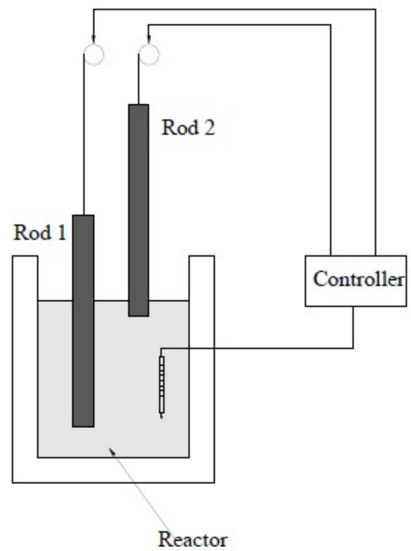
Analysis and Control of Cyber-Physical Systems

Final exam — 11 June 2025

Problem 1. [7 pts] A system designed to cool a nuclear reactor consists of two independently movable control rods as in figure. Due to mechanical constraints, a rod can only be inserted into the reactor core if it has not been inside the core during the past 20 seconds.

Initially, the coolant temperature, denoted by $x(t)$ is 510 degrees, and both rods are outside the reactor core. The temperature inside the reactor increases according to the following (linearized) differential equation: $\dot{x}(t) = 2.5$.



When the temperature reaches 550 degrees, the reactor must be cooled inserting a rod into the core. At this point, one of the following two cases can occur:

1. One of the available rods is inserted into the core.
2. No rod is available for insertion: the reactor overheats causing a meltdown.

The cooling due to the rods follows the laws:

Rod 1: $\dot{x}(t) = -2.5$ Rod 2: $\dot{x}(t) = -6.5$.

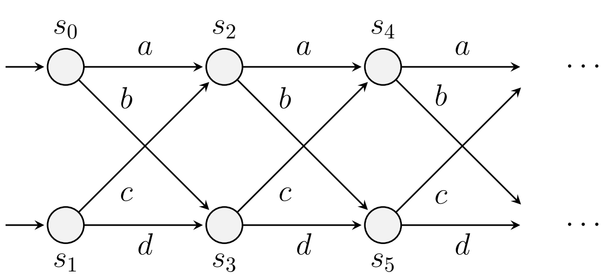
Once the temperature decreases to 510 degrees, the inserted rod is removed from the reactor core.

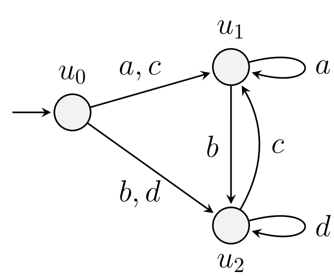
(a) *(4 pts)* Model the this plant as a hybrid system providing both its algebraic and graphical description. *Hint: the automaton should have three locations.*

(b) *(1 pts)* Is this system deterministic?

(c) *(bonus, 2 pts)* Compute two different hybrid trajectories of duration 40 s. Is a meltdown (bad state) reachable?

Problem 2. [7 pts] Consider the infinite state transition system $T = (S, \Sigma, \longrightarrow, S_0, S)$ in the figure below (left) and the finite one $T' = (U, \Sigma, \longrightarrow, U_0, U)$ on its right.





(a) *(3 pts)* Discuss if T and T' are: language equivalent, bisimilar or isomorph. If not bisimilar, does one of the two simulate the other one?

(b) *(4 pts)* Determine a minimal bisimulation $\sim$ on the states of T and compute the equivalent quotient automaton $T/\sim$ showing all steps of your procedure.

Problem 3. [4 pts] Consider the symmetric matrix

$$M = \begin{bmatrix} 8 & -1 \\ -1 & 4 \end{bmatrix}$$

(a) *(2 pts)* Compute the eigenvalues of M and discuss if the matrix is positive/negative definite.

(b) *(2 pts)* Determine the quadratic form $V(x) = x^T M x$ and discuss is this function of $x \in \mathbb{R}^2$ is positive/negative definite? In which point?

Problem 4. [9 pts] Consider the linear switched system $\{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) *(2 pts)* What does is mean when we say that a linear switched system is quadratically stabilizable?

(b) *(2 pts)* Show that this system is quadratically stabilizable determining an equivalent linear systems with matrix A.

(c) *(2 pts)* Find a Lyapunov function $V(x) = x^T P x$ for the equivalent linear system $\dot{x}(t) = Ax(t)$, where $P \succ 0$ is the solution of $A^T P + PA = -Q$, for an arbitrary symmetric matrix $Q \succ 0$.
Hint: You may want to choose a diagonal matrix Q.

(d) *(3 pts)* Find a quadratic stabilizing switching law $\ell(x(t))$ for this system.