A NEW DUAL-MODE HYBRID MPC ALGORITHM WITH A ROBUST STABILITY GUARANTEE

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Abstract: This paper employs the Input-to-State Stability (ISS) framework to investigate the robustness of discrete-time Piece-Wise Affine (PWA) systems in closed-loop with Model Predictive Controllers (MPC), or hybrid MPC for short. We show via an example taken from literature that stabilizing hybrid MPC can generate MPC values functions *that are not ISS Lyapunov functions* for arbitrarily small additive disturbances. As a consequence, it is not easy to prove that nominally stabilizing hybrid MPC schemes are robust. This motivates the need to design MPC schemes for hybrid systems with an a priori robust stability guarantee. A possible solution to this problem was recently developed by the authors for a particular class of PWA systems, i.e. when the origin lies in the interior of one of the regions in the partition. The main contribution of this paper is a novel dual-mode MPC algorithm for hybrid systems with an a priori ISS guarantee. This MPC scheme is applicable to general PWA systems, i.e. when the origin may lie on the boundaries of multiple regions in the partition.

Keywords: Predictive control, Hybrid systems, Robust stability.

1. INTRODUCTION

A certain maturity is reached in the field of Model Predictive Control (MPC) for hybrid systems, regarding computational and nominal stability aspects. This is illustrated by the existing tools for solving hybrid MPC optimization problems, the Hybrid Toolbox (Bemporad, 2003) and the Multi Parametric Toolbox (MPT) (Kvasnica *et al.*, 2004), and by the stability results published in the literature, for example, see (Bemporad and Morari, 1999), (Borrelli, 2003) for attractivity results and (Kerrigan and Mayne, 2002), (Lazar *et al.*, 2005*b*) for asymptotic stability results for Piece-Wise Affine (PWA) systems. In this paper we focus on *inherent robustness* of PWA systems in closed-loop with MPC controllers (or hybrid MPC for short), which is a problem that was not addressed before in the literature. By the inherent robustness property we mean that a nominally stabilizing controller has some robustness in the presence of perturbations. Its importance cannot be overstated, since all controllers designed to be nominally stable are affected by perturbations when applied in practice.

Inherent robustness has been studied in MPC for linear and smooth nonlinear systems. In (Grimm *et al.*, 2004) the authors proved that linear systems in closed-loop with stabilizing MPC are inherently robust due to the presence of a *continuous* MPC value function, which is an Input-to-State Stable (ISS) Lyapunov function (Jiang and Wang, 2001) in this case. However, they also showed via examples that continuous and necessarily nonlinear systems in closed-loop with MPC can actually have zero robustness to arbitrarily small disturbances, in the absence of a continuous MPC

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value function [2]. The first contribution of this paper is to issue a warning by presenting an example of a PWA system in closed-loop with a stabilizing MPC controller that generates an MPC value function that is *discontinuous* and, more importantly, it is *not* an ISS Lyapunov function. This indicates that the natural way to ensure ISS (robustness) in MPC fails for PWA systems.

The aim of this paper is to design an MPC scheme for hybrid systems with an a priori robust stability guarantee. Several solutions that rely on *continuous* (or even *Lipschitz continuous*) system dynamics are available in the literature, e.g. see (Limon *et al.*, 2002), (Grimm *et al.*, 2003). However, since hybrid systems are inherently nonlinear and discontinuous, these methods are not applicable to MPC of hybrid systems. Recently, in (Lazar *et al.*, 2005*a*) the authors developed a hybrid MPC scheme with an a priori ISS guarantee, under the assumption that the origin lies in the interior of one of the regions in the state-space partition. However, this method cannot be applied to general PWA systems, i.e. when the origin may lie on the boundaries of multiple regions in the state-space partition. The main contribution of this paper is a new ISS dual-mode MPC algorithm for hybrid systems, which extends the results of (Lazar *et al.*, 2005*a*) to these general PWA systems. The dual-mode MPC scheme uses tightened constraints and it does not require continuity of the MPC value function, nor of the PWA system dynamics. Note that tightened constraints were used before in order to *ensure robust feasibility only*, in smooth nonlinear MPC (Limon *et al.*, 2002). In this paper, however, an extension of this technique is employed for discontinuous PWA systems to achieve *both robust feasibility and ISS* (and thus, robustness to additive disturbance inputs).

A special remark is dedicated to the results presented in (Kerrigan and Mayne, 2002) and (Raković and Mayne, 2004), which deal with dynamic programming and tube based, respectively, approaches for solving feedback *min-max* MPC problems for *continuous* PWA systems, and also provide a robust stability guarantee. These results are opening roads towards feedback *min-max* MPC of hybrid systems. However, in this paper we use a different approach that does not resort to computationally expensive min-max formulations and we specifically include *discontinuous* PWA systems with the origin lying on the boundaries of multiple regions in the partition, which is not the case for the before-mentioned references.

Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. Let $\|\cdot\|$ denote an arbitrary Hölder vector p-norm and let $|\cdot|$ denote the absolute value. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| \triangleq \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. For a sequence $\{z_p\}_{p\in\mathbb{Z}_+}$ with $z_p \in \mathbb{R}^l$ let $\|\{z_p\}_{p\in\mathbb{Z}_+}\| \triangleq \sup\{\|z_p\| \mid p \in \mathbb{Z}_+\}$. Let $z_{[k]}$ denote the truncation of $\{z_p\}_{p\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k],p} = z_p$, $p \leq k$. Also, let $z_{[k_1,k_2]}$ denote the truncation of $\{z_p\}_{p\in\mathbb{Z}_+}$ at times $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq k_1}$, i.e. $z_{[k_1,k_2],p} = z_p$, $k_1 \leq p \leq k_2$. For a set $\mathscr{P} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{P}$ the boundary of $\mathscr{P}$, by $\mathrm{int}(\mathscr{P})$ its interior and by $\mathrm{cl}(\mathscr{P})$ its closure. For two arbitrary sets $\mathscr{P}_1 \subseteq \mathbb{R}^n$ and $\mathscr{P}_2 \subseteq \mathbb{R}^n$, let $\mathscr{P}_1 \sim \mathscr{P}_2 \triangleq \{x \in \mathbb{R}^n \mid x + \mathscr{P}_2 \subseteq \mathscr{P}_1\}$ and $\mathscr{P}_1 \oplus \mathscr{P}_2 \triangleq \{x+y \mid x \in \mathscr{P}_1, y \in \mathscr{P}_2\}$ denote their Pontryagin difference and Minkowski sum, respectively. For any real $\lambda \geq 0$, the set $\lambda\mathscr{P}$ is defined as $\{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathscr{P}\}$. A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set. A polyhedron (or a polyhedral set) is a set obtained as the intersection of a finite number of open and/or closed half-spaces.

2. INPUT-TO-STATE STABILITY PRELIMINARIES

Consider the discrete-time perturbed autonomous nonlinear system described by

$$x_{k+1} = G(x_k, w_k), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^l$ is an unknown disturbance input and $G : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. For simplicity of notation, we assume that the origin is an equilibrium in (1) for zero disturbance input, meaning that $G(0,0) = 0$.

Definition 1. For a given $0 \leq \lambda \leq 1$, a set $\mathscr{P} \subseteq \mathbb{R}^n$ with $0 \in \mathrm{int}(\mathscr{P})$ is called a *robust λ-contractive set* for system (1) if for all $x \in \mathscr{P}$ it holds that $G(x,w) \in \lambda\mathscr{P}$ for all $w \in \mathbb{W}$. For $\lambda = 1$ a robust λ-contractive set is called a *Robust Positively Invariant (RPI) set*.

Definition 2. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class $\mathscr{K}$* if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class $\mathscr{K}_\infty$* if $\varphi \in \mathscr{K}$ and it is radially unbounded (i.e. $\varphi(s) \to \infty$ as $s \to \infty$). A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class $\mathscr{KL}$* if for each fixed k, $\beta(\cdot,k) \in \mathscr{K}$ and for each fixed s, $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s,k) = 0$.

Next, we introduce the notion of input-to-state stability, as defined in (Jiang and Wang, 2001), for the discrete-time nonlinear system (1).

Definition 3. The perturbed system (1) is *globally Input-to-State Stable (ISS)* if there exist a $\mathscr{KL}$-function β and a $\mathscr{K}$-function γ such that, for each initial condition $x_0 \in \mathbb{R}^n$ and all $\{w_p\}_{p\in\mathbb{Z}_+}$ with $w_p \in \mathbb{R}^l$

[2] The value function corresponding to the MPC cost is usually used as the candidate Lyapunov function to prove nominal stability.

for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$ for all $k \in \mathbb{Z}_{\geq 1}$. Let $\mathbb{X}$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively, with $0 \in \mathrm{int}(\mathbb{X})$. We call system (1) *ISS for initial conditions in* $\mathbb{X}$ *and disturbances in* $\mathbb{W}$ if there exist a $\mathscr{K}\mathscr{L}$-function β and a $\mathscr{K}$-function γ such that, for each $x_0 \in \mathbb{X}$ and all $\{w_p\}_{p\in\mathbb{Z}_+}$ with $w_p \in \mathbb{W}$ for all $p \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies $\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|w_{[k-1]}\|)$ for all $k \in \mathbb{Z}_{\geq 1}$.

The following sufficient conditions for ISS will be used throughout the paper to establish ISS for the particular case of MPC of hybrid systems.

Theorem 4. Let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ for some $a, b, c, \lambda > 0$ and let $\sigma \in \mathscr{K}$. Let $\mathbb{W}$ be a subset of $\mathbb{R}^l$ that contains the origin. Let $\mathbb{X}$ with $0 \in \mathrm{int}(\mathbb{X})$ be a RPI set for system (1) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider now the following inequalities:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2a}$$

$$V(G(x,w)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|). \tag{2b}$$

If inequalities (2) hold for all $x \in \mathbb{X}$ and all $w \in \mathbb{W}$, then system (1) is ISS for initial conditions in $\mathbb{X}$ and disturbances in $\mathbb{W}$. Moreover, the ISS property of Definition 3 holds with

$$\beta(s,k) \triangleq \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{3}$$

where $\rho \triangleq \frac{c}{b} \in [0, 1)$.

PROOF. The proof of this theorem can be based on the proof of Lemma 3.5 in (Jiang and Wang, 2001). Note that although continuity of the candidate ISS Lyapunov function V is assumed in Lemma 3.5 of (Jiang and Wang, 2001), the continuity property is not actually used in the proof. A complete proof, including how the specific form of the β and γ functions given in (3) is obtained, is given in (Lazar *et al.*, 2005*a*). □

Remark 5. The hypothesis of Theorem 4 allows that both G and V are discontinuous. It *only* implies continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$.

Definition 6. A function V that satisfies the hypothesis of Theorem 4 is called an *ISS Lyapunov function*.

3. MODEL PREDICTIVE CONTROL OF PWA SYSTEMS PRELIMINARIES

In this paper we consider nominal and perturbed discrete-time PWA systems of the form:

$$x_{k+1} = g(x_k, u_k) \triangleq A_j x_k + B_j u_k + f_j \quad \text{when } x_k \in \Omega_j, \tag{4a}$$

$$\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, u_k, w_k) \triangleq A_j \tilde{x}_k + B_j u_k + f_j + w_k \quad \text{when } \tilde{x}_k \in \Omega_j, \tag{4b}$$

where $w_k \in \mathbb{W} \subset \mathbb{R}^n$, $k \in \mathbb{Z}_+$, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ and $f_j \in \mathbb{R}^n$, $j \in \mathscr{S}$ with $\mathscr{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indexes. We assume that $\mathbb{W}$ is a bounded polyhedral set that contains the origin, and the state and the input are constrained in some polyhedral C-sets $\mathbb{X}$ and $\mathbb{U}$. The collection $\{\Omega_j \mid j \in \mathscr{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathscr{S}} \Omega_j = \mathbb{X}$ and $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathscr{S}_0 \triangleq \{j \in \mathscr{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathscr{S}_1 \triangleq \{j \in \mathscr{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1$. We assume that the origin is an equilibrium state for (4) with $u = 0$ and therefore we require that $f_j = 0$ for all $j \in \mathscr{S}_0$. Note that this does not exclude PWA systems which are *discontinuous over the boundaries*.

Although we focus on PWA systems of the form (4), the results developed in this paper have a wider applicability since it is known (Heemels *et al.*, 2001) that PWA systems are equivalent under certain mild assumptions with other relevant classes of hybrid systems, such as mixed logical dynamical systems (Bemporad and Morari, 1999) and linear complementarity systems (van der Schaft and Schumacher, 1998).

Next, consider the case when the MPC methodology is used to generate the control input u_k, $k \in \mathbb{Z}_+$, in (4). For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by the nominal PWA system (4a) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a desired polyhedral target set that contains the origin in its interior. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$, $k \in \mathbb{Z}_+$, is $\mathscr{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N, x_{N|k} \in \mathbb{X}_T\}$. For the rest of the paper let $\|\cdot\|$ denote the ∞-norm for shortness. Consider now the functions $F(x) \triangleq \|P_j x\|$ when $x \in \Omega_j$ and $L(x,u) \triangleq \|Qx\| + \|Ru\|$, where $P_j \in \mathbb{R}^{p_j\times n}$, $j \in \mathscr{S}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times n}$ are assumed to be known matrices that have full-column rank.

Problem 7. Let $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost $J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k})$, with prediction model (4a), over all sequences $\mathbf{u}_k$ in $\mathscr{U}_N(x_k)$.

In the MPC literature, F, L and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x_0 \in \mathbb{X}$ *feasible* if $\mathscr{U}_N(x_0) \neq \emptyset$. Similarly, Problem 7 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathscr{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to Problem 7 and let $\widehat{V} : \mathbb{X}_f(N) \to \mathbb{R}_+$, $\widehat{V}(x_k) \triangleq \inf_{\mathbf{u}_k\in\mathscr{U}_N(x_k)} J(x_k, \mathbf{u}_k)$ denote the MPC value function corresponding to Problem 7. Suppose there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for Problem 7 and any state $x_k \in \mathbb{X}_f(N)$. Then, $\widehat{V}(x_k) = J(x_k, \mathbf{u}_k^*)$ and the *MPC control law* is obtained as

$$\hat{u}(x_k) \triangleq u_{0|k}^*; \quad k \in \mathbb{Z}_+. \tag{5}$$

Consider now an auxiliary state feedback control law $h_{\text{aux}} : \mathbb{R}^n \to \mathbb{R}^m$ with $h_{\text{aux}}(0) = 0$, which is usually employed in proving stability of *terminal cost and constraint set* MPC. In the PWA setting we take this state feedback PWL, i.e. $h_{\text{aux}}(x) \triangleq K_j x$ when $x \in \Omega_j$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathscr{S}$. Let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h_{\text{aux}}(x) \in \mathbb{U}\}$ denote the *safe set* with respect to both state and input constraints for this controller. Let $\mathbb{X}_{\text{PI}}$ with $0 \in \text{int}(\mathbb{X}_{\text{PI}})$ be a Positively Invariant (PI) set for system (4a) in closed-loop with h_{aux} that is contained in $\mathbb{X}_{\mathbb{U}}$. Consider now the following assumption.

Assumption 8. There exist $\{P_j, K_j \mid j \in \mathscr{S}\}$ such that

$$\begin{aligned}\|P_i(A_j + B_j K_j)x + P_i f_j\| - \|P_j x\| \\ + \|Qx\| + \|RK_j x\| \leq 0,\end{aligned} \tag{6}$$

for all $x \in \mathbb{X}_{\text{PI}}$ and all $(j,i) \in \mathscr{S} \times \mathscr{S}$.

Theorem 9. (Lazar *et al.*, 2005*a*) Suppose that Assumption 8 holds and take $\mathbb{X}_T = \mathbb{X}_{\text{PI}}$. Then, the PWA system (4a) in closed-loop with the MPC controller (5) is asymptotically stable in the Lyapunov sense for initial conditions in $\mathbb{X}_f(N)$.

The proof of Theorem 9 relies on the fact that Assumption 8 is equivalent to

$$F(g(x, h_{\text{aux}}(x))) - F(x) + L(x, h_{\text{aux}}(x)) \leq 0, \quad \forall x \in \mathbb{X}_T.$$

This in turn ensures that the hybrid MPC value function $\widehat{V}$ is a *Lyapunov function* for the closed-loop system (4a)-(5), i.e. there exist $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$, with $a, b, c, \lambda > 0$, such that $\alpha_1(\|x\|) \leq \widehat{V}(x) \leq \alpha_2(\|x\|)$ and $\widehat{V}(g(x, \hat{u}(x))) - \widehat{V}(x) \leq -\alpha_3(\|x\|)$ for all $x \in \mathbb{X}_f(N)$.

In the above setting, Theorem 8.4 of (Borrelli, 2003) states that the MPC control law $\hat{u}$ defined in (5) is a PWA state-feedback. Hence, the resulting *hybrid MPC closed-loop system* is a PWA system, i.e.

$$\begin{aligned} x_{k+1} &= g(x_k, \hat{u}(x_k)) = A_j x_k + B_j \hat{u}(x_k) + f_j, \\ &\quad \hat{u}(x_k) = L_i x_k + l_i \text{ when } x_k \in \Omega_j \cap \overline{\Omega}_i, \end{aligned} \tag{7a}$$

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{g}(\tilde{x}_k, \hat{u}(\tilde{x}_k), w_k) = A_j \tilde{x}_k + B_j \hat{u}(\tilde{x}_k) + f_j + w_k, \\ &\quad \hat{u}(\tilde{x}_k) = L_i \tilde{x}_k + l_i \text{ when } \tilde{x}_k \in \Omega_j \cap \overline{\Omega}_i, \end{aligned} \tag{7b}$$

with $(j,i) \in \mathscr{S} \times \overline{\mathscr{S}}$ ($\overline{\mathscr{S}}$ is a finite set of indexes), $k \in \mathbb{Z}_+$, where $L_i \in \mathbb{R}^{m\times n}$, $l_i \in \mathbb{R}^m$, and $\cup_{i\in\overline{\mathscr{S}}} \overline{\Omega}_i = \mathbb{X}_f(N)$ (with $\text{int}(\overline{\Omega}_i) \cap \text{int}(\overline{\Omega}_j) = \emptyset$ for $i \neq j$) is a new partition corresponding to the explicit MPC control law. Moreover, the MPC value function $\widehat{V}$ is a PWA function (recall that $\|\cdot\|$ denotes the ∞-norm), i.e.

$$\widehat{V}(x) \triangleq E_j x + e_j \text{ when } x \in \widehat{\Omega}_j, \tag{8}$$

where $E_j \in \mathbb{R}^{1\times n}$, $e_j \in \mathbb{R}$, j takes values in some finite set of indexes $\widehat{\mathscr{S}}$, and $\cup_{j\in\widehat{\mathscr{S}}} \widehat{\Omega}_j = \mathbb{X}_f(N)$ (with $\text{int}(\widehat{\Omega}_i) \cap \text{int}(\widehat{\Omega}_j) = \emptyset$ for $i \neq j$) is a new partition corresponding to the MPC value function.

4. NOMINALLY STABILIZING HYBRID MPC VALUE FUNCTIONS ARE NOT NECESSARILY ISS LYAPUNOV FUNCTIONS

In the linear and continuous nonlinear case, nominally stable systems typically have some robustness properties. Note that, as done classically, if V is a *uniformly continuous* (or even stronger, *Lipschitz continuous*) Lyapunov function for the nominal dynamics, i.e. $x_{k+1} = H(x_k)$, and the disturbance acts additively on the state, i.e. $x_{k+1} = H(x_k) + w_k$, then it is easy to prove that the hypothesis of Theorem 4 is satisfied, which ensures ISS. Indeed, uniform continuity implies that for any compact subset $\mathscr{P}$ of $\mathbb{R}^n$ there exists a $\mathscr{K}$-function σ such that for any $x, y \in \mathscr{P}$ it holds that $|V(y) - V(x)| \leq \sigma(\|y - x\|)$. Hence,

$$\begin{aligned} V(H(x) + w) - V(x) &\leq V(H(x)) + \sigma(\|w\|) - V(x) \\ &\leq -\alpha_3(\|x\|) + \sigma(\|w\|) \end{aligned}$$

and thus, V is an ISS Lyapunov function. For more general robust stability results that use *continuous* candidate (ISS) Lyapunov functions see (Grimm *et al.*, 2004). Clearly, the above continuity based robustness (ISS) argument no longer holds if the function V is discontinuous at some points.

Note that discontinuity of the candidate Lyapunov function V does not necessarily obstruct the ISS inequality (2b) to hold. However, we show via an example from literature that stabilizing hybrid MPC can generate discontinuous value functions that are not ISS Lyapunov functions for perturbed systems of the form (7b). As in (7) and (8), the following notation will be used: for $i \in \overline{\mathscr{S}}$ and $j \in \widehat{\mathscr{S}}$, $\hat{u}_i(x) \triangleq L_i x + l_i$ and $\widehat{V}_j(x) \triangleq E_j x + e_j$ for any $x \in \mathbb{X}$.

Example 10. Consider the following discontinuous PWA system, taken from (Mignone *et al.*, 2000):

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad E_1 x_k > 0 \\ A_2 x_k + B u_k & \text{if} \quad E_2 x_k \geq 0 \\ A_3 x_k + B u_k & \text{if} \quad E_3 x_k > 0 \\ A_4 x_k + B u_k & \text{if} \quad E_4 x_k \geq 0 \end{cases} \tag{9}$$

where all inequalities hold componentwise, $A_1 = \begin{bmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{bmatrix}$, $A_3 = \begin{bmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{bmatrix}$, $A_4 = \begin{bmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, $E_1 = -E_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $E_2 = -E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The state and the input of system (9) are constrained at all times in the sets $\mathbb{X} = [-10, 10] \times [-10, 10]$ and $\mathbb{U} = [-1, 1]$, respectively. The method presented in (Lazar *et al.*, 2005*b*) was employed to compute a common terminal weight matrix $P = P_1 = P_2 = P_3 = P_4$ and feedbacks $\{K_j \mid j = 1, \ldots, 4\}$ such that inequality (6) of Assumption 8 holds for the stage cost weights $Q = \text{diag}([1\ 1])$ and $R = 0.1$.

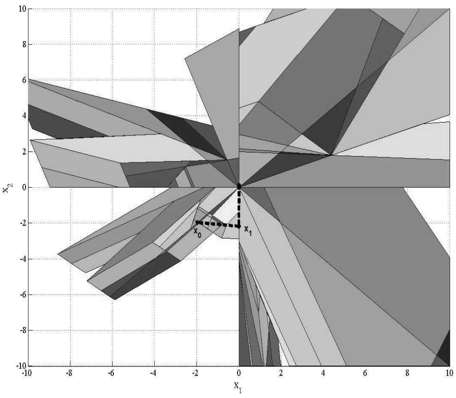

Fig. 1. The feasible set $\mathbb{X}_f(1)$ and state trajectory for the PWA MPC closed-loop system (9)-(5) with $x_0 = [-1.9649\ -1.9649]^\top$.

The following matrices were obtained:

$$P = \begin{bmatrix} 6.7001 & 3.1290 \\ -2.1107 & 4.1998 \end{bmatrix}, K_1 = \begin{bmatrix} 0.2703 & -0.1136 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.8042 & -0.2560 \end{bmatrix}, K_3 = \begin{bmatrix} 1.0122 & -0.7513 \end{bmatrix},$$
$$K_4 = \begin{bmatrix} -0.5548 & -1.1228 \end{bmatrix}. \quad (10)$$

Then, we used the MPT (Kvasnica *et al.*, 2004), which implements the algorithm of (Raković *et al.*, 2004), in order to calculate the terminal constraint set $\mathbb{X}_T$ as the maximal positively invariant set contained in $\mathbb{X}_\mathbb{U}$ for system (9) in closed-loop with h_{aux} with the feedbacks given in (10), and where $\mathbb{X}_\mathbb{U} = \cup_{j=1,\ldots,4}\{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$. By Theorem 9, this is sufficient to guarantee that the MPC closed-loop system (9)-(5) is asymptotically stable in the Lyapunov sense for all $x \in \mathbb{X}_f(N)$, $N \in \mathbb{Z}_{\geq 1}$. Then, the MPT was used to calculate the MPC control law (5) for $N = 1$ as an explicit PWA state-feedback, and to simulate the resulting PWA MPC closed-loop system (7a). The explicit MPC controller is defined over 86 state-space regions $\overline{\Omega}_i$, $i \in \overline{\mathscr{S}} \triangleq \{1,\ldots,86\}$ that satisfy $\cup_{i\in\overline{\mathscr{S}}}\overline{\Omega}_i = \mathbb{X}_f(1)$. The set of feasible states $\mathbb{X}_f(1)$ is plotted in Figure 1 together with the partition corresponding to the explicit MPC control law. $\square$

Lemma 11. For the MPC closed-loop system (7) corresponding to system (9) of Example 10 it holds that:

(i) The value function $\widehat{V}$ and the closed-loop dynamics (7a) *are not continuous*;

(ii) $\widehat{V}$ *is a Lyapunov function* for the closed-loop dynamics (7a) showing asymptotic stability in $\mathbb{X}_f(1)$;

(iii) For any $\varepsilon > 0$, $\widehat{V}$ *is not an ISS Lyapunov function* for the closed-loop dynamics (7b) and disturbances $w \in \mathscr{B}_\varepsilon \triangleq \{w \in \mathbb{W} \mid \|w\| \leq \varepsilon\}$.

PROOF. *(i)* We have chosen the state $x^* = [0\ -2.1830]^\top \in \{\partial\overline{\Omega}_{38} \cap \partial\overline{\Omega}_{58}\} \cap \overline{\Omega}_{38}$ to show that the MPC closed-loop system (7a) and $\widehat{V}$ for the above example are not continuous on $\text{int}(\mathbb{X}_f(1))$. We have obtained the following values:

$$A_2x^* + B\hat{u}_{38}(x^*) = [0.0130\ \ 0.1070]^\top;$$
$$A_3x^* + B\hat{u}_{58}(x^*) = [-0.7791\ \ -1.3535]^\top;$$
$$\widehat{V}(x^*) = \widehat{V}_{38}(x^*) = 2.6766; \lim_{x\to x^*,\, x\in\overline{\Omega}_{58}} \widehat{V}(x) = \widehat{V}_{58}(x^*) = 11.7383.$$

(ii) As Assumption 8 is satisfied via the procedure of (Lazar *et al.*, 2005*b*), the statement follows from Theorem 9.

(iii) The MPC closed-loop system (7a) corresponding to system (9) is such that the dynamics active in region $\overline{\Omega}_{38}$ is employed for $x_k = x^*$. The nominal state trajectory obtained for the initial state $x_0 = [-1.9649\ -1.9649]^\top \in \{\partial\overline{\Omega}_{47} \cap \partial\overline{\Omega}_{53}\} \cap \overline{\Omega}_{47}$ reaches the state $x_1 = x^*$ in one step (see Figure 1 for the trajectory plot). Then, for any $\mathscr{K}$-function σ we can take an arbitrarily small disturbance w such that $x^* + w \in \overline{\Omega}_{58}$, for which $\widehat{V}(x^*+w) - \widehat{V}(x_0) = \widehat{V}_{58}(x^*+w) - \widehat{V}(x_0) \approx 0.5970 > \sigma(\|w\|) \geq -\alpha_3(\|x_0\|) + \sigma(\|w\|)$ for any $\alpha_3 \in \mathscr{K}_\infty$. Hence, the ISS inequality (2b) of Theorem 4 does not hold for arbitrarily small w and thus, $\widehat{V}$ is not an ISS Lyapunov function for the closed-loop dynamics (7b). $\square$

The result of Lemma 11-(iii) implies that the most likely and natural candidate (i.e. the MPC value function $\widehat{V}$) for proving ISS for the closed-loop system (7b) fails. Hence, one should be careful in drawing conclusions on robustness from nominal stability (established via $\widehat{V}$) when dealing with hybrid MPC. At least, there is no obvious way to infer ISS from nominal stability in hybrid MPC, or to modify nominally stabilizing MPC schemes for hybrid systems such that ISS is ensured a priori.

5. MAIN RESULT

In this section we present a new technique for setting up ISS MPC schemes for hybrid systems, which uses a dual-mode approach. In the sequel, the nomenclature of Section 3 is employed, i.e. $h_{\text{aux}}(x) = K_j x$ when $x \in \Omega_j$ and, let $\mathbb{X}_{\text{RPI}} \subseteq \mathbb{X}_\mathbb{U}$ with $0 \in \text{int}(\mathbb{X}_{\text{RPI}})$ be a RPI set for system (4b) in closed-loop with h_{aux}. Let $\xi \triangleq \max_{j\in\mathscr{S}} \|P_j\|$, let $\eta \triangleq \max_{j\in\mathscr{S}} \|A_j\|$ and, for any $i \in \mathbb{Z}_{\geq 1}$, let $\mathscr{L}^i_\mu \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \mu \sum_{p=0}^{i-1} \eta^p\}$.

Next, choose the terminal set as $\mathbb{X}_T \triangleq \mathbb{X}_{\text{RPI}} \cap \mathbb{X}_N \subset \mathbb{X}_{\text{RPI}}$, where $\mathbb{X}_N \triangleq \cup_{j\in\mathscr{S}}\{\Omega_j \sim \mathscr{L}^N_\mu\} \subseteq \mathbb{X}$, and let $\mathscr{Q}_1(\mathbb{X}_T) \triangleq \{x \in \mathbb{X}_\mathbb{U} \mid g(x, h_{\text{aux}}(x)) \in \mathbb{X}_T\}$. Consider now the following (tightened) set of admissible input sequences:

$$\widetilde{\mathscr{U}}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid x_{i|k} \in \mathbb{X}_i,\ i = 1,\ldots,N-1,$$
$$x_{N|k} \in \mathbb{X}_T\},\ k \in \mathbb{Z}_+, \quad (11)$$

where $\mathbb{X}_i \triangleq \cup_{j\in\mathscr{S}}\{\Omega_j \sim \mathscr{L}^i_\mu\} \subseteq \mathbb{X}$ for all $i \in \mathbb{Z}_{[1,N-1]}$ and $(x_{1|k},\ldots,x_{N|k})$ is a state sequence generated from initial state $x_{0|k} \triangleq \tilde{x}_k$ and by applying the input sequence $\mathbf{u}_k$ to the nominal PWA model (4a). Let $\widetilde{\mathbb{X}}_f(N)$

denote the set of feasible states for Problem 7 with $\widetilde{\mathscr{U}_N}(x_k)$ instead of $\mathscr{U}_N(x_k)$, and let $\widehat{V}$ and $\hat{u}$ denote the corresponding MPC value function and MPC control law, respectively.

We define a dual-mode MPC control law as follows:

$$\hat{u}^{\mathrm{DM}}(x_k) \triangleq \begin{cases} \hat{u}(x_k) \text{ if } x_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}} \\ h_{\mathrm{aux}}(x_k) \text{ if } x_k \in \mathbb{X}_{\mathrm{RPI}} \end{cases} ; k \in \mathbb{Z}_+. \tag{12}$$

Therefore, the set of feasible states corresponding to $\hat{u}^{\mathrm{DM}}$ is $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$, which contains the origin in its interior due to $0 \in \mathrm{int}(\mathbb{X}_{\mathrm{RPI}})$.

Remark 12. Usually, e.g. see (Kerrigan and Mayne, 2002), in dual-mode robust MPC the terminal set is taken as $\mathbb{X}_{\mathrm{RPI}}$. The terminal state is restricted here to a disconnected subset of $\mathbb{X}_{\mathrm{RPI}}$, i.e. $\mathbb{X}_T = \mathbb{X}_{\mathrm{RPI}} \cap \mathbb{X}_N \subset \mathbb{X}_{\mathrm{RPI}}$, with $0 \notin \mathbb{X}_T$, in order to guarantee robust feasibility of Problem 7 with $\widetilde{\mathscr{U}_N}(x_k)$ instead of $\mathscr{U}_N(x_k)$ and ISS, as it will be shown next. If the state trajectory reaches either $\mathbb{X}_T$ or $\mathbb{X}_{\mathrm{RPI}} \setminus \mathbb{X}_T$, the dual-mode control law switches to the PWL local controller and then the state trajectory remains in $\mathbb{X}_{\mathrm{RPI}}$ (*and not necessarily in* $\mathbb{X}_T$) forever, due to robust positive invariance of $\mathbb{X}_{\mathrm{RPI}}$.

Theorem 13. Take $\mu > 0$ and $N \in \mathbb{Z}_{\geq 1}$ such that $\mathbb{X}_T = \mathbb{X}_{\mathrm{RPI}} \cap \mathbb{X}_N \neq \emptyset$ and let $\mathscr{B}_\mu \triangleq \{w \in \mathbb{W} \mid \|w\| \leq \mu\}$. Suppose that h_{aux} and the terminal cost satisfy (6) for all $x \in \mathbb{X}_{\mathrm{RPI}}$. Then it holds that:

(i) If $\mathbb{X}_T \oplus \mathscr{L}_\mu^N \subseteq \mathscr{Q}_1(\mathbb{X}_T)$ and Problem 7 with $\widetilde{\mathscr{U}_N}(x_k)$ instead of $\mathscr{U}_N(x_k)$ is feasible at time $k \in \mathbb{Z}_+$ for state $\tilde{x}_k \in \mathbb{X}$, then Problem 7 with $\widetilde{\mathscr{U}_N}(x_k)$ instead of $\mathscr{U}_N(x_k)$ is feasible at time $k+1$ for state $\tilde{x}_{k+1} = A_j\tilde{x}_k + B_j\hat{u}^{\mathrm{DM}}(\tilde{x}_k) + f_j + w_k$ for all $w_k \in \mathscr{B}_\mu$ and all $k \in \mathbb{Z}_+$;

(ii) The perturbed PWA system (4b) in closed-loop with $\hat{u}^{\mathrm{DM}}$ is ISS for initial conditions in $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$ and disturbances in $\mathscr{B}_\mu$.

PROOF. *(i)* There are two situations possible: either $\tilde{x}_k \in \mathbb{X}_{\mathrm{RPI}}$ or $\tilde{x}_k \notin \mathbb{X}_{\mathrm{RPI}}$. If $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ for some $k \in \mathbb{Z}_+$, let $(x^*_{1|k}, \ldots, x^*_{N|k})$ denote an optimal predicted state sequence obtained at time k from initial state $x_{0|k} \triangleq \tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ and by applying the input sequence $\mathbf{u}^*_k = (u^*_{0|k}, \ldots, u^*_{N-1|k})$ to the PWA model (4a). Let $(x_{1|k+1}, \ldots, x_{N|k+1})$ denote the state sequence obtained from the perturbed initial state $x_{0|k+1} \triangleq \tilde{x}_{k+1} = x_{k+1} + w_k = x^*_{1|k} + w_k$ and by applying the input sequence $\mathbf{u}_{k+1} \triangleq (u^*_{1|k}, \ldots, u^*_{N-1|k}, h_{\mathrm{aux}}(x_{N-1|k+1}))$ to the nominal PWA model (4a). The state constraints imposed in (11) ensure that: (P1) $(x_{i|k+1}, x^*_{i+1|k}) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}}$, $j_{i+1} \in \mathscr{S}$ for all $i = 0, \ldots, N-2$ and, $\|x_{i|k+1} - x^*_{i+1|k}\| \leq \eta^i \mu$ for $i = 0, \ldots, N-1$. Then, as shown in the proof of Theorem 4.3-(i) of (Lazar *et al.*, 2005*a*), we have that $x_{i|k+1} \in \Omega_{j_{i+1}} \sim \mathscr{L}^i_\mu \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. Next, $x_{N-1|k+1} = x^*_{N|k} + \prod_{i=1}^{N-1} A_{j_i} w_k$ and $x^*_{N|k} \in \mathbb{X}_T$ imply that $x_{N-1|k+1} \in \mathbb{X}_T \oplus \mathscr{L}^N_\mu$. Since $\mathbb{X}_T \oplus \mathscr{L}^N_\mu \subseteq \mathscr{Q}_1(\mathbb{X}_T)$, it follows that $h_{\mathrm{aux}}(x_{N-1|k+1}) \in \mathbb{U}$ and $x_{N|k+1} \in \mathbb{X}_T$. Hence, $\mathbf{u}_{k+1}$ is feasible at time $k+1$ and the optimization problem as given in Problem 7 with $\widetilde{\mathscr{U}_N}(x_k)$ instead of $\mathscr{U}_N(x_k)$ remains feasible. Consider now the other situation, i.e. $\tilde{x}_k \in \mathbb{X}_{\mathrm{RPI}}$. If the state trajectory enters (or starts in) $\mathbb{X}_{\mathrm{RPI}} \subseteq \mathbb{X}_{\mathbb{U}}$ (note that $\mathbb{X}_T \subset \mathbb{X}_{\mathrm{RPI}}$), feasibility of $\hat{u}^{\mathrm{DM}}(x_k) = h_{\mathrm{aux}}(x_k)$ is ensured due to robust positive invariance of $\mathbb{X}_{\mathrm{RPI}}$ for system (4b) in closed-loop with $u_k = h_{\mathrm{aux}}(x_k)$, $k \in \mathbb{Z}_+$.

(ii) The result of part (i) implies that $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$ is a RPI set for system (4b) in closed-loop with the dual-mode MPC control $\hat{u}^{\mathrm{DM}}$ and disturbances in $\mathscr{B}_\mu$. To prove ISS, we consider three situations: in Case 1 we assume that $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ for all $k \in \mathbb{Z}_+$, in Case 2 we assume that $\tilde{x}_0 \in \mathbb{X}_{\mathrm{RPI}}$, and in Case 3 we assume that $\tilde{x}_0 \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ and there exists a $p \in \mathbb{Z}_{\geq 1}$ such that $\tilde{x}_k \notin \mathbb{X}_{\mathrm{RPI}}$ for all $k \in \mathbb{Z}_{<p}$ and $\tilde{x}_p \in \mathbb{X}_{\mathrm{RPI}}$.

In Case 1, the hypothesis already ensures that the MPC value function $\widehat{V}$ satisfies the ISS condition (2a) for some $a, b, c > 0$ and $\lambda = 1$ (see Theorem 4.3 of (Lazar *et al.*, 2005*a*) for a proof). Then, it follows that $\alpha_1(\|x\|) \leq \widehat{V}(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathbb{X}}_f(N)$. Let $\tilde{x}_{k+1}$ denote the solution of the perturbed system (4b) in closed-loop with $\hat{u}^{\mathrm{DM}}$ obtained as indicated in part (i) of the proof and let $x^*_{0|k} \triangleq \tilde{x}_k$. Due to full-column rank of Q there exists $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$ for all x. Then, as shown in the proof of Theorem 4.3-(ii) of (Lazar *et al.*, 2005*a*) it holds that

$$\begin{aligned}\widehat{V}(\tilde{x}_{k+1}) - \widehat{V}(\tilde{x}_k) &\leq J(\tilde{x}_{k+1}, \mathbf{u}_{k+1}) - J(\tilde{x}_k, \mathbf{u}^*_k) \\ &\leq -\alpha_3(\|\tilde{x}_k\|) + \sigma(\|w_k\|),\end{aligned}$$

with $\sigma(s) \triangleq (\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p)s$ and $\alpha_3(s) \triangleq \gamma s$. Hence, it follows that $\widehat{V}$ satisfies the hypothesis of Theorem 4, thereby establishing ISS in this particular case for the closed-loop system (4b)-(12), for initial conditions in $\widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ and disturbances in $\mathscr{B}_\mu$.

In Case 2, we prove that the closed-loop system is ISS by showing that the candidate (discontinuous) ISS Lyapunov function $F(x) = \|P_jx\|$ when $x \in \Omega_j$ satisfies the hypothesis of Theorem 4. Since P_j has full-column rank for all $j \in \mathscr{S}$ there exist positive constants a_j and $b_j \triangleq \|P_j\|$ such that $a_j\|x\| \leq \|P_jx\| \leq b_j\|x\|$ for all $j \in \mathscr{S}$. Hence, the $\mathscr{K}_\infty$-functions $\alpha_1(s) \triangleq \min_{j \in \mathscr{S}} a_j s$ and $\alpha_2(s) \triangleq \max_{j \in \mathscr{S}} b_j s$ satisfy $\alpha_1(\|x\|) \leq F(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{R}^n$. Next, from the hypothesis we have that inequality (6) holds for all $x \in \mathbb{X}_{\mathrm{RPI}}$ and all $(j, i) \in \mathscr{S} \times \mathscr{S}$, which yields:

$$\begin{aligned}&F((A_j + B_jK_j)x + f_j + w) - F(x) \\ &\quad = \|P_i((A_j + B_jK_j)x + f_j + w)\| - \|P_jx\| \\ &\quad \leq \|P_i(A_j + B_jK_j)x + P_if_j\| + \|P_iw\| - \|P_jx\| \\ &\quad \leq -\|Qx\| + \max_{i \in \mathscr{S}} \|P_i\|\|w\| \leq -\alpha_3(\|x\|) + \sigma(\|w\|),\end{aligned}$$

for all $x \in \mathbb{X}_{\mathrm{RPI}}$, $(j, i) \in \mathscr{S} \times \mathscr{S}$ and disturbances in $\mathscr{B}_\mu$, where $\alpha_3(s) \triangleq \gamma s$ (with $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$) and $\sigma(s) \triangleq \max_{i \in \mathscr{S}} \|P_i\|s$. Then, due to robust positive invariance of $\mathbb{X}_{\mathrm{RPI}}$, ISS for initial conditions in $\mathbb{X}_{\text{RPI}}$ and disturbances in $\mathscr{B}_\mu$ follows from Theorem 4.

In Case 3 there exists a finite $p \in \mathbb{Z}_{\geq 1}$ such that $\tilde{x}_k \notin \mathbb{X}_{\text{RPI}}$ for all $k \in \mathbb{Z}_{<p}$ and $\tilde{x}_p \in \mathbb{X}_{\text{RPI}}$. Then, from Theorem 4, Case 1 and Case 2, it follows that there exist $\mathscr{K}\mathscr{L}$-functions β_1, β_2 and $\mathscr{K}$-functions γ_1, γ_2 such that for all $p \in \mathbb{Z}_{\geq 1}$ it holds:

$$\|\tilde{x}_k\| \leq \beta_1(\|\tilde{x}_0\|, k) + \gamma_1(\|w_{[k-1]}\|),\ \forall k \in \mathbb{Z}_{\leq p},$$
$$\|\tilde{x}_k\| \leq \beta_2(\|\tilde{x}_p\|, k-p) + \gamma_2(\|w_{[k-p,k-1]}\|),\ \forall k \in \mathbb{Z}_{>p},$$

for all $w_{[k-1]} \in \{\mathscr{B}_\mu\}^k$ and all $w_{[k-p,k-1]} \in \{\mathscr{B}_\mu\}^p$, respectively. The functions $\beta_1 \in \mathscr{K}\mathscr{L}$, $\gamma_1 \in \mathscr{K}$ and $\beta_2 \in \mathscr{K}\mathscr{L}$, $\gamma_2 \in \mathscr{K}$ are obtained as in (3) for some constants $\overline{\rho}, \underline{\rho} \in [0,1)$ and some $\mathscr{K}_\infty$-functions $\overline{\alpha}_1(\underline{s}) \triangleq \overline{a}s$, $\overline{\alpha}_2(s) \triangleq \overline{b}s$ and $\alpha_1(s) \triangleq as$, $\alpha_2(s) \triangleq bs$, with $\overline{a}, \overline{b}, a, b > 0$, respectively. Then, for all $k \in \mathbb{Z}_{>p}$ and all $p \in \mathbb{Z}_{\geq 1}$ it follows that

$$\begin{aligned}
\|\tilde{x}_k\| &\leq \beta_2(\beta_1(\|\tilde{x}_0\|, p) + \gamma_1(\|w_{[p-1]}\|), k-p) \\
&\quad + \gamma_2(\|w_{[k-p,k-1]}\|) \\
&\leq \beta_2(2\beta_1(\|\tilde{x}_0\|, p), k-p) \\
&\quad + \beta_2(2\gamma_1(\|w_{[p-1]}\|), k-p) + \gamma_2(\|w_{[k-p,k-1]}\|) \\
&\overset{(13)}{\leq} \beta_3(\|\tilde{x}_0\|, k) + \beta_2(2\gamma_1(\|w_{[p-1]}\|), 1) \\
&\quad + \gamma_2(\|w_{[k-p,k-1]}\|) \\
&\leq \beta_3(\|\tilde{x}_0\|, k) + \beta_2(2\gamma_1(\|w_{[k-1]}\|), 1) \\
&\quad + \gamma_2(\|w_{[k-1]}\|) \\
&\leq \beta_3(\|\tilde{x}_0\|, k) + \gamma_3(\|w_{[k-1]}\|),
\end{aligned}$$

where $\gamma_3(s) \triangleq \beta_2(2\gamma_1(s), 1) + \gamma_2(s)$ and we used the fact that

$$\begin{aligned}
&\beta_2(2\beta_1(s,p), k-p) \\
&\qquad \overset{(3)}{=} \alpha_1^{-1}(2\rho^{k-p}\alpha_2(2\overline{\alpha}_1^{-1}(2\overline{\rho}^p\overline{\alpha}_2(s))) \\
&\qquad \leq 8\frac{b\overline{b}}{a\overline{a}}\tilde{\rho}^k s \triangleq \beta_3(s,k), \qquad (13)
\end{aligned}$$

and $\tilde{\rho} \triangleq \max(\rho, \overline{\rho}) \in [0,1)$. Hence, $\beta_3 \in \mathscr{K}\mathscr{L}$ and, since $\beta_2 \in \mathscr{K}\mathscr{L}$ and $\gamma_1, \gamma_2 \in \mathscr{K}$, we obtain that $\gamma_3 \in \mathscr{K}$. Applying Case 1 and Case 2 and combining with the result obtained above for Case 3 it follows that:

$$\|x_k\| \leq \beta(\|\tilde{x}_0\|, k) + \gamma(\|w_{[k-1]}\|),$$

for all $\tilde{x}_0 \in \widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\text{RPI}}$, $w_{[k-1]} \in \{\mathscr{B}_\mu\}^k$ and all $k \in \mathbb{Z}_{\geq 1}$, where

$$\beta(s,k) \triangleq \max(\beta_1(s,k), \beta_2(s,k), \beta_3(s,k))$$

is a $\mathscr{K}\mathscr{L}$-function and $\gamma(s) \triangleq \max(\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a $\mathscr{K}$-function. Hence, ISS is proven for system (4b) in closed-loop with $\hat{u}^{\text{DM}}$ for all initial conditions in $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\text{RPI}}$ and disturbances in $\mathscr{B}_\mu$. $\square$

Illustrative example

Next, we demonstrate the ISS properties of the dual-mode MPC control law (12) on the PWL system (9) of Example 10, introduced in Section 4. The terminal weight matrices $P_j = P$ for $j = 1, \ldots, 4$ and the feedbacks $\{K_j \mid j \in \mathscr{S}\}$ given in (10) are such that inequality (6) holds for all $x \in \mathbb{R}^n$. In order to implement the dual-mode MPC control law one has to compute the terminal set $\mathbb{X}_T$. The MPT (Kvasnica *et al.*, 2004) was employed in order to calculate the maximal RPI set $\mathbb{X}_{\text{RPI}}$ contained in $\mathbb{X}_{\mathbb{U}}$. We choose $\mu = 0.1$ and $N = 1$, for which the terminal constraint set $\mathbb{X}_T = \mathbb{X}_{\text{RPI}} \cap \mathbb{X}_1 \neq \emptyset$ (see Figure 2), where $\mathbb{X}_1 = \cup_{j=1,\ldots,4}\{\Omega_j \sim \mathscr{L}_\mu^1\}$, satisfies the hypothesis of Theorem 13. An explicit solution of Problem 7 with $\widetilde{\mathscr{U}}_N(x_k)$ instead of $\mathscr{U}_N(x_k)$ was calculated with the MPT. The feasible set $\widetilde{\mathbb{X}}_f(1) \cup \mathbb{X}_{\text{RPI}}$ of the dual-mode MPC control law and the state-space partition (138 regions) corresponding to the explicit MPC control law are plotted in Figure 3.

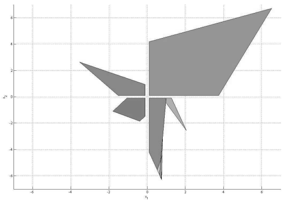

Fig. 2. The terminal constraint set $\mathbb{X}_T = \mathbb{X}_{\text{RPI}} \cap \mathbb{X}_1$.

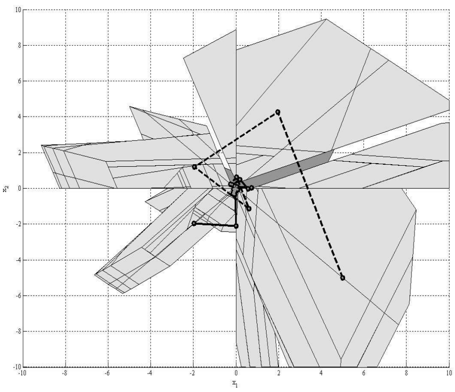

Fig. 3. The feasible set $\widetilde{\mathbb{X}}_f(1) \cup \mathbb{X}_{\text{RPI}}$: $\widetilde{\mathbb{X}}_f(1)$ - light grey; a part of $\mathbb{X}_{\text{RPI}}$ - dark grey.

Note that, by Theorem 13, ISS is ensured for the closed-loop system for initial states in $\widetilde{\mathbb{X}}_f(1) \cup \mathbb{X}_{\text{RPI}}$ and disturbances in $\mathscr{B}_\mu$, without employing a *continuous MPC value function*. Indeed, the dual-mode MPC value function $\widehat{V}$ is discontinuous at any $x \in \partial\overline{\Omega}_{32} \cap \partial\overline{\Omega}_{80}$. For example, $\widehat{V}_{32}(x^*) = 2.9038$ and $\widehat{V}_{80}(x^*) = 11.7383$ for $x^* = [0\ -2.1830]^\top$, i.e. the critical point at which the nominal MPC value function of Example 10 is not an ISS Lyapunov function.

In order to illustrate the ISS property of the dual-mode MPC controller we simulated system (9) in closed-loop with $\hat{u}^{\text{DM}}$ for initial states $x_{01} = [-1.9649\ -1.9649]^\top$ (solid line) and $x_{02} = [5\ -5]^\top$ (dashed line) and the disturbance values depicted in Figure 4 - (a), (b) for both x_{01} and x_{02}. The control inputs are also plotted in Figure 4 - (c), (d) for initial states x_{01} and x_{02}, respectively. Once the disturbance converges to zero, the state trajectories also converge to the origin

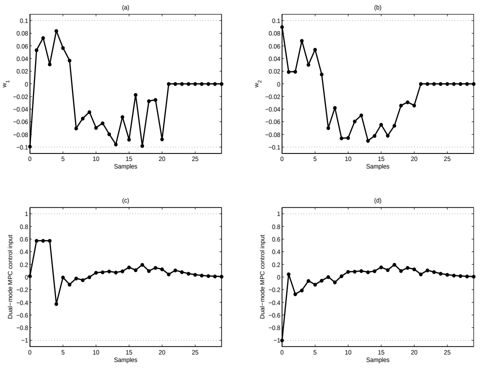

Fig. 4. Disturbance inputs $w = [w_1\ w_2]^\top$ - (a) and (b); $\hat{u}^{\mathrm{DM}}$ for x_{01} - (c); $\hat{u}^{\mathrm{DM}}$ for x_{02} - (d).

for both initial states, due to the ISS property. It is also worth to point out that the initial state x_{01}, which was a problematic initial condition, as shown in the proof of Lemma 11, is contained in the feasible set of the ISS dual-mode MPC controller. This illustrates the effectiveness of the proposed methodology.

6. CONCLUSIONS

This paper considered robust asymptotic stability in terms of ISS for *discontinuous* PWA systems controlled by MPC strategies, as this is an important property from a practical point of view. We presented an example of a PWA system (with the origin lying on the boundaries of multiple regions in the partition) taken from literature for which a nominally stabilizing MPC scheme generates an MPC value function that is not an ISS Lyapunov function. In such cases, there are no systematic ways available for modifying hybrid MPC schemes such that robustness (ISS) is a priori ensured. Therefore, a new method for setting up MPC schemes for general discontinuous PWA systems, with an a priori ISS guarantee, was developed via a dual-mode approach. The dual-mode hybrid MPC algorithm uses tightened constraints and does not require continuity of the system, the MPC control law nor of the MPC value function. An example demonstrated the effectiveness of the developed methodology.

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