PRACTICAL STABILIZATION OF DISCRETE–TIME LINEAR SISO SYSTEMS UNDER ASSIGNED INPUT AND OUTPUT QUANTIZATION [1]

Bruno Picasso [*,**,2] **Antonio Bicchi** [*]

[*] *Università di Pisa, Centro Interdipartimentale di Ricerca E. Piaggio", Via Diotisalvi 2, 56100 Pisa, Italy*
[**] *Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy*

Abstract: This work is concerned with the practical stabilization of discrete–time SISO linear systems under assigned quantization of the input and output spaces. A controller is designed ensuring practical stability properties.
Unlike most of the existing literature, quantization is supposed to be a problem datum rather than a degree of freedom in design. Moreover, in the framework of control under assigned quantization, results are concerned with state quantization only and do not include the quantized output feedback case considered here.
While standard stability analysis techniques are based on Lyapunov theory and invariant ellipsoids, our study involves a particularly suitable family of sets, which are hypercubes in controller form coordinates. *Copyright, © 2006, IFAC*

Keywords: Quantized systems, stabilizability, dynamic output feedback.

1. INTRODUCTION

The need of studying quantized control systems (i.e., dynamical systems with discrete input and/or output variables) arises by many control applications. Commonly encountered examples include the presence of digital sensors and actuators or of finite capacity communication links in the control loop. Quantized control systems have been attracting increasing attention of the control community in the past twenty years (see for instance (Delchamps, 1990; Wong–Brockett, 1999; Brockett–Liberzon, 2000; Elia–Mitter, 2001; Baillieul, 2001; Bicchi et al., 2002; Tatikonda–Mitter, 2004; Fagnani–Zampieri, 2004)).
This paper deals with the control of the linear, time–invariant dynamical system

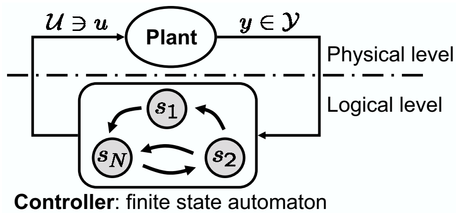

Fig. 1. Graphical illustration of the quantized control system considered in this paper.

$$\begin{cases} x(t+1) = Ax(t) + b\,u(t) \\ y(t) = q\big(Cx(t)\big) \\ x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}, \quad y \in \mathcal{Y}, \quad t \in \mathbb{N} \\ A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n, \quad C \in \mathbb{R}^{1\times n}, \end{cases} \tag{1}$$

where $\mathcal{U}$ is a *given* closed discrete set and $q : \mathbb{R} \to \mathcal{Y}$ is an *assigned* output map taking values in a countable set $\mathcal{Y}$ (finite or infinite). A pictorial representation of the control problem is illustrated in Fig. 1: the system has a hybrid structure and is organized into two levels. At the logical level, the controller manipulates output and input strings from discrete alphabets. At the physical level, the plant is modelled by Eqn. (1).
We focus on the stabilization problem. It has been clarified in (Delchamps, 1990) that "practical"

[1] This work was supported by European Commission through the IST-2004-004536 (IP) "RUNES - Reconfigurable Ubiquitous Networked Embedded Systems" and the IST-2004-511368 (NoE) "HYCON - HYbrid CONtrol: Taming Heterogeneity and Complexity of Networked Embedded Systems".
[2] Corresponding author, e–mail: `b.picasso@sns.it`

stability notions are the suitable stability properties to be considered for quantized systems. Accordingly, our study concerns the construction of symbolic feedback controllers capable of steering the system to within small neighborhoods of the equilibrium, starting from large attraction basins. In most of the existing literature on stabilization, quantization is considered as a degree of freedom in control synthesis: the designer can choose the elements of the control set $\mathcal{U}$, as well as the output map q. Results in this vein have a strong theoretical interest as they allow to identify fundamental limitations in quantized control (Wong–Brockett, 1999; Elia–Mitter, 2001; Ishii–Francis, 2002; Nair–Evans, 2004; Tatikonda–Mitter, 2004; Fagnani–Zampieri, 2004). On the contrary, in this paper we assume that the input and output sets $\mathcal{U}, \mathcal{Y}$, as well as the output map q, are data of the problem: control synthesis for practical stabilization is then subdued to these data. This kind of study provides tools for the analysis of achievable control objectives by using a *given* technology, as e.g., actuators (modelled by $\mathcal{U}$) or sensors (modelled by q).

The main contribution of this paper consists in providing a novel, simple and general technique to solve the practical stabilization problem for quantized SISO systems. The controller is synthesized so as to encapsulate the trajectories into increasingly smaller hypercubes in the controller form coordinates. This approach results in a controller in the form of a finite state automaton. The fundamental observation is that most of the significant information related to the input set $\mathcal{U}$, to the output map q and to the dynamics of the system, are contained in a pair (ρ, H) of scalar functions providing an effective representation of the resolution (or *dispersion*) of the quantizers.
The quantization schemes typically encountered in the literature include either innovation (Elia–Mitter, 2001; Tatikonda–Mitter, 2004) or state quantization (Delchamps, 1990; Wong–Brockett, 1999; Liberzon, 2003; Picasso–Bicchi, 2003). In the latter case, the output map $y = q(x)$ is such that $q^{-1}(y)$ is a bounded set. In this work instead, quantized outputs are considered: the output map $q \circ C : \mathbb{R}^n \to \mathcal{Y}$ induces a state space partition $\mathbb{R}^n = \bigcup_{y\in\mathcal{Y}} (q\circ C)^{-1}(y)$ made of *unbounded* sets and this makes the problem considerably different. We also remark that the only contributions addressing quantization as a problem datum do so by fixing either input or state quantization. In our work, we expressly consider the case in which both input and output quantizations are assigned.

The paper is organized as follows: the problem is formulated in Sec. 2, the main result about practical stability is in Sec. 3 (including an example) and its proof is given in Sec. 4. The easy proofs of the most technical results are omitted.

Notation: $Q_n(\Delta) := \left[-\frac{\Delta}{2}; \frac{\Delta}{2}\right]^n = \left\{x \in \mathbb{R}^n \,|\, \|x\|_\infty \le \frac{\Delta}{2}\right\}$. Let $E \subseteq \mathbb{R}^k$: E^{ch} and $\# E$ denote respectively its convex hull and its cardinality; $\mathrm{diam}(E) := \sup_{x,y\in E} \|x-y\|_2$ is the diameter of E. Given $v \in \mathbb{R}^k$, $E + v := \{x \in \mathbb{R}^k \,|\, x - v \in E\}$. Let x_i be the i^{th} coordinate of x: given $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Pr}_i(\Omega) := \{\omega_i \,|\, \omega \in \Omega\}$ and $\mathrm{diam}_i(\Omega) := \mathrm{diam}\big(\mathrm{Pr}_i(\Omega)\big)$. If $A \in \mathbb{R}^{n\times n}$, $A\Omega = \{A\omega \,|\, \omega \in \Omega\}$ while $(Ax)_i$ is the i^{th} coordinate of the vector Ax. x' denotes the transpose of the vector x, x^+ stands for $x(t+1)$.

2. PROBLEM STATEMENT

We deal with the discrete–time, quantized SISO system given in Eqn. (1). We assume that the pair (A, b) is reachable and that the pair (A, C) is observable. Without loss of generality, we assume that the pair (A, b) is in *controller form*:

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ is the characteristic polynomial of A. Because

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^n |A_{i,j}| = \max\Big\{1, \sum_{i=1}^n |\alpha_i|\Big\},$$

if $\sum_{i=1}^n |\alpha_i| \le 1$ then the system is stable, we hence assume $\sum_{i=1}^n |\alpha_i| > 1$ and we let $\alpha := \sum_{i=1}^n |\alpha_i|$.
The output quantizer $q : \mathbb{R} \to \mathcal{Y}$ is characterized by the induced output space partition: $\mathbb{R} = \bigcup_{y\in\mathcal{Y}} q^{-1}(y)$. The partition is supposed to be locally finite, namely, if $B \subset \mathbb{R}$ is bounded, $\#\{y \in \mathcal{Y} \,|\, B \cap q^{-1}(y) \neq \emptyset\} < +\infty$. We also assume that $\forall\, y \in \mathcal{Y}$, $q^{-1}(y) \subseteq \mathbb{R}$ is a connected set. In this case, by suitably redefining q without varying the induced output space partition (hence, without loss of generality), we can assume that $\mathcal{Y} \subset \mathbb{R}$ and $q : \mathbb{R} \to \mathcal{Y}$ is such that $\forall\, y \in \mathcal{Y}$ the closure of $q^{-1}(y)$ is either an interval of length λ_y of the type $[\, y - \frac{\lambda_y}{2}\,;\, y + \frac{\lambda_y}{2}\,]$ or a half–line. Let $\mathcal{Y}_\star := \{y \in \mathcal{Y} \,|\, q^{-1}(y) \text{ is an interval of finite length}\}$.
With regard to the control set, we assume that $0 \in \mathcal{U}$ so that $(x = 0, u = 0)$ is an equilibrium pair. Given $\mathcal{U}_* \subseteq \mathcal{U}$, let $\rho_{\mathcal{U}_*}$ represent the dispersion (or maximal gap) of $\mathcal{U}_*$, that is:

$$\rho_{\mathcal{U}_*} := \begin{cases} \sup\ \big\{\mathrm{diam}\big(]a;b[\big) \,\big|\,]a;b[\subseteq \mathcal{U}_*^{\mathrm{ch}} \text{ and } \\ \qquad]a;b[\cap\, \mathcal{U}_* = \emptyset\,\big\} \quad \text{if } \#\mathcal{U}_* > 1 \\ +\infty \ \text{(conventionally)} \quad \text{otherwise.} \end{cases} \tag{2}$$

Definition 1. Given system (1), let

$$\nu : \mathbb{R} \longrightarrow \mathcal{U} \tag{3}$$

be a map which associates to each real number r an element of $\mathcal{U}$ minimizing the Euclidean distance from r. The feedback law $k : \mathbb{R}^n \to \mathcal{U}$ defined by

$$k(x) := \nu\big(-\textstyle\sum_{i=1}^n \alpha_i x_i\big) = \nu\big(-(Ax)_n\big)$$

is called *state feedback quantized deadbeat controller* (state feedback qdb–controller).

The definition of the map ν is well posed because $\mathcal{U}$ is a closed set.

Problem statement: given system (1), we are interested in the synthesis of a controller which,

based on quantized output measurements $y \in \mathcal{Y}$, selects a quantized control $u \in \mathcal{U}$ and ensures practical stability properties. We will consider dynamical controllers and we will study the practical stability notion of (X_0, X_1, Ω)–*stability*.
More precisely, let the controller be described by the following system defined on some set $\mathcal{W}$:

$$\begin{cases} w(t+1) = \gamma\big(w(t), y(t), t\big) \\ u(t) = \phi\big(w(t), y(t), t\big), \end{cases} \tag{4}$$

where $\gamma : \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{W}$ and $\phi : \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{U}$. The closed–loop dynamics induced by the feedback interconnection of such a controller with system (1) is:

$$\begin{cases} x(t+1) = Ax(t) + b\,\phi\Big(w(t), q\big(Cx(t)\big), t\Big) \\ w(t+1) = \gamma\,\Big(w(t), q\big(Cx(t)\big), t\Big). \end{cases} \tag{5}$$

Definition 2. (Cf. (Fagnani–Zampieri, 2004)) Let Ω, X_0 and X_1 be subsets of $\mathbb{R}^n$ such that Ω and X_0 are neighborhoods of the origin and $X_1 \supseteq X_0$ is bounded. The controller (4) is said to be (X_0, X_1, Ω)–*stabilizing* iff the corresponding closed–loop dynamics (5) is so that $\forall\, x(0) \in X_0$ and $\forall\, w(0) \in \mathcal{W}$, $x(t) \in X_1 \ \ \forall\, t \geq 0$ and $\exists\, \bar{t} \in \mathbb{N}$ such that $\forall\, t \geq \bar{t}$, $x(t) \in \Omega$.

3. INPUT AND OUTPUT QUANTIZATION

The use of a controller endowed with memory, hence taking the general form in Eqn. (4), allows to treat the quantized output case by taking advantage of some techniques introduced for the quantized state case. In fact, by storing the past inputs and outputs, it is possible to reconstruct a bounded region within which the current state is confined. Nevertheless, the state quantization obtained in this way is *time–varying* so that the results from the quantized state case need to be further elaborated in order to be applied to the quantized output problem.
Following (Picasso et al., 2002; Picasso–Bicchi, 2003), the stabilization problem is studied taking into consideration sets X_0, X_1 and Ω in the form of hypercubes in the controller form coordinates. This choice makes the analysis particularly simple so that explicit results can be provided for arbitrarily assigned input and output quantized sets.

3.1 Preliminaries

Suppose that at time t the current state of the system $x(t)$ is only known to belong to a certain bounded set, more precisely assume that $x(t) \in \mathcal{C}_{x(t)} \subset Q_n(\Delta)$. The existence of a control value $u \in \mathcal{U}$ ensuring that $x(t+1) \in Q_n(\Delta)$ is tantamount to requiring that $\exists\, u \in \mathcal{U}$ such that $A\mathcal{C}_{x(t)} + b\,u \subseteq Q_n(\Delta)$. For $u \in \mathcal{U}$, by the controller form of (A, b), it holds that

$$\begin{aligned} x^+ = (x_2, \ldots, x_n, \textstyle\sum_i \alpha_i x_i + u) \in Q_n(\Delta) \\ \iff \big|\textstyle\sum_i \alpha_i x_i + u\big| \leq \tfrac{\Delta}{2}\,. \end{aligned} \tag{6}$$

Therefore, $x(t+1) \in Q_n(\Delta)$ if and only if

$$\mathrm{Pr}_n(A\mathcal{C}_{x(t)}) + u \subseteq \big[-\tfrac{\Delta}{2}\,;\, \tfrac{\Delta}{2}\big]. \tag{7}$$

It is then important to have an estimate of $\mathrm{diam}_n(A\mathcal{C}_{x(t)})$. Furthermore, the control acts only on the n^{th} component while the others shift upward (see Eqn. (6)) : hence, trajectories with both the properties of converging to a small neighborhood Ω of the equilibrium and a high speed of convergence towards Ω can be obtained by selecting the control value u so that the middle point of the set $\mathrm{Pr}_n(A\mathcal{C}_{x(t)}) + u$ is as near as possible to 0. This task is accomplished by the state feedback qdb–controller.
Accordingly, the dynamic qdb–controller proposed below is based on the following paradigm:

1- a bounded set within which the current state lies is located;
2- an estimate $\hat{x}$ of the current state is obtained;
3- the control action is selected by the state feedback qdb–controller, namely $u = k(\hat{x})$.

Before explicitly defining the dynamic qdb–controller, we need the following preliminary result.

Lemma 1. If $x \in Q_n(\Delta)$ and u is such that $x^+ \in Q_n(\Delta)$, then $u \in \big[-\frac{\Delta}{2}(\alpha+1)\,;\, \frac{\Delta}{2}(\alpha+1)\big]$.

Hence, if the goal is to find $u \in \mathcal{U}$ such that x^+ remains in $Q_n(\Delta)$, the set of the control values that are relevant to this problem is

$$\mathcal{U}(\Delta) := \mathcal{U} \cap \big[-\tfrac{\Delta}{2}(\alpha+1)\,;\, \tfrac{\Delta}{2}(\alpha+1)\big]. \tag{8}$$

Notice that $\#\mathcal{U}(\Delta) < +\infty$ because $\mathcal{U}$ is a closed discrete set. Let

$$\begin{cases} m(\Delta) := \min \mathcal{U}(\Delta) \\ M(\Delta) := \max \mathcal{U}(\Delta) \end{cases}$$

and, according to (2), let

$$\rho(\Delta) := \rho_{\mathcal{U}(\Delta)} \tag{9}$$

be the dispersion of $\mathcal{U}(\Delta)$.

3.2 The dynamic qdb–controller

In this section the dynamic qdb–controller is defined following the steps listed in the paradigm described in the previous section. The practical stability properties of the corresponding closed loop system are then analyzed in Theorem 1.

Step 1- *Derivation of* $\mathcal{C}_{x(t)}$

The function $\boldsymbol{q} : \mathbb{R}^n \to \mathcal{Y}^n$ defined by $\boldsymbol{q}(z) := \big(q(z_1), \ldots, q(z_n)\big)$ induces a partition of $\mathbb{R}^n$ such that $\forall\, \boldsymbol{y} \in \mathcal{Y}^n_\star$ the closure of $\boldsymbol{q}^{-1}(\boldsymbol{y})$ is $\boldsymbol{y} + \mathcal{P}_{\boldsymbol{y}}$, where $\mathcal{P}_{\boldsymbol{y}} = \prod_{i=1}^n [-\frac{\lambda_{\boldsymbol{y}_i}}{2}\,;\, \frac{\lambda_{\boldsymbol{y}_i}}{2}]$. Let

$$S := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ Cb & 0 & \cdots & 0 \\ CAb & Cb & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}b & CA^{n-3}b & \cdots & Cb \end{pmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

Denote by $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ the vectors collecting respectively the last $n-1$ inputs and the last n outputs at time t ($t \geq n-1$), that is $\boldsymbol{u}(t) := \big(u(t-n+1), \ldots, u(t-1)\big)'$, and $\boldsymbol{y}(t) := \big(y(t-n+1), \ldots, y(t)\big)'$. Let $R := \big[A^{n-2}b\,|\cdots|Ab\,|b\big] \in \mathbb{R}^{n\times(n-1)}$ and $\mathcal{O} \in \mathbb{R}^{n\times n}$ be the observability

matrix (i.e., the matrix whose i^{th} row is CA^{i-1}): $\mathcal{O}$ is invertible by hypothesis.
By standard theory on observability it holds that

$$\boldsymbol{y}(t) = \boldsymbol{q}\left(\mathcal{O}\, x(t-n+1) + S\boldsymbol{u}(t)\right),$$

hence

$$x(t-n+1) \in \mathcal{O}^{-1}\Big(\boldsymbol{q}^{-1}\big(\boldsymbol{y}(t)\big) - S\boldsymbol{u}(t)\Big)$$

and

$$x(t) \in A^{n-1}\mathcal{O}^{-1}\Big(q^{-1}\big(\boldsymbol{y}(t)\big)\Big) - -A^{n-1}\mathcal{O}^{-1}S\boldsymbol{u}(t) + R\boldsymbol{u}(t).$$

If moreover $\boldsymbol{y}(t) \in \mathcal{Y}_\star^n$, as the closure of $\boldsymbol{q}^{-1}(\boldsymbol{y})$ is $\boldsymbol{y} + \mathcal{P}_{\boldsymbol{y}}$, then the current state belongs to the following bounded set:

$$x(t) \in \mathcal{C}_{x(t)} := A^{n-1}\mathcal{O}^{-1}\big(\mathcal{P}_{\boldsymbol{y}(t)}\big) + +A^{n-1}\mathcal{O}^{-1}\boldsymbol{y}(t) + (R - A^{n-1}\mathcal{O}^{-1}S)\,\boldsymbol{u}(t). \tag{10}$$

Step 2– State estimation

Let the map $\psi : \mathcal{Y}^n \times \mathcal{U}^{n-1} \to \mathbb{R}^n$ be defined by

$$\psi(\boldsymbol{y}, \boldsymbol{u}) := A^{n-1}\mathcal{O}^{-1}\boldsymbol{y} + (R - A^{n-1}\mathcal{O}^{-1}S)\boldsymbol{u}. \tag{11}$$

If $\boldsymbol{y}(t) \in \mathcal{Y}_\star^n$, then $\hat{x}(t) := \psi\big(\boldsymbol{y}(t), \boldsymbol{u}(t)\big)$ is the centroid of the parallelogram $\mathcal{C}_{x(t)}$.

Step 3– Control selection

The controller is defined by selecting the control action as if the current state was $\psi\big(\boldsymbol{y}(t), \boldsymbol{u}(t)\big)$. Naturally, such controller needs to be initialized for $t \le n-2$, we hence define the *dynamic qdb–controller* as follows: denote by $k(x)$ the state feedback qdb–controller and let [3]

$$u(t) := \begin{cases} 0 & \text{if } t \le n-2 \\ (k \circ \psi)\big(\boldsymbol{y}(t), \boldsymbol{u}(t)\big) & \text{if } t \ge n-1. \end{cases} \tag{12}$$

Practical stability analysis

Let us analyze the resulting closed–loop dynamics. For $t \le n-1$, by the controller form of A, it holds that $\forall x(0) \in Q_n(\Delta)$ and $\forall t \le n-1$, $x(t) \in Q_n(\Delta \|A^{n-1}\|_\infty)$. For $t \ge n$, let us determine an upper bound $H(\Delta)$ for $\operatorname{diam}_n(A\mathcal{C}_{x(t)})$: for any $\Delta > 0$, consider $\mathcal{Y}(\Delta) := \big(q \circ C\big)\big(Q_n(\Delta)\big)$. If $\mathcal{Y}(\Delta) \subseteq \mathcal{Y}_\star$, let $\Lambda_\Delta := \max_{y \in \mathcal{Y}(\Delta)} \lambda_y$ and $H(\Delta) := \operatorname{diam}_n\Big(A^n\mathcal{O}^{-1}\big(Q_n(\Lambda_\Delta)\big)\Big)$, else $H(\Delta) := +\infty$. Let $\Delta > 0$ be such that $\mathcal{Y}(\Delta) \subseteq \mathcal{Y}_\star$ and suppose that $\boldsymbol{y}(t) \in \mathcal{Y}(\Delta)^n$: since $\mathcal{C}_{x(t)}$ is a translation of the set $A^{n-1}\mathcal{O}^{-1}\big(\mathcal{P}_{\boldsymbol{y}(t)}\big)$ (see Eqn. (10)), and $\mathcal{P}_{\boldsymbol{y}(t)} \subseteq Q_n(\Lambda_\Delta)$, then

$$\operatorname{diam}_n(A\mathcal{C}_{x(t)}) = \operatorname{diam}_n\big(A^n\mathcal{O}^{-1}(\mathcal{P}_{\boldsymbol{y}(t)})\big) \le \le \operatorname{diam}_n\Big(A^n\mathcal{O}^{-1}\big(Q_n(\Lambda_\Delta)\big)\Big) = H(\Delta). \tag{13}$$

Theorem 1. Let $\Delta_1 > 0$ be such that

$$\begin{cases} m(\Delta_1) < -\dfrac{\Delta_1}{2}(\alpha - 1) & (14) \\ M(\Delta_1) > \dfrac{\Delta_1}{2}(\alpha - 1) & (15) \\ \rho(\Delta_1) + H(\Delta_1) < \Delta_1, & (16) \end{cases}$$

and $\Delta_0 := \frac{\Delta_1}{\|A^{n-1}\|_\infty}$. Consider the following algorithm:

- ∘ Input: $\boldsymbol{\Delta} := \Delta_1$
- • $h := 1$;
- • while $\big(\rho(\Delta_h) + H(\Delta_h) < \Delta_h\big)$ do
  $\big(\Delta_{h+1} := \rho(\Delta_h) + H(\Delta_h)$;
  $\boldsymbol{\Delta} := (\boldsymbol{\Delta}, \Delta_{h+1}) \in \mathbb{R}^{h+1}$; $h := h+1\big)$ (17)
- ∘ Output: $\boldsymbol{\Delta}$.

The output $\boldsymbol{\Delta} \in \mathbb{R}^f$ for some $f < +\infty$ and $\Delta_h > \Delta_{h+1}\ \forall h = 1, \ldots, f-1$. Moreover, let $k(x)$ be the state feedback qdb–controller with saturated inputs $\mathcal{U} = \mathcal{U}(\Delta_1)$, then the dynamic qdb–controller (12) is $\big(Q_n(\Delta_0), Q_n(\Delta_1), Q_n(\Delta_f)\big)$–stabilizing.

Proof. The proof is given in next Section 4. ■

Example 1. Consider the unstable system

$$\begin{cases} x^+ = \begin{pmatrix} 0 & 1 \\ 5/4 & 1/4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y = q(Cx), \end{cases}$$

where $C = (\,3/2\ \ 1/3\,)$, $u \in \mathcal{U} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24\}$ and the extremes of the intervals forming the output space partition induced by q are $\big\{\pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{25}{2}, \pm\frac{39}{2}\big\}$. According to the developed theory, let $\mathcal{Y} = \mathcal{Y}_\star \cup \{\pm y_{\mathrm{s}}\} = \{0, \pm 3, \pm 6, \pm 10, \pm 16, \pm y_{\mathrm{s}}\}$ (where $\mathcal{Y}_\star$ collects the middle points of the output quantization intervals and q takes the values $\pm y_{\mathrm{s}}$ for $|Cx| > \frac{39}{2}\big)$. The values of λ_y for $y \in \mathcal{Y}_\star$ are: $\lambda_0 = \lambda_{\pm 3} = \lambda_{\pm 6} = 3$, $\lambda_{\pm 10} = 5$ and $\lambda_{\pm 16} = 7$. The infinity norm of A is $\alpha = \frac{3}{2}$. By direct computations it holds that $\rho(\Delta) + H(\Delta) < \Delta \Leftrightarrow \rho(\Delta) + \frac{6}{7}\Lambda_\Delta < \Delta \Leftrightarrow \Delta \in \big]\frac{25}{7}\,;\,\frac{234}{11}\big] := \mathcal{I}$. Also, $M\big(\frac{234}{11}\big) = 24 > \frac{234}{11} \cdot \frac{\alpha - 1}{2} \simeq 5.32$ and inequalities (14–15) are satisfied $\forall \Delta \in \mathcal{I}$ (see Lemma 4 in Sec. 4), hence Theorem 1 guarantees that $\forall \Delta \in \mathcal{I}$, the dynamic qdb–controller with saturated inputs $\mathcal{U} = \mathcal{U}(\Delta)$ is $\big(Q_2(\frac{\Delta}{\alpha}), Q_2(\Delta), Q_2(\Delta_f)\big)$–stabilizing with $\Delta_f = \frac{25}{7}$ (see Fig. 2).

4. PROOF OF THEOREM 1

In order to prove Theorem 1 some preliminary results are needed. We will refer to the following notation: $\forall \Delta > 0$ such that $\rho(\Delta) < +\infty$, define the partition $\mathbb{R} = \mathcal{S}_{M(\Delta)} \cup \mathcal{N}_\Delta \cup \mathcal{S}_{m(\Delta)}$, where $\mathcal{S}_{M(\Delta)} :=]-\infty\,;\,-M(\Delta) - \frac{\rho(\Delta)}{2}[$, $\mathcal{N}_\Delta := [-M(\Delta) - \frac{\rho(\Delta)}{2}\,;\,-m(\Delta) + \frac{\rho(\Delta)}{2}]$ and $\mathcal{S}_{m(\Delta)} :=]-m(\Delta) + \frac{\rho(\Delta)}{2}\,;\,+\infty[$. Let $\mathcal{S}_\Delta := \mathcal{S}_{M(\Delta)} \cup \mathcal{S}_{m(\Delta)}$. Let us analyze the main properties of the map ν defining the state feedback qdb–controller.

Lemma 2. (Basic properties of ν). Let $\Delta > 0$:
ı) if inequalities (14–15) hold, then $\forall z \in \operatorname{Pr}_n\big(AQ_n(\Delta)\big)$, $\nu(z) \in \mathcal{U}(\Delta)$;
ıı) if $\rho(\Delta) < +\infty$ and $z \in \mathcal{N}_\Delta$, then $|z + \nu(-z)| \le \frac{\rho(\Delta)}{2}$;
ııı) assume $\rho(\Delta) < +\infty$ and let z be such that

[3] This controller can be modelled in the form of Eqn. (4) with $\mathcal{W} := \mathcal{Y}^n \times \mathcal{U}^{n-1}$: see in Appendix.

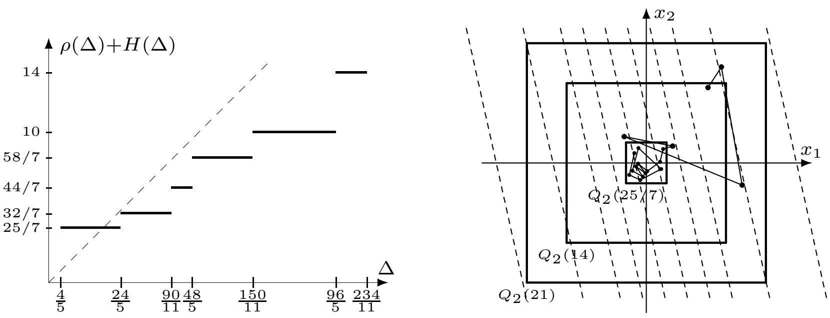

Fig. 2. *Left*: graph of $\rho(\Delta)+H(\Delta)$. *Right*: a trajectory generated by the dynamic qdb–controller for $\Delta_0 = 14$ and $x(0) = (5.42\ 6.60)$. Broken lines identify the state space partition induced by $q \circ C$.

$\nu(-z) \in \mathcal{U}(\Delta)$. If $z \in \mathcal{S}_{M(\Delta)}$, then $\nu(-z) = M(\Delta)$ and $|z+\nu(-z)| = -\big(z+\nu(-z)\big) > \frac{\rho(\Delta)}{2}$; if $z \in \mathcal{S}_{m(\Delta)}$, then $\nu(-z) = m(\Delta)$ and $z + \nu(-z) > \frac{\rho(\Delta)}{2}$.

The core of the proof of Theorem 1 is represented by the following result:

Lemma 3. (Main tool). Let $\Delta > 0$ be such that $\rho(\Delta) < +\infty$ and inequalities (14–15) hold. Assume that $x \in Q_n(\Delta)$ and $\hat{x} \in \mathbb{R}^n$ are so that $\big|\big(A(x-\hat{x})\big)_n\big| \le \frac{\mathscr{H}}{2}$ (for some $\mathscr{H} \ge 0$). Let $k(x)$ be the state feedback qdb–controller and suppose that $k(\hat{x}) \in \mathcal{U}(\Delta)$, then $x^+ = Ax + b\,k(\hat{x})$ is such that $|x_n^+| \le \max\big\{\frac{\rho(\Delta)+\mathscr{H}}{2}, \|x\|_\infty - \varphi(\Delta)\big\}$, where

$$\varphi(\Delta) := \min\big\{ M(\Delta) - \tfrac{\Delta}{2}(\alpha-1), \\ -\tfrac{\Delta}{2}(\alpha-1) - m(\Delta) \big\}. \qquad (18)$$

Proof. By definition of k, $x_n^+ = (Ax)_n + \nu\big(-(A\hat{x})_n\big)$. Notice also that, by Lemma 2.$\imath$, $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$. Three cases can occur:

I) Suppose that $(A\hat{x})_n \in \mathcal{N}_\Delta$, then

$$|x_n^+| = \Big|\big(A(x-\hat{x})\big)_n + (A\hat{x})_n + \nu\big(-(A\hat{x})_n\big)\Big| \le \\ \le \Big|\big(A(x-\hat{x})\big)_n\Big| + \big|(A\hat{x})_n + \nu\big(-(A\hat{x})_n\big)\big| \le \\ \le \frac{\mathscr{H}}{2} + \frac{\rho(\Delta)}{2},$$

where the last inequality follows by the hypothesis on $\hat{x}$ and by Lemma 2.$\imath\imath$.

II) Suppose that $(A\hat{x})_n \in \mathcal{S}_\Delta$ and x is such that $(Ax)_n \in \mathcal{N}_\Delta$. If $(A\hat{x})_n \in \mathcal{S}_{m(\Delta)}$, then $k(\hat{x}) = m(\Delta)$ thanks to Lemma 2.$\imath\imath\imath$ which can be applied because, by assumption, $k(\hat{x}) \in \mathcal{U}(\Delta)$. Hence, $x_n^+ = (Ax)_n + m(\Delta) \le (Ax)_n + \nu\big(-(Ax)_n\big) \le \frac{\rho(\Delta)}{2}$, where the first inequality holds because $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$ and the latter by Lemma 2.$\imath\imath$. Moreover, by Lemma 2.$\imath\imath\imath$, $(A\hat{x})_n + \nu\big(-(A\hat{x})_n\big) > \frac{\rho(\Delta)}{2}$, and by assumption $\big(A(x-\hat{x})\big)_n \ge -\frac{\mathscr{H}}{2}$, therefore $x_n^+ = \big(A(x-\hat{x})\big)_n + (A\hat{x})_n + \nu\big(-(A\hat{x})_n\big) > -\frac{\mathscr{H}}{2} + \frac{\rho(\Delta)}{2} > -\frac{\mathscr{H}+\rho(\Delta)}{2}$. To sum up, $|x_n^+| \le \frac{\rho(\Delta)+\mathscr{H}}{2}$. The case $(A\hat{x})_n \in \mathcal{S}_{M(\Delta)}$ is similar.

III) Suppose that $(A\hat{x})_n \in \mathcal{S}_\Delta$ and $(Ax)_n \in \mathcal{S}_\Delta$. If $(A\hat{x})_n \in \mathcal{S}_{m(\Delta)}$, we know by part II that $k(\hat{x}) = m(\Delta)$ and $x_n^+ > -\frac{\mathscr{H}+\rho(\Delta)}{2}$. Assume that $(Ax)_n \in \mathcal{S}_{M(\Delta)}$, since $\nu\big(-(Ax)_n\big) \in \mathcal{U}(\Delta)$, then $x_n^+ = (Ax)_n + m(\Delta) < (Ax)_n + M(\Delta) = (Ax)_n + \nu\big(-(Ax)_n\big) < -\frac{\rho(\Delta)}{2}$, where both the last equality and the last inequality hold by Lemma 2.$\imath\imath\imath$. Hence, $|x_n^+| < \frac{\mathscr{H}+\rho(\Delta)}{2}$. If instead $(Ax)_n \in \mathcal{S}_{m(\Delta)}$, then $|x_n^+| \le \|x\|_\infty - \varphi(\Delta)$. In fact: in this case $k(x) = k(\hat{x}) = m(\Delta)$ and, thanks to inequalities (14–15), we can write $m(\Delta) = -\frac{\Delta}{2}(\alpha-1) - \varphi(\Delta) - \theta$, with $\theta \ge 0$. Again by Lemma 2.$\imath\imath\imath$, $x_n^+ = (Ax)_n + m(\Delta) > \frac{\rho(\Delta)}{2} > 0$, hence $|x_n^+| = (Ax)_n + m(\Delta) \le \sum_i |\alpha_i|\,|x_i| + m(\Delta) \le \alpha \cdot \|x\|_\infty + m(\Delta) = \alpha \cdot \|x\|_\infty - \frac{\Delta}{2}(\alpha-1) - \varphi(\Delta) - \theta \le \|x\|_\infty - \varphi(\Delta)$ because $\big(\|x\|_\infty - \frac{\Delta}{2}\big)(\alpha-1) - \theta \le 0$. The case $(A\hat{x})_n \in \mathcal{S}_{M(\Delta)}$ is similar. ■

Remark 1. The motivation for assuming $k(\hat{x}) \in \mathcal{U}(\Delta)$ (which corresponds to the restriction of the state feedback qdb–controller to the saturated input set $\mathcal{U} = \mathcal{U}(\Delta_1)$ in Theorem 1) is that, by Lemma 1, if $k(\hat{x}) \notin \mathcal{U}(\Delta)$, then $x^+ \notin Q_n(\Delta)$.

Lemma 4. If $\Delta > 0$ satisfies inequalities (14–15), and Δ' is such that $\rho(\Delta) \le \Delta' < \Delta$, then Δ' satisfies inequalities (14–15).

Proof of Theorem 1. The sequence defined by the algorithm (17) is decreasing by construction, let us show that it is finite: first notice that, by definition, $H(\Delta)$ is a piecewise constant and non–decreasing function. As for $\rho(\Delta)$, $\exists\,\bar{\Delta} > 0$ such that $\rho(\Delta) = +\infty\ \forall\,\Delta < \bar{\Delta}$, whilst for $\Delta \ge \bar{\Delta}$, $\rho(\Delta)$ is piecewise constant and non–decreasing with Δ. If $\rho(\Delta_{h+1}) + H(\Delta_{h+1}) < \rho(\Delta_h) + H(\Delta_h)$, then $\rho(\Delta_{h+1}) < \rho(\Delta_h)$ or $H(\Delta_{h+1}) < H(\Delta_h)$: in the first case $\#\mathcal{U}(\Delta_{h+1}) < \#\mathcal{U}(\Delta_h)$, in the latter $\#\mathcal{Y}(\Delta_{h+1}) < \#\mathcal{Y}(\Delta_h)$. Therefore, $f \le \#\mathcal{U}(\Delta_1) + \#\mathcal{Y}(\Delta_1) < +\infty$ because $\mathcal{U}$ is a closed discrete set and the output space partition induced by q is supposed to be locally finite.

We have already noticed that $\forall\,x(0) \in Q_n(\Delta_0)$ and $\forall\,t \le n-1$, $x(t) \in Q_n(\Delta_1)$, therefore $\boldsymbol{y}(n-1) \in \mathcal{Y}(\Delta_1)^n$. Since $H(\Delta_1) < +\infty$ (see inequality (16)), then $\mathcal{Y}(\Delta_1) \subseteq \mathcal{Y}_\star$ and $\mathrm{diam}_n(A\mathcal{C}_{x(n-1)}) \le H(\Delta_1)$. Because $\hat{x}(n-1)$ is the centroid of the parallelogram $\mathcal{C}_{x(n-1)}$, then $\Big|\Big(A\big(x(n-1) - \hat{x}(n-1)\big)\Big)_n\Big| \le \frac{H(\Delta_1)}{2}$. Moreover, $k\big(\hat{x}(n-1)\big) \in \mathcal{U}(\Delta_1)$ by assumption, therefore

Lemma 3 guarantees that

$$|x_n(n)| \leq \max\left\{\tfrac{\Delta_2}{2} = \tfrac{H(\Delta_1)+\rho(\Delta_1)}{2}, \|x(n-1)\|_\infty - \varphi(\Delta_1)\right\} < \tfrac{\Delta_1}{2} \quad (19)$$

(where $\varphi(\Delta_1) > 0$ is defined in Eqn. (18)). Since $x^+ = (x_2, \ldots, x_n, x_n^+)$, then $x(n) \in Q_n(\Delta_1)$ and $\boldsymbol{y}(n) \in \mathcal{Y}(\Delta_1)^n$: therefore, the arguments which have allowed us to prove that $x(n) \in Q_n(\Delta_1)$ can be repeated so that $\forall\, t > n$, $x(t) \in Q_n(\Delta_1)$. Furthermore, because $\varphi(\Delta_1) > 0$ does not depend on t, $\exists\, t_1 > 0$ such that $\forall\, t \geq t_1$, $x(t) \in Q_n(\Delta_2)$. Therefore $\boldsymbol{y}(t_1+n-1) \in \mathcal{Y}(\Delta_2)^n$ and $\mathrm{diam}_n(A\mathcal{C}_{x(t_1+n-1)}) \leq H(\Delta_2)$. If $f > 2$, the arguments above can be iterated until t_{f-1} is found such that $\forall\, t \geq t_{f-1}$, $x(t) \in Q_n(\Delta_f)$: we have only to check that $\forall\, h = 2 \ldots, f-1$, Δ_h satisfies the hypotheses of Lemma 3.
Indeed, let $h \in \{2, \ldots, f-1\}$. By the algorithm (17), $\rho(\Delta_h) = \Delta_{h+1} - H(\Delta_h) < +\infty$. Inequalities (14–15) are satisfied by Δ_h (this guarantees that $\varphi(\Delta_h) > 0$), in fact: such inequalities hold for Δ_1 by assumption and, since $\rho(\Delta_{h-1}) \leq \Delta_h = \rho(\Delta_{h-1}) + H(\Delta_{h-1}) < \Delta_{h-1}$, then the result follows by recursive application of Lemma 4. As far as the remaining two hypotheses are concerned, that is $k(\hat{x}(t_{h-1}+n-1)) \in \mathcal{U}(\Delta_h)$ and $\left|\Big(A\big(x(t_{h-1}+n-1) - \hat{x}(t_{h-1}+n-1)\big)\Big)_n\right| \leq \frac{H(\Delta_h)}{2}$, we have already noticed that $\mathrm{diam}_n(A\mathcal{C}_{x(t_1+n-1)}) \leq H(\Delta_2)$ implies that $\left|\Big(A\big(x(t_1+n-1) - \hat{x}(t_1+n-1)\big)\Big)_n\right| \leq \frac{H(\Delta_2)}{2}$. Moreover, since $\forall\, t \geq t_1$, $x(t) \in Q_n(\Delta_2)$, then $\forall\, t \geq t_1$, $k(\hat{x}(t)) \in \mathcal{U}(\Delta_2)$ thanks to Lemma 1. We then conclude by a recursive argument. ■

5. CONCLUSION

We have introduced a novel technique for the stabilizability analysis of quantized SISO systems. The results hold under very general hypotheses and are of direct applicability. Interesting questions are open for future investigations, especially in the framework of sampled continuous–time systems under communication constraints.

REFERENCES

J. Baillieul, *Feedback Designs in Information–Based Control*, Proc. of the Workshop on Stochastic Theory and Control, Kansas, pages: 35–57. Springer-Verlag. 2001.

A. Bicchi, A. Marigo and B. Piccoli, *On the Reachability of Quantized Control Systems*, IEEE Trans. Autom. Control, 47(4); pages: 546–563. 2002.

R. Brockett and D. Liberzon, *Quantized feedback stabilization of linear systems*, IEEE Trans. Autom. Control, 45(7); pages: 1279–1289. 2000.

D.F. Delchamps, *Stabilizing a Linear System with Quantized State Feedback*, IEEE Trans. Autom. Control, 35(8); pages: 916–924. 1990.

N. Elia and S. Mitter, *Stabilization of Linear Systems With Limited Information*, IEEE Trans. Autom. Control, 46(9); pages: 1384–1400. 2001.

F. Fagnani and S. Zampieri *Quantized stabilization of linear systems: complexity versus performance, IEEE Trans. Autom. Control, 49 (9) special issue on "Networked Control Systems"*, pages: 1534–1548. 2004.

H. Ishii and B.A. Francis, *Stabilizing a Linear Systems by Switching Control With Duel Time*, IEEE Trans. Autom. Control, 47(12); pages: 1962–1973. 2002.

D. Liberzon, *Hybrid feedback stabilization of systems with quantized signals*, Automatica, 39; pages: 1543–1554. 2003.

G.N. Nair and R.J. Evans, *Stabilizability of stochastic linear systems with finite feedback data rates*, SIAM Journal on Contr. Optim. Vol. 43 (2), Pages: 413–436. 2004.

B. Picasso, F. Gouaisbaut and A. Bicchi, *Construction of invariant and attractive sets for quantized–input linear systems*, Proc. of the 41st IEEE Conference on Decision and Control "CDC 2002" pages: 824–829. 2002.

B. Picasso and A. Bicchi, *Stabilization of LTI Systems with Quantized State – Quantized Input Static Feedback*, In A. Pnueli and O. Maler, editors, Hybrid Systems: Computation and Control, volume LNCS 2623 of *Lecture Notes in Computer Science*, pages: 405–416. Springer–Verlag, Heidelberg, Germany. 2003.

D.E. Quevedo, G.C. Goodwin, J.A. De Doná, *Finite constraint set receding horizon control*, Int. J. of Robust and Nonlinear Control, 14(4); pages: 355–377. 2004.

S.C. Tatikonda and S. Mitter, *Control Under Communication Constraints*, IEEE Trans. Autom. Control, 49(7); pages: 1056–1068.. 2004.

W. Wong and R. Brockett, *Systems with finite communication bandwidth constraints - part II: Stabilization with limited information feedback*, IEEE Trans. Autom. Control, 44(5); pages: 1049–1053. 1999.

6. APPENDIX

Let us represent the dynamic qdb–controller defined in Eqn. (12) in the form of Eqn. (4). Let

$$\mathcal{W} := \mathcal{Y}^n \times \mathcal{U}^{n-1},$$

the elements $w \in \mathcal{W}$ are denoted either by $w = (\boldsymbol{y}, \boldsymbol{u})$ or $w = (w_1, \cdots, w_{2n-1})$. Let

$$\tilde{\phi}:\ \mathcal{W} \times (\mathbb{N} \cup \{-1\}) \to \mathcal{U}$$

be defined by

$$\tilde{\phi}\big((\boldsymbol{y}, \boldsymbol{u}), t\big) = \begin{cases} 0 & \text{if} \quad t \leq n-2 \\ (k \circ \psi)(\boldsymbol{y}, \boldsymbol{u}) & \text{if} \quad t \geq n-1, \end{cases}$$

and

$$\gamma:\ \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{W}$$

be defined by

$$\gamma(w, y, t) = \big(y, w_1, \cdots, w_{n-1}, \tilde{\phi}(w, t-1), w_{n+1}, \cdots, w_{2n-2}\big).$$

Finally,

$$\begin{aligned} \phi:\ & \mathcal{W} \times \mathcal{Y} \times \mathbb{N} \to \mathcal{U} \\ & (w, y, t) \quad \mapsto \tilde{\phi}\big(\gamma(w, y, t), t\big). \end{aligned}$$