Analysis and Control of Cyber-Physical Systems

Homework 5 — 22 May 2025

Problem 1. Consider a (non-switched) spring-mass-damper system whose state equation is:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

where x_1 is the position of the mass ($x_1 = 0$ is the equilibrium position), x_2 is the velocity of the mass, $k > 0$ is the spring constant, $b > 0$ is the friction constant of the damper.

(a) The total energy of the system is $E = E_p + E_k$ where the potential energy (stored in the spring) is $E_p = 1/2 \cdot kx_1^2$ and the kinetic energy is $E_k = 1/2 \cdot mx_2^2$. Verify that E is a Lyapunov function satisfying:

$$(1)\ E(0) = 0; \qquad (2)\ E(x) > 0 \text{ for all } x \neq 0; \qquad (3)\ \frac{d}{dt} E(x(t)) \leq 0 \text{ for all } x \neq 0.$$

Note that the last condition of the Lyapunov function E is slightly different from the one we discussed in class: $d/dt\, E(x(t)) = 0$ for some $x \neq 0$. In fact, $d/dt\, E(x(t)) = 0$ for the line $x_2 = 0$. However, since the system cannot stay on the line $x_2 = 0$ with $x_1 \neq 0$, the energy will be continuously decreasing so that system is indeed asymptotically stable.

(b) Assume $m = k = b = 1$. Determine a Lyapunov function $V(x) = x^T P x$ by solving the Lyapunov equation: $A^T P + PA = -Q$ where Q is an arbitrary positive-definite matrix you choose. If you solve the Lyapunov equation manually, you may choose $Q = I$.

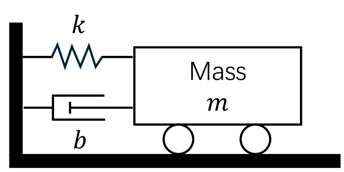

Figure 1: A spring-mass-damper system.

Problem 2. Consider a linear switched system $H = \{A_1, \ldots, A_n\}$ in which all matrices A_i are stable and *normal*, i.e., $A_i^T A_i = A_i A_i^T$.

(a) Prove that such a system is stable under arbitrary switching law $\ell(t)$. To prove this you should find a suitable positive-definite matrix P such that $V(x) = x^T P x$ is a common quadratic Lyapunov function for all A_i's.

[Hint] There is a Lemma for *normal* matrices. "If a continuous-time system $\dot{x} = Ax$ is normal and stable, then $A^T + A \prec 0$, i.e., $Z = A^T + A$ is negative-definite."

(b) Consider $\{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & 4 \\ -4 & -3 \end{bmatrix}.$$

Are these matrices stable and normal? If so find a common quadratic Lyapunov function.

(c) Do these results apply to diagonal matrices?

Problem 3. Consider the linear switched system $\{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}.$$

(a) Show that $V(x_1, x_2) = \max\{x_1^2 + 2x_2^2, 2x_1^2 + x_2^2\}$ is a common Lyapunov function.

(b) Note that V is not a quadratic function. Use an LMI solver to find a quadratic common Lyapunov function if it exists.

Problem 4. Consider an LSS that consists of two unstable modes:

$$A_1 = \begin{bmatrix} 15 & -6 \\ 25 & -6 \end{bmatrix} \quad A_2 = \begin{bmatrix} -8 & -5 \\ -5 & 2 \end{bmatrix}.$$

1. Show that the system is quadratically stabilizable finding a stable linear combination of matrices A_1, A_2. Design the corresponding state feedback switching law $\ell(x(t))$ that stabilizes the system.
2. Draw the regions of the state space where each mode is active according to ℓ. Show the system evolution for two different initial conditions (chattering is allowed).