# STABILIZING DYNAMIC CONTROLLER OF SWITCHED LINEAR SYSTEMS

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Abstract: This paper is devoted to the problem of design a dynamic controller of switched linear systems. In the rst part of this paper, we give some remarks about the in uences of switching signal on the asymptotic stability of switched systems. We give a practical example to generate a large class of switching signals. The second part is devoted to the switched dynamic controller design, based on a common Lyapunov function approach. A su cient condition are formulated as an LMI problem for the switched controller design under arbitrary switching. A stabilizing switched controller with regional pole placements is also formulated as a convex problem, an LMI approach is used to derive the switched dynamic controller with performance limitations. 

Keywords: Switched linear systems, Common Lyapunov function, Dynamic controller, LMI, Pole placements.

## 1. INTRODUCTION

Switched systems are hybrid dynamical systems consisting of a family of continuous-time subsystems and a switching rule that orchestrates the switching among them. The primary motivation for studying switched systems in control theory comes partly from the fact that switched systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft, tra c control, biology, network and many other elds [8], [9]. Stability of switched systems is not systematic and we can meet certain strange phenomena, even when all the subsystems are asymptotically stable. For example, the switched system can be unstable under certain switching signals [2], [11]. Thus, the stability of switched systems depends not only on the dynamics of each subsystem but also on the behavior of the switching signals.

The common Lyapunov approach is one of the principal methods to study stability and design a controller of switched systems. This approach is based on the existence of a common quadratic Lyapunov function for all subsystems of the switched system. There have been various attempts to derive conditions for the existence of a common quadratic Lyapunov function. Under the asymptotic stability of each subsystem, a common Lyapunov function exists when the subsystems

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matrices are pairwise commutative [13]. In [1], [10], the authors proposed a generalization of the commutativity notion, based on the solvability of the Lie algebra generated by the subsystems state matrices, i.e., state matrices are upper-triangularizable in the same reference frame. Note that many other results are based on the multiple Lyapunov (or like Lyapunov) functions approach [2], [5], [11].
In the context of switched systems with linear continuous-time subsystems, the issues of stabilization and control have been studied in many works [5], [11], [12], [14], [15], [16] and [17].
In this paper, we will focus on the study of a dynamic stabilization of switched linear systems. The paper is then organized as follows: In the next Section, we present some remarks on the in uence of the switching signal on the stability of switched systems. In section III, a switched dynamic controller is studied based on an LMI approach. A dynamic switched controller with regional pole placements is investigated in section IV. Su cient conditions for the existence of a dynamic switched controller with performance limitations are then given.

## 2. STABILITY UNDER SOME SWITCHING SIGNALS

In this section, we are interested in switched linear systems of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) \qquad (1)$$

where $A_\sigma \in R^{n \quad n}$, $\quad \in \mathcal{Q} \triangleq \{1, ..., N\}$, $x \in R^n$ and $\quad (t) : [0, \ \infty) \rightarrow \mathcal{Q}$ is a piecewise constant switching signal.
As mentioned in the introduction, the stability of switched systems are strongly related to the switching signal behavior, except in the case when all subsystems share the same Lyapunov function. Hereafter we will analyze the switched systems by using di erent class of switching signals. For this, we give four basic classes of switching signals, each class is characterized by some behavior, which can have a signi cant in uence on the asymptotic stability of the switched system. These classes can be summarized by:
*Class 0.* This class contains all switching signals with a nite number of switchings. We denote this class by $\mathcal{P}_0$. Theoretically for this class, if all the subsystems are asymptotically stable, then at a certain nite time, the switched system evolves as only one subsystem. Therefore the switched system is asymptotically stable.
The remaining classes considered in this paper are those containing an in nite number of switchings.
*Class 1.* This class de nes all switching signals for which any consecutive switching times $t_k$ and $t_{k+1}$

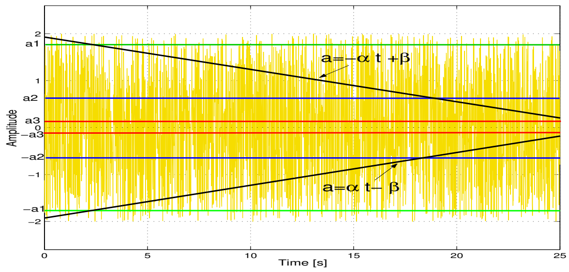


Fig. 1. The random signal used for the generation of $\sigma(t)$

are separated by some *dwell-time* $\ _D$. Denote by $\mathcal{P}_1$ the set of this class,

$$\mathcal{P}_1 = \{ \ \in \mathcal{Q} : \ t_{k+1} \quad t_k \quad \ _D\} \qquad (2)$$

*Class 2.* This class can be de ned as

$$\mathcal{P}_2 = \{ \ \in \mathcal{Q} : \ \exists \quad > 0 : \forall T > 0, \ \exists\, i > 0 \text{ such that } t_{i+1} \qquad t_i \quad T\} \qquad (3)$$

This class includes the class for which the number of discontinuities of in any interval of time is bounded [7]

$$N_\sigma(t, \ ) \quad N_0 + \frac{t}{\ _D}, \quad \forall t \qquad 0 \qquad (4)$$

where $N_\sigma(t, \ )$ is the number of discontinuities of in the open interval $( \ , t)$, $\ _D$ is called the *average dwell time* and $N_0$ the *chatter bound*.
*Class 3.* This class is the class of all switching signals who dot not belong to $\mathcal{P}_0$, $\mathcal{P}_1$ or $\mathcal{P}_2$. This class contains the chattering and Zeno switching signals. We can write that

$$\mathcal{P}_1 \quad \mathcal{P}_2 \quad \mathcal{P}_3 \qquad (5)$$

Now, we give a procedure to generate each class of switching signals given above. Consider a random signal, with some sample time $\Delta T$, and amplitude $a \in [\ \ a_m, a_m]$. Fig.1 shows an example of such signal. The three classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ can be generated for $\mathcal{Q} = \{1, 2\}$ as

$$\begin{aligned} \mathcal{P}_i = \{ \ \in \mathcal{Q} : \\ = 1 \text{ if } [a \qquad a_i] \vee [( \ \ a_i < a < a_i) \wedge ( \qquad = 1)] \\ = 2 \text{ if } [a \quad \ a_i] \vee [( \ \ a_i < a < a_i) \wedge ( \qquad = 2)]\} \end{aligned} \qquad (6)$$

where $i = 1, 2, 3$ and $a_3 < a_2 < a_1 < a_m$ are the parameters characterizing each class (see Fig.1), and is the previous value of . The same procedure can be used for $\mathcal{Q} = \{1, ..., N\}$, $N > 2$. We can also generate more complex classes by this procedure. Like a Zeno signal can be generated by setting

$$a_i = \begin{cases} \quad t + \quad \text{if } t \quad - \\ 0 \qquad \text{if } t > - \end{cases}, \quad , \quad > 0 \qquad (7)$$

for an analytic example see [7].
Now, we give some remarks about the stability

under these classes. For this, we limit our remarks to the case of a Lyapunov function equal to $x^T x$. Rewrite each matrix $A_\sigma$ as $A_\sigma = S_\sigma + M_\sigma$, where $S_\sigma$ and $M_\sigma$ are the symmetric and the skew-symmetric part of $A_\sigma$ respectively, given by $S_\sigma = \frac{1}{2}(A_\sigma + A_\sigma^T)$ and $M_\sigma = \frac{1}{2}(A_\sigma \quad A_\sigma^T)$. The following theorem is a su cient condition for stability of the switched system (1).

**Theorem 1.** *If the following conditions*

**i)** *$S_\sigma$ is semi negative for all $\in \mathcal{Q}$,*
**ii)** *$(S_\sigma, A_\sigma)$ is observable for all $\in \mathcal{Q}$,*

*hold, then the switched system (1) is globally uniformly stable under arbitrary switching. Moreover,*

**iii)** *If $(t) \in \mathcal{P}_2$, then the switched system is globally uniformly asymptotically stable.*

*Proof.* The proof can be found in [4]. ∎

**Corollary 1.**
1) *If the matrices $A_\sigma$ are Hurwitz, then the pair $(S_\sigma, A_\sigma)$ is observable, therefore condition ii) of theorem 1 is ful lled.*
2) *If the symmetric part $S_\sigma$ is negative de nite for all $\in \mathcal{Q}$, then the switched system is globally uniformly asymptotically stable under arbitrary switching, and condition iii) of theorem 1 can be relaxed.*
3) *If the linear algebra generated by $\{A_\sigma, A_\sigma^T\}$ is solvable and $A_\sigma$ are Hurwitz, then $S_\sigma$ are negative de nite, therefore the switched system (1) is globally uniformly asymptotically stable and condition iii) of theorem 1 can be relaxed.*

**Example 1.** Consider the switched system $\dot{x} = A_{\sigma(t)} x$, with

$$A_1 = \begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix} , \qquad A_2 = \begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix} \tag{8}$$

The symmetric part of the matrices $A_\sigma$, $= 1, 2$ are given by

$$S_1 = S_2 = \begin{matrix} 0 & 0 \\ 0 & 2 \end{matrix} \tag{9}$$

which are semi-negative de nite. Condition $i)$ and $ii)$ of theorem 1 are satis ed, then the switched system is globally uniformly stable. Now we present some situations in which the switched system can be stable or asymptotically stable depending on condition $iii)$ of theorem 1.

***a)*** We use a switching signal of class 3. As stated in theorem 1, in Fig.2, the switched system is stable but does not converge to zero (see [7] for an analytic proof).
***b)*** We use a switching signal $(t) \in \mathcal{P}_2$. In Fig.3 the switched system is asymptotically stable as mentioned in $iii)$ of theorem 1.
***c)*** In the third case, we use a switching signal of class 1. Fig.4 shows that the switched system is asymptotically stable. However the convergence is faster than situation $b)$.

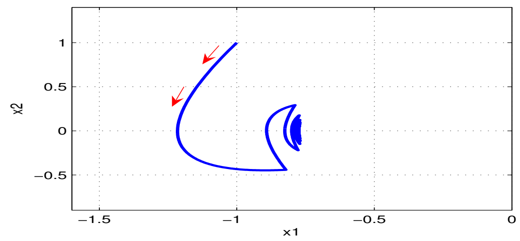


Fig. 2. Trajectory of the switched system under a switching signal of class 3, $\sigma \in \mathcal{P}_3$.

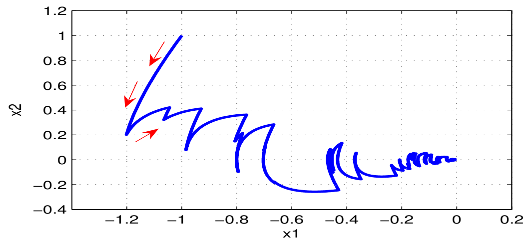


Fig. 3. Trajectory of the switched system under a switching signal of class 2, $\sigma \in \mathcal{P}_2$.

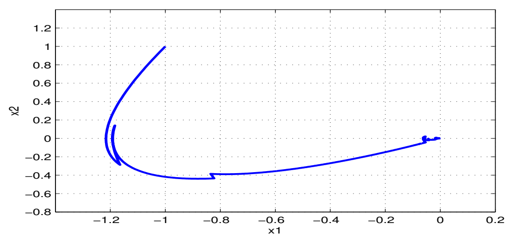


Fig. 4. Trajectory of the switched system under a switching signal of class 1, $\sigma \in \mathcal{P}_1$.

This example con rms that the asymptotic stability depends critically on the class of switching signals considered.
In the sequel, the stability based on the common Lyapunov function will be used to design a stabilizing dynamic controller of switched linear systems.

## 3. SWITCHED DYNAMIC CONTROLLER

Consider the continuous-time switched linear system described by

$$\begin{aligned} x(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t), \\ y(t) &= C_{\sigma(t)} x(t), \\ (t) &: R^+ \to \mathcal{Q}, \quad \mathcal{Q} := \{1, ..., N\}. \end{aligned} \tag{10}$$

where $x(t) \in R^n$ is the continuous state, $u(t) \in R^{n_u}$ is the control input, $y(t) \in R^{n_y}$ is the output, $(t)$ is the switching signal, and $A_\sigma \in R^{n \quad n}, B_\sigma \in R^{n \quad n_u}, C_\sigma \in R^{n_y \quad n}$ are the subsystem matrices. We assume that the switching signal $(t)$ is available in real time. In this section, we search a dynamic stabilizing controller having the following state representation

$$\begin{cases} \dot{x}_c(t) = \mathscr{A}_{\sigma(t)}x_c(t) + \mathscr{B}_{\sigma(t)}y(t) \\ u(t) = \mathscr{C}_{\sigma(t)}x_c(t) + \mathscr{D}_{\sigma(t)}y(t) \\ \ (t) \in \mathcal{Q},\ x_c(0) = x_0 \end{cases} \tag{11}$$

where $x_c(t) \in R^{n_c}$ is the state of the controller, $\mathscr{A}_\sigma \in R^{n_c \ n_c}$, $\mathscr{B}_\sigma \in R^{n_c \ n_y}$, $\mathscr{C}_\sigma \in R^{n_u \ n_c}$ and $\mathscr{D}_\sigma \in R^{n_u \ n_y}$, $\ \in \mathcal{Q}$ are the state parameters of the switched dynamic controller. A su cient conditions for the existence of the stabilizing controller of the form (11) are provided by the following theorem.

**Theorem 2.** *If there exist matrices* $P_1, Q_1$, $(F^1, F^2, F^3, F^4)_\sigma$, $\ \in \mathcal{Q}$, *and matrices* $P_2, P_3, Q_2$ *such that the following LMIs/equation*

$$\begin{bmatrix} A_\sigma P_1 + P_1 A_\sigma^T + B_\sigma F_\sigma^2 + F_\sigma^{2T} B_\sigma^T & A_\sigma + B_\sigma F_\sigma^1 C_\sigma + F_\sigma^{4T} \\ A_\sigma^T + C_\sigma^T F_\sigma^{1T} B_\sigma^T + F_\sigma^4 & Q_1 A_\sigma + A_\sigma^T Q_1 + F_\sigma^3 C_\sigma + C_\sigma^T F_\sigma^{3T} \end{bmatrix} < 0, \tag{12}$$

$$Q_1 P_1 + P_2 Q_2^T = I, \tag{13}$$

*and*

$$\begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} > 0 \tag{14}$$

*hold, then the dynamic controller (11), with the parameters* $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})_\sigma$ *given by*

$$\begin{bmatrix} \mathscr{A}_\sigma & \mathscr{B}_\sigma \\ \mathscr{C}_\sigma & \mathscr{D}_\sigma \end{bmatrix} = \begin{bmatrix} Q_2 & Q_1 B_\sigma \\ 0 & I \end{bmatrix}^{-1} \left( \begin{bmatrix} F_\sigma^4 & F_\sigma^3 \\ F_\sigma^2 & F_\sigma^1 \end{bmatrix} + \begin{bmatrix} Q_1 A_\sigma P_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} P_2 & 0 \\ C_\sigma P_1 & I \end{bmatrix}^{-1} \tag{15}$$

*ensures the asymptotic stability of the closed loop switched system under arbitrary switching, and all closed loop subsystems share the common Lyapunov function* $V(\tilde{x}) = \tilde{x}^T P^{-1} \tilde{x}$, $P := \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix}$.

*Proof.* The closed loop switched system is given by

$$\dot{\tilde{x}}(t) = A_{\sigma(t)}^{cl} \tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0 \tag{16}$$

with

$$A_\sigma^{cl} := \begin{bmatrix} A_\sigma + B_\sigma \mathscr{D}_\sigma C_\sigma & B_\sigma \mathscr{C}_\sigma \\ \mathscr{B}_\sigma C_\sigma & \mathscr{A}_\sigma \end{bmatrix}; \ \tilde{x} := \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{17}$$

a su cient condition for quadratic stability of (16) under arbitrary switching is the existence of a common Lyapunov function $V(\tilde{x}) = \tilde{x}^T W \tilde{x}$, such that

$$W = W^T > 0, \ \ W A_\sigma^{cl} + A_\sigma^{clT} W < 0, \quad \ \in \mathcal{Q} \tag{18}$$

Multiplying the second inequality in (18) on the left and right by $W^{-1}$, and de ning a new variable $P = W^{-1}$, we may rewrite (18) as

$$P = P^T > 0, \ \ \mathcal{V}_\sigma := A_\sigma^{cl} P + P A_\sigma^{clT} < 0, \quad \ \in \mathcal{Q} \tag{19}$$

This dual inequality is an equivalent condition for quadratic stability. Now, consider that there exists a symmetric matrix $P > 0$, such that (19) holds. De ne $P$ and its inverse as

$$P := \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix}, \ \ P^{-1} := \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \tag{20}$$

The development of (19) gives

$$\mathcal{V}_\sigma = \begin{bmatrix} \mathcal{V}_\sigma(1,1) & \mathcal{V}_\sigma(1,2) \\ \mathcal{V}_\sigma(2,1) & \mathcal{V}_\sigma(2,2) \end{bmatrix} < 0 \tag{21}$$

where

$$\begin{aligned} \mathcal{V}_\sigma(1,1) &= A_\sigma P_1 + P_1 A_\sigma^T + P_1 C_\sigma^T \mathscr{D}_\sigma^T B_\sigma^T \\ &\quad + P_2^T \mathscr{C}_\sigma^T B_\sigma^T + B_\sigma \mathscr{D}_\sigma C_\sigma P_1 + B_\sigma \mathscr{C}_\sigma P_2 \\ \mathcal{V}_\sigma(1,2) &= A_\sigma P_2^T + B_\sigma \mathscr{D}_\sigma C_\sigma P_2^T + B_\sigma \mathscr{C}_\sigma P_3 \\ &\quad + P_1 \mathscr{C}_\sigma^T B_\sigma^T + P_2^T \mathscr{A}_\sigma^T \\ \mathcal{V}_\sigma(2,1) &= \mathscr{B}_\sigma C_\sigma P_1 + \mathscr{A}_\sigma P_2 + P_2 A_\sigma^T \\ &\quad + P_2 C_\sigma^T \mathscr{D}_\sigma^T B_\sigma^T + P_3 \mathscr{C}_\sigma^T B_\sigma^T \\ \mathcal{V}_\sigma(2,2) &= \mathscr{B}_\sigma C_\sigma P_2^T + \mathscr{A}_\sigma P_3 + P_2 \mathscr{C}_\sigma^T B_\sigma^T \\ &\quad + P_3 \mathscr{A}_\sigma^T \end{aligned}$$

We show well that $\mathcal{V}_\sigma$ with the parameters de ned in the above equations, is not a ne in the variables $P_1, P_2, P_3$ and $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})_\sigma$, then $\mathcal{V}_\sigma < 0$ is not a convex problem. For this we must transform (21) to an a ne form in synthesis variables. First, we apply to (21) a transformation, (21) is equivalent to

$$\mathcal{V}_\sigma < 0 \ \Leftrightarrow \ \ ^T \mathcal{V}_\sigma \ \ < 0 \tag{22}$$

where is a matrix transformation to be determined, such that

$$^T A_\sigma^{cl} P \ \ + \ \ ^T P A_\sigma^{clT} \ \ < 0$$

and $\ ^T A_\sigma^{cl} P \ $, $\ ^T P A_\sigma^{clT} \ $ = indirectly a ne ($P_1$, $P_2, P_3, \mathscr{A}_\sigma, \mathscr{B}_\sigma, \mathscr{C}_\sigma, \mathscr{D}_\sigma$)

De ne the matrix transformation as

$$:= \begin{bmatrix} I & Q_1 \\ 0 & Q_2^T \end{bmatrix} \ \Rightarrow \ P \ \ = \begin{bmatrix} P_1 & I \\ P_2 & 0 \end{bmatrix} \tag{23}$$

where $I$ is the identity matrix.

De ne the following change of variables

$$F_\sigma^1 := \mathscr{D}_\sigma, \tag{24}$$

$$F_\sigma^2 := \mathscr{C}_\sigma P_2 + \mathscr{D}_\sigma C_\sigma P_1, \tag{25}$$

$$F_\sigma^3 := Q_2 \mathscr{B}_\sigma + Q_1 B_\sigma \mathscr{D}_\sigma, \tag{26}$$

$$\begin{aligned} F_\sigma^4 := & \ Q_2 \mathscr{A}_\sigma P_2 + Q_1 A_\sigma P_1 + Q_2 \mathscr{B}_\sigma C_\sigma P_1 \\ & + Q_1 B_\sigma \mathscr{C}_\sigma P_2 + Q_1 B_\sigma \mathscr{D}_\sigma C_\sigma P_1 \end{aligned} \tag{27}$$

by this change of variables, $\ ^T A_\sigma^{cl} P \ $, $\ ^T P A_\sigma^{clT}$ become

$$^T A_\sigma^{cl} P \ \ = \begin{bmatrix} A_\sigma P_1 + B_\sigma F_\sigma^2 & A_\sigma + B_\sigma F_\sigma^1 C_\sigma \\ F_\sigma^4 & Q_1 A_\sigma + F_\sigma^3 C_\sigma \end{bmatrix}$$

$$^T P A_\sigma^{clT} = \begin{bmatrix} P_1 A_\sigma^T + F_\sigma^{2T} B_\sigma^T & F_\sigma^{4T} \\ A_\sigma^T + C_\sigma^T F_\sigma^{1T} B_\sigma^T & A_\sigma^T Q_1 + C_\sigma^T F_\sigma^{3T} \end{bmatrix}$$

which are a ne in the variables $F_\sigma^1$, $F_\sigma^2$, $F_\sigma^3$, $F_\sigma^4$, $P_1$, $Q_1$ and the nonlinear inequality of the synthesis variables $P_1$,$P_2$,$P_3$ and $(\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D})_\sigma$ can be transformed to an LMI form, with $F_\sigma^1$, $F_\sigma^2$, $F_\sigma^3$, $F_\sigma^4$, $P_1$ and $Q_1$ as variables of the LMI. Then the problem which consists to nd $\mathscr{A}_\sigma,\mathscr{B}_\sigma,\mathscr{C}_\sigma,\mathscr{D}_\sigma$ is transformed to an LMI problem, which consists to compute the variables $F_\sigma^1$, $F_\sigma^2$, $F_\sigma^3$, $F_\sigma^4$, $P_1$, $Q_1$. These LMIs are given by

$$\begin{bmatrix} A_\sigma P_1 + P_1 A_\sigma^T & A_\sigma + B_\sigma F_\sigma^1 C_\sigma \\ +B_\sigma F_\sigma^2 + F_\sigma^{2T} B_\sigma^T & +F_\sigma^{4T} \\ A_\sigma^T + C_\sigma^T F_\sigma^{1T} B_\sigma^T & Q_1 A_\sigma + A_\sigma^T Q_1 \\ +F_\sigma^4 & +F_\sigma^3 C_\sigma + C_\sigma^T F_\sigma^{3T} \end{bmatrix} < 0, \tag{28}$$

the change of variables (24)-(27) can be putted in the compact form

$$\begin{matrix} F_\sigma^4 & F_\sigma^3 \\ F_\sigma^2 & F_\sigma^1 \end{matrix} = \begin{matrix} Q_2 & Q_1 B_\sigma \\ 0 & I \end{matrix} \quad \begin{matrix} \mathscr{A}_\sigma & \mathscr{B}_\sigma \\ \mathscr{C}_\sigma & \mathscr{D}_\sigma \end{matrix} \quad \begin{matrix} P_2 & 0 \\ C_\sigma P_1 & I \end{matrix}$$
$$+ \begin{matrix} Q_1 \\ 0 \end{matrix} \; A_\sigma \; \begin{matrix} P_1 & 0 \end{matrix} \tag{29}$$

LMI (28) do not depend on $P_2$, $Q_2$, but these two variables appear in (29), then these variables are necessary to compute the switched controller parameters $(\mathscr{A}_\sigma,\mathscr{B}_\sigma,\mathscr{C}_\sigma,\mathscr{D}_\sigma)$. For this the variables $P_2$ and $Q_2$ should be computed a posteriori. Given matrices $Q_1 > 0$ and $P_1 > 0$, the matrices $P_2 > 0$ and $Q_2 > 0$ must be computed from 13, and nally the dynamic switched controller gains are computed using (15). ∎

**Remark 1.** If $\dim x = \dim x_c$, i.e., the controller has the same order of the plant, then $P_2$, $Q_2$ are square and nonsingular matrices. Thus the controller parameters can be calculated by the equation

$$\begin{matrix} \mathscr{A}_\sigma & \mathscr{B}_\sigma \\ \mathscr{C}_\sigma & \mathscr{D}_\sigma \end{matrix} = \begin{matrix} Q_2^{\ 1} & Q_2^{\ 1} Q_1 B_\sigma \\ 0 & I \end{matrix}$$
$$. \quad \begin{matrix} F_\sigma^4 & Q_2 A_\sigma P_1 & F_\sigma^3 \\ & F_\sigma^2 & F_\sigma^1 \end{matrix} \quad \begin{matrix} P_2^{\ 1} & 0 \\ C_\sigma P_1 P_2^{\ 1} & I \end{matrix}$$

**Corollary 2.** *In the case of a switched static controller of the form $u(t) = K_\sigma(t)x(t)$, where $K_\sigma$, for $\in \mathcal{Q}$ are the controller gains. Assume that $x(t)$ is available, the conditions in theorem 1 are reduced to the LMI with $X_\sigma$ as variables of synthesis*

$$PA_\sigma^T + A_\sigma P + B_\sigma X_\sigma + X_\sigma^T B_\sigma^T < 0, \quad \in \mathcal{Q} \tag{30}$$

*where $P$ is a symmetric positive de nite matrix. The switched static controller gains $K_\sigma$, for $\in \mathcal{Q}$ are given by*

$$K_\sigma = X_\sigma P^{\ 1}, \quad \in \mathcal{Q} \tag{31}$$

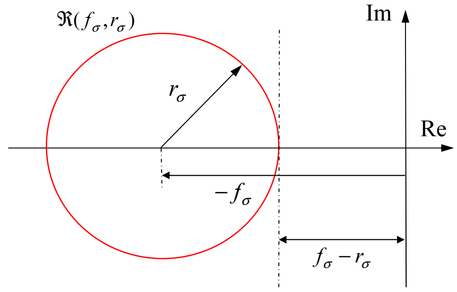


Fig. 5. Circular region $\mathscr{R}(f_\sigma, r_\sigma)$

*and the closed loop switched system stability is ensured by the common Lyapunov function $V(x) = x^T P^{\ 1} x$.*

## 4. DYNAMIC SWITCHED CONTROLLER WITH REGIONAL POLE PLACEMENTS

The goal of this section is to combine the rst result of theorem 2 and pole placements, our objective consists then to determine a stabilizing dynamic switched controller of the switched system with some constraint speci cations on the poles of the closed loop system, which is de ned previously as

$$A_\sigma^{cl} := \begin{matrix} A_\sigma + B_\sigma \mathscr{D}_\sigma C_\sigma & B_\sigma \mathscr{C}_\sigma \\ \mathscr{B}_\sigma C_\sigma & \mathscr{A}_\sigma \end{matrix} \tag{32}$$

The rst objective is then to place the poles of the closed loop switched system inside a circular region $\mathscr{R}(f_\sigma, r_\sigma)$ in the complex plane, with a center at $(\ f_\sigma, 0)$, radius $r_\sigma \quad f_\sigma, \quad \in \mathcal{Q}$, and distance $(f_\sigma \quad r_\sigma)$ from the imaginary axis. Fig.5 shows the circular region $\mathscr{R}(f_\sigma, r_\sigma)$ for pole location. The full problem can then be formulated as

$$\text{nd} \quad \begin{array}{c|c} \mathscr{A}_\sigma & \mathscr{B}_\sigma \\ \hline \mathscr{C}_\sigma & \mathscr{D}_\sigma \end{array}$$

subject to:

eigenvalues$(A_\sigma^{cl}) \in \mathscr{R}(f_\sigma, r_\sigma)$

$\dot{\tilde{x}}(t) = A_{\sigma(t)}^{cl} \tilde{x}(t)$ is asym-stable $\forall \quad \in \mathcal{Q}$

The asymptotic stability of $A^{cl}$ is ensured by the existence of a common Lyapunov function $V(\tilde{x}) = \tilde{x}^T W \tilde{x}$, $W = W^T > 0$. The problem concerning the eigenvalues of $A^{cl}$ is resolved as : In [6] a necessary and su cient condition assuring that all eigenvalues of a given matrix $A_\sigma^{cl}$ lie inside a circular region $\mathscr{R}(f_\sigma, r_\sigma)$ is provided by the existence of a symmetric positive de nite matrix $P$ such that

$$[A_\sigma^{cl} + (f_\sigma \quad r_\sigma)I]P + P[A_\sigma^{cl} + (f_\sigma \quad r_\sigma)I]^T$$
$$+\frac{1}{r_\sigma}[A_\sigma^{cl} + (f_\sigma \quad r_\sigma)I]P[A^{cl} + (f_\sigma \quad r_\sigma)I]^T < 0 \tag{33}$$

**Theorem 3.** *If there exist matrices $P_1, Q_1$, $(F^1, F^2, F^3, F^4)_\sigma$, $\;\in \mathcal{Q}$, and matrices $P_2, P_3, Q_2$ such that the following LMIs/equation*

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} \\ \ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} \\ \ell_{13}^T & \ell_{23}^T & r_\sigma P_1 & r_\sigma I \\ \ell_{14}^T & \ell_{24}^T & r_\sigma I & r_\sigma Q_1 \end{bmatrix} < 0, \qquad (34)$$

$$Q_1 P_1 + P_2 Q_2^T = I, \qquad (35)$$

*hold, where*

$$\begin{aligned}
\ell_{11} &= A_\sigma P_1 + B_\sigma F_\sigma^2 + P_1 A_\sigma^T + F_\sigma^{2T} B_\sigma^T \\
&\quad +2(f_\sigma \quad r_\sigma)P_1 \\
\ell_{12} &= A_\sigma + B_\sigma F_\sigma^1 C_\sigma + F_\sigma^{4T} + 2(f_\sigma \quad r_\sigma)I \\
\ell_{13} &= A_\sigma P_1 + B_\sigma F_\sigma^2 + (f_\sigma \quad r_\sigma)P_1 \\
\ell_{14} &= A_\sigma + B_\sigma F_\sigma^1 C_\sigma + (f_\sigma \quad r_\sigma)I \\
\ell_{21} &= F_\sigma^4 + A_\sigma^T + C_\sigma^T F_\sigma^{1T} B_\sigma^T + 2(f_\sigma \quad r_\sigma)I \\
\ell_{22} &= Q_1 A_\sigma + F_\sigma^3 C_\sigma + A_\sigma^T Q_1 + C_\sigma^T F_\sigma^{3T} \\
&\quad +2(f_\sigma \quad r_\sigma)Q_1 \\
\ell_{23} &= F_\sigma^4 + (f_\sigma \quad r_\sigma)I \\
\ell_{24} &= Q_1 A_\sigma + F_\sigma^3 C_\sigma + (f_\sigma \quad r_\sigma)Q_1
\end{aligned}$$

*then the dynamic controller (11), with the parameters $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})_\sigma$ given by (15), is a stabilizing controller for (10), and the stability of the closed loop switched system is ensured by the common Lyapunov function $V(\tilde{x}) = \tilde{x}^T P \;\, {}^1 \tilde{x}$, where $P$ is de ned as in theorem 2.*

*Proof.* If there exists a matrix transformation $\tilde{\ }$ which will be determined, such that $[A_\sigma^{cl} + (f_\sigma \quad r_\sigma)I]P$, $P[A_\sigma^{cl} + (f_\sigma \quad r_\sigma)I]^T$ and $r_\sigma P$ can be transformed to an a ne form of all the variables synthesis like $(\mathscr{A}_\sigma, \mathscr{B}_\sigma, \mathscr{C}_\sigma, \mathscr{D}_\sigma)$. Then the Schur complement can be applied to nonlinear equation (33), which is equivalent by using the Schur complement to

$$\begin{matrix} A_\sigma^{cl} P + P A_\sigma^{clT} + 2(f_\sigma - r_\sigma)P & A_\sigma^{cl} P + (f_\sigma - r_\sigma)P \\ P A_\sigma^{clT} + (f_\sigma - r_\sigma)P & -r_\sigma P \end{matrix} < 0 \qquad (36)$$

De ne $P, P \;\, {}^1$ and the transformation as in (20) and (23), and de ne the transformation $\tilde{\ }$ := diag( , ). Apply this transformation to (36), we obtain (34), which is an LMI of the variables synthesis $F_\sigma^1$, $F_\sigma^2$, $F_\sigma^3$, $F_\sigma^4$, $P_1$, $Q_1$, where $(F^1, F^2, F^3, F^4)_\sigma$ are the change of variables de ned as in (24)-(27). ∎

## 5. CONCLUDING REMARKS

The problem related to the in uence of switching signal on the asymptotic stability of switched system is investigated in the rst part of this paper, some remarks are driven, based on identity Lyapunov function, an illustrative example is given, to show that stability depends critically on the behavior of the switching signal. The problem of synthesis of a switched dynamic controller has been addressed through LMI approach. A generalization of this result is given, consisting to regional pole placements, an LMI formulation is also given.

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