mathcal{S}))$. Observe also that $\Psi_\mathcal{S}$ and $\Gamma_\mathcal{S}$ are both suffix-closed and prefix-closed, and if the map UpDt is total, then both $\Psi_\mathcal{S}$ and $\Gamma_\mathcal{S}$ are monoidal, extension-preserving, non-anticipating, totally free in input, ω-responsive and output-time-unbounded.

Example 5.8. Over continuous time, an input-state-output system of *differential inclusions* is a structure $\mathcal{DI} = (X, U, Y, \mathcal{U}, F, G)$ where $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^p$, $\mathcal{U} \subseteq \mathsf{IPath}(\mathbb{R}_0^+, U)$ is a set of input paths, and the maps $F : (X \times U) \rightsquigarrow \mathbb{R}^n$ and $G : (X \times U) \rightsquigarrow Y$ are subject to regularity assumptions to guarantee existence of solutions without finite escape time (Aubin *et al.*, 2002). The resulting solution map $\mathsf{Sol}_\mathcal{DI} : (X \times \mathcal{U}) \rightsquigarrow \mathsf{IPath}(\mathbb{R}_0^+, X)$ is an initialized time-homogeneous reactive system which is suffix-closed, prefix-closed and strictly non-anticipating. The time-homogeneous external I/O map $\Gamma_\mathcal{DI} : \mathsf{IPath}(\mathbb{R}_0^+, U) \rightsquigarrow \mathsf{IPath}(\mathbb{R}_0^+, Y)$ also incorporates the G output constraint, and is suffix-closed, prefix-closed, and non-anticipating. These two maps can be combined to define the input-state-output general flow $\Phi_\mathcal{DI} : (U \times X \times Y) \rightsquigarrow \mathsf{IPath}(\mathbb{R}_0^+, U \times X \times Y)$.

As our basic example of differing time lines, consider *analog-to-digital conversion* based on *event-driven* sampling rather than fixed periodic sampling.

Example 5.9. Let $Y \subseteq \mathbb{R}^n$ be a continuous *measurement space*, and the input signals will be continuous-time interval paths $\eta \in \mathsf{IPath}(\mathbb{R}_0^+, Y)$. Let $Z \neq \varnothing$ be a finite set of *event symbols*, and let $\mathbf{A} : Z \rightsquigarrow Y$ be any total map from Z to Y which associates with each event symbol $z \in Z$ a non-empty subset $\mathbf{A}(z) \subseteq Y$ of measurement values y which trigger the event symbol z. We require $\mathbf{A}(z) \cap \mathbf{A}(z') \neq \varnothing$ for some $z \neq z'$, as it is in these overlap regions (e.g. common boundaries) that the discrete event output signal $\gamma \in \mathsf{IPath}(\mathbb{N}, Z)$ can switch from value z to value z' (or vice-versa).

We define reactive systems $\mathsf{AnIn}_\mathbf{A}$ and $\mathsf{DigOut}_\mathbf{A}$:

$$\begin{aligned} \mathsf{AnIn}_\mathbf{A} &: \mathsf{IPath}(\mathbb{R}_0^+, Y) \rightsquigarrow \\ &\qquad (\mathsf{HPath}(Z \times Y) \times \mathsf{TT}(\mathbb{H}, \mathbb{R}_0^+)) \\ \mathsf{DigOut}_\mathbf{A} &: \mathsf{HPath}(Z \times Y) \rightsquigarrow \\ &\qquad (\mathsf{IPath}(\mathbb{N}, Z) \times \mathsf{TT}(\mathbb{N}, \mathbb{H})) \end{aligned}$$

with

$$\begin{aligned} &\mathsf{AnIn}_\mathbf{A}(\eta) \\ := \; &\{ (\xi, \bar{\tau}) \in \mathsf{HPath}(Z \times Y) \times \mathsf{TT}(\mathbb{H}, \mathbb{R}_0^+) \mid \\ &\quad \mathrm{ran}(\bar{\tau}) \subseteq \mathrm{dom}(\eta) \;\wedge\; \mathrm{dom}(\bar{\tau}) = \mathrm{dom}(\xi) \;\wedge \\ &\quad (\forall (i,t) \in \mathrm{dom}(\xi))\; \big(\bar{\tau}(i,t) = t + \textstyle\sum_{j<i} \Delta_j \\ &\quad \text{where } \Delta = \mathrm{ds}(\xi) \;\wedge\; \pi_Y \xi(i,t) = \eta(\bar{\tau}(i,t)) \big) \\ &\quad \wedge (\forall i < \mathrm{dl}(\xi))(\exists z_i \in Z)(\forall t \in [0, \Delta_i]\,) \\ &\quad \pi_Z \xi(i,t) = z_i \;\wedge\; \eta(\bar{\tau}(i,t)) \in \mathbf{A}(z_i) \qquad \} \\ &\mathsf{DigOut}_\mathbf{A}(\xi) \\ := \; &\{ (\gamma, \hat{\tau}) \in \mathsf{IPath}(\mathbb{N}, Z) \times \mathsf{TT}(\mathbb{N}, \mathbb{H}) \mid \\ &\quad \mathrm{ran}(\hat{\tau}) \subseteq \mathrm{dom}(\xi) \;\wedge\; \mathrm{dom}(\hat{\tau}) = \mathrm{dom}(\gamma) \;\wedge \\ &\quad \mathrm{length}(\gamma) = \mathrm{dl}(\xi) \;\wedge\; (\forall k \in \mathrm{dom}(\gamma)) \\ &\quad \hat{\tau}(k) = (k, 0) \;\wedge\; \gamma(k) = \pi_Z \xi(\hat{\tau}(k)) \quad \} \end{aligned}$$

Applied to a real-time input path η with values in Y, the map $\mathsf{AnIn}_\mathbf{A}$ will produce as output hybrid paths ξ over the product space $X = Z \times Y$, where the Y-projection $\pi_Y \xi$ reproduces the input η on the hybrid output time line, in the sense that $\pi_Y \xi = \bar{\tau} \circ \eta$, and the time translation $\bar{\tau} : \mathrm{dom}(\xi) \to \mathrm{dom}(\eta)$ maps a hybrid time point (i, t) back to $\bar{\tau}(i, t)$, the (real-valued) total duration of the hybrid path ξ to this point. The further constraint on ξ is that the Z-projection $\pi_Z \xi$ is constant with some value z_i for all positions (i, t) between $(i, 0)$ and (i, Δ_i), and $(\bar{\tau} \circ \eta)(i, t) \in \mathbf{A}(z_i)$. This means for each $i < \mathrm{dl}(\xi)$, the input path η remains continuously within the region $\mathbf{A}(z_i) \subseteq Y$ for all times $s \in \mathrm{dom}(\eta)$ such that $\tau(i, 0) \leqslant s \leqslant \tau(i, \Delta_i)$, and $\eta(s) \in \mathbf{A}(z_i) \cap \mathbf{A}(z_{i+1})$ at a switching time $s = \tau(i, \Delta_i) = \tau(i + 1, 0)$. The second map $\mathsf{DigOut}_{\mathbf{A}}$ takes as input hybrid paths ξ with values in $X = Z \times Y$ and returns as output a discrete-time path γ with values in Z obtained from ξ by simply projecting on to discrete time and discrete values in Z.

The overall analog-to-digital conversion system is a map $\mathsf{AnDig}_{\mathbf{A}} : \mathsf{IPath}(\mathbb{R}_0^+, Y) \rightsquigarrow (\mathsf{IPath}(\mathbb{N}, Z) \times \mathsf{TT}(\mathbb{N}, \mathbb{R}_0^+))$, induced by the map $\mathbf{A}$, and obtained from the two components $\mathsf{AnIn}_{\mathbf{A}}$ and $\mathsf{DigOut}_{\mathbf{A}}$ by a compound sequential composition operation. The systems $\mathsf{AnIn}_{\mathbf{A}}$ and $\mathsf{DigOut}_{\mathbf{A}}$ and $\mathsf{AnDig}_{\mathbf{A}}$ are each extension-preserving, non-anticipating, prefix-closed, suffix-closed and monoidal.

6. COMPARISON WITH OTHER INPUT-OUTPUT SYSTEM MODELS

Consider an input-output behaviour $\mathfrak{B} \subseteq [L \to (U \times Y)]$; we write $(\alpha, \beta) \in \mathfrak{B}$ where $\alpha \in [L \to U]$ and $\beta \in [L \to Y]$. An input-output behaviour $\mathfrak{B}$ has *free input* if $\pi_U(\mathfrak{B}) = [L \to U]$. The output of $\mathfrak{B}$ is *non-anticipating* of the input, if for all $(\alpha', \beta'), (\alpha'', \beta'') \in \mathfrak{B}$, and for all $t \in L$, if $\alpha'|_t = \alpha''|_t$, then there exists $\beta \in [L \to Y]$ such that $\beta|_t = \beta'|_t$ and $(\alpha'', \beta) \in \mathfrak{B}$. The following result establishes a basic correspondence, for interval paths with time lines $L = \mathbb{N}$ and $L = \mathbb{R}_0^+$, between these behavioural properties and the corresponding properties of reactive systems.

Theorem 6.1. Let the time line be either $L = \mathbb{N}$ or $L = \mathbb{R}_0^+$ and let $\Gamma : \mathsf{IPath}_\epsilon(L, U) \rightsquigarrow \mathsf{IPath}_\epsilon(L, Y)$ be a time-homogeneous, ω-extendible, output-time-unbounded, interval-path reactive system, in which case, $\mathrm{dom}(\mathsf{M}^\omega \Gamma) \subseteq [L \to U]$ and $\mathrm{ran}(\mathsf{M}^\omega \Gamma) \subseteq [L \to Y]$. Further suppose that $\mathfrak{B} = \mathsf{M}^\omega \Gamma = \{(\alpha, \beta) \mid \beta \in (\mathsf{M}^\omega \Gamma)(\alpha)\}$. Then:

(1.) $\mathfrak{B}$ is a non-anticipating behaviour with free input iff Γ non-anticipating with totally free input;

(2.) $\mathfrak{B}$ is a time-invariant behaviour iff Γ is suffix-closed/time-invariant.

7. CONCLUSION

We have developed a general formulation of non-deterministic input-output reactive systems, based on finite length paths, that is uniform for discrete, continuous and hybrid time, and that allows for the time line of output paths to differ from that of input paths. The work is intended as a first installment of a larger project on the study of compositions of input-output reactive systems by sequential, parallel and feedback constructions, with the aim of providing explicit set-theoretic semantics for non-deterministic dynamics and for time-translations within block-diagram based modeling and description frameworks such as CHARON (Alur *et al.*, 2000), among others.

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