Analysis and Control of Cyber-Physical Systems

Exercises on Hybrid Systems

May 2021

Problem 1. Consider a time-driven system modelled by the differential equation

$$\dot{x}(t) = f(x(t))$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and the piece-wise constant activity is

$$f(x_1, x_2) = \begin{cases} (x_1^3, -x_2) & \text{if } x_1 < 4 \\ (5, -1) & \text{if } x_1 = 4 \\ (-x_1, -x_2) & \text{if } x_1 > 4 \end{cases}$$

(a) Determine the discontinuity points of this activity function and show them in the (x_1, x_2) plane.

(b) Determine if in the discontinuity points there exists a Cauchy solution. It may help if you plot the flow, i.e., the vector $f(x_1, x_2)$, in a ball centered around a point of discontinuity.

(c) If not determine a Filippov solution.

(d) Determine a hybrid automaton that describes this system.

(e) Determine analytically the evolution of the hybrid automaton for $t \geq 0$ starting from the initial condition $x(0) = (0, 1)$. You should represent this solution as a hybrid signal.

(f) Discuss if the solution determined at the previous point is affected by chattering and if so determine by space regularization, a chattering-free model.

Problem 2. Consider a time-driven system modelled by the differential equation

$$\dot{x}(t) = f(x(t))$$

where $x \in \mathbb{R}$ and the activity is

Case A: $f(x) = |x|^3$; Case B: $f(x) = -x^3$.

(a) Determine the Lipschitz properties of these activity functions and based on this discuss what can be concluded on the existence, uniqueness and globality their solutions.

(b) Determine for all possible initial states the solution of these equations and discuss if these results are consistent with your answer to point (a).

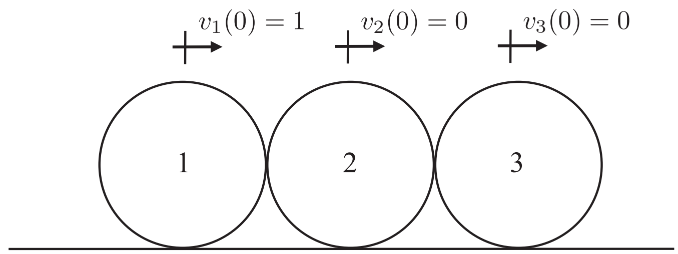
Problem 3. A system consists of three balls as shown in figure. The initial velocity of the three balls (positive if directed to the right) are $v_1(0) = 1\ ms^{-1}$, $v_2(0) = v_3(0) = 0$, and this corresponds to the case in which ball 1 is impacting on ball 2 which is in contact with ball 3, while balls 2 and 3 are at rest. The three balls have the same mass.

A physics professor explains to his class that the system evolves as follows. For $i = 1, 2$, if at time t ball i is in contact with ball $i + 1$ and it holds that $v_i(t) > v_{i+1}(t)$, an inelastic collision will occur and both balls will update their velocity to

$$v_i(t^+) = v_{i+1}(t^+) = \frac{v_i(t) + v_{i+1}(t)}{2},$$

while the remaining ball is not involved in the collision.

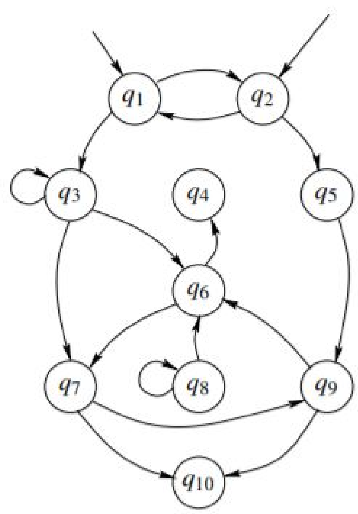
The professor claims that in $3s$ all balls will have moved one meter to the right. A Greek student claims the balls will never move.



(a) Describe the evolution of the system up to the first collisions in terms of hybrid temporal trajectory and hybrid signals.

(b) Who do you believe is right: the professor or the student? Based on which argument?

Problem 4. Consider the automaton in figure with set of states $S = \{q_1, q_2, \ldots q_{10}\}$.



(a) Give the algebraic structure of a state transition system T representing it.

(b) Compute the set of states $Reach_k(T)$ reached in k steps or less and the reachability set $Reach(T)$.

(c) Give a collection of subsets of S that is not a partition.

(d) Give a collection of subsets of S that is a partition.

(e) Assume now a set of final states $S_F = \{q_4, q_{10}\}$ is given and call the corresponding state transition system T'. Is the partition you have given in item (e) a bisimulation? Why (not)?

(f) Give a minimal bisimulation of T' and determine its quotient automaton. What is the number of equivalence classes $n'_{\min}$ of this bisimulation?

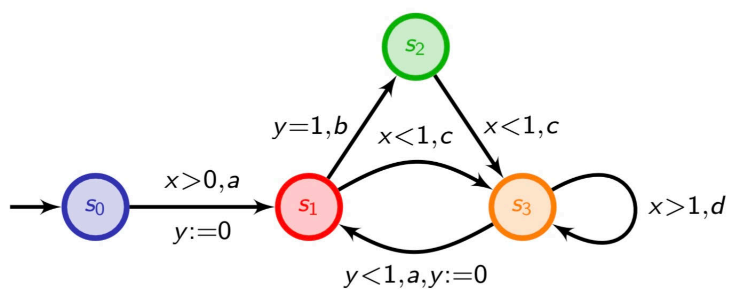
(g) Give a bisimulation of T' with the maximum possible number of sets and determine its quotient automaton. What is the number of equivalence classes $n'_{\max}$ of this bisimulation?

(h) Assume you arbitrary label the transitions of the automaton with symbols from alphabet $E = \{a, b\}$ and let T'' be the corresponding state transition model with set of final states $S_F = \{q_{10}\}$.

Let the number of the equivalence classes of the minimal, resp., maximal, bisimulation of T'' be $n''_{\min}$, resp., $n''_{\max}$.

How do you expect $n''_{\min}$ and $n''_{\max}$ compare with $n'_{\min}$ and $n'_{\max}$?

Problem 5. Consider the timed automaton in figure with two clocks denoted x and y and initial state $(s_0, (0, 0))$. Note that each edge is labeled with a symbol from alpahabet $E = \{a, b, c, d\}$.



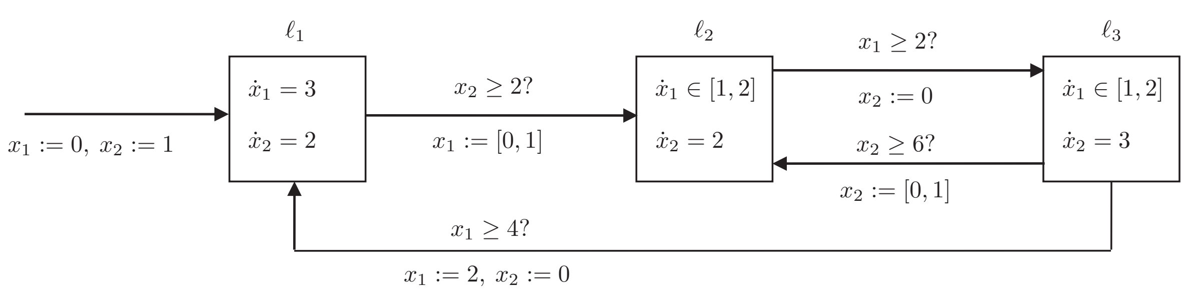
(a) Construct the region graph this automaton: some of its edges will be labeled with symbols in E.

(b) Determine if it is possbile to visit first s_2 and then s_3.

(c) Determine if it is possbile to reach a state where $y < 1 < x$.

(d) Determine if the sequence $acdac$ can be generated.

Problem 6. Consider the hybrid automaton H in figure.



(a) Discuss the known classes of hybrid automata to which H belongs.

(b) Is it possible to define a finite bisimulation on its states? Is the reachability problem decidable for this automaton? Motivate your answer.

(c) Determine a time-abstract state transition system describing H.

(d) Determine, if possible, a multirate automaton equivalent to H.

(e) Let automaton H' be obtained from H changing the two differential inclusions $\dot{x}_1 \in [1, 2]$ into $\dot{x}_1 = 2$. Determine, if possible, a timed automaton equivalent to H'.