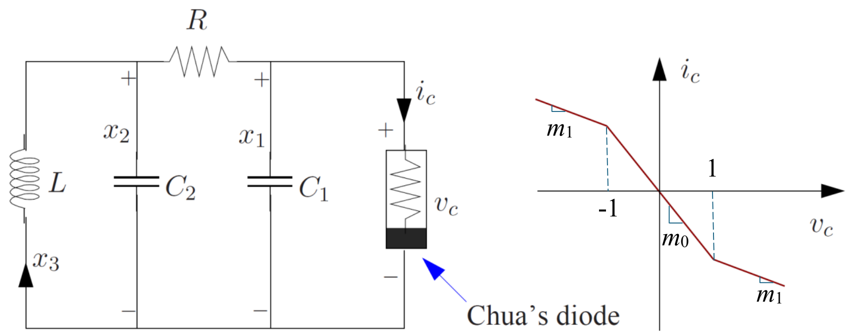
Analysis and Control of Cyber-Physical Systems

Homework 4 — 28 April 2025

Problem 1. Chua's circuit (see figure) is a simple electronic circuit exhibiting a wide variety of bifurcation and chaotic phenomena. The peculiar component of this circuit is an element with nonlinear negative resistance called *Chua's diode*.



The state of Chua's circuit is (x_1, x_2, x_3) and its components are:

- $x_1(t) = v_1(t) = v_c(t)$: the voltage across capacitor C_1 (and across Chua's diode);
- $x_2(t) = v_2(t)$: the voltage across capacitor C_2 (and across the inductor);
- $x_3(t) = i_L(t)$: the current through the inductor L.

The circuit's dynamics are characterized by the following equations:

$$\begin{cases} C_1\dot{x}_1(t) = \dfrac{1}{R}[x_2(t) - x_1(t)] - g(x_1(t)) \\ C_2\dot{x}_2(t) = \dfrac{1}{R}[x_1(t) - x_2(t)] + x_3(t) \\ L\dot{x}_3(t) = -x_2(t) \end{cases} \tag{1}$$

where $g(x_1) = i_c$ is the current through Chua's diode, whose nonlinear characteristic is given by:

$$g(x_1) = \begin{cases} -m_0 + m_1(x_1 + 1) & \text{if} \quad x_1 < -1; \\ m_0 x_1 & \text{if} \quad -1 \le x_1 < 1; \\ m_0 + m_1(x_1 - 1) & \text{if} \quad x_1 \ge 1. \end{cases} \tag{2}$$

We assume the following parameter values:

$$C_1 = \frac{1}{10}, \quad C_2 = 2, \quad L = \frac{1}{7}, \quad R = \frac{10}{7}, \quad m_0 = -4, \quad m_1 = -0.1.$$

(a) Model the system as a hybrid automaton, providing both its algebraic and graphical representation.

(b) Discuss whether the automaton is deterministic, autonomous, whether it exhibits dynamics switching and jumps in the continuous state space. Classify the edges as autonomous or controlled.

(c) Simulate the evolution of the automaton starting from the following initial conditions:

$$\text{Case 1:} \quad \begin{cases} v_1(0) = A \\ v_2(0) = -1.4 \\ i_L(0) = -59 \end{cases} \qquad \text{Case 2:} \quad \begin{cases} v_1(0) = 10 \\ v_2(0) = -1.8 \\ i_L(0) = -24 \end{cases}$$

where $A \in \{1, \dots, 31\}$ is your day of birth.

For each simulation, print the following plots: state trajectory in the (x_1, x_2) subspace, time evolution of the state variables, time evolution of the discrete state, simulating up to time $t = 1000\ s$.

Problem 2. Consider the bouncing ball model discussed in class, where x_1 denotes the height of the ball above the horizontal plane and x_2 denotes its velocity.

(a) Assuming that the initial continuous state is $x_1(0) = 15$ and $x_2(0) = 0$, and considering an elastic coefficient $\alpha = 0.9$, determine the time $\bar{T}$ after which this model no longer admits time evolution.

(b) Simulate the evolution of this system for a time greater than $\bar{T}$, plotting the time evolution of the state variables and the trajectory in the state space. Provide an interpretation of the result obtained.

(c) Describe the model obtained from the previous one through spatial regularization, and simulate the evolution of the new system under the same conditions as the previous case.

Problem 3. Consider the time-driven system described by the differential equation

$$\dot{x}(t) = f(x(t)),$$

where $x(t) \in \mathbb{R}$ and the piecewise linear activity function takes the form

$$f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ -2x & \text{if} \quad x \geq 1 \end{cases}$$

(a) Plot the activity function and discuss for which values of the initial condition it is certainly possible to guarantee that the equation admits a solution.

(b) Determine the evolution of the system for $t \geq 0$, both analytically and by simulation, starting from an initial condition $x(0) < 0$. Is this solution global?

(c) Determine the evolution of the system for $t \geq 0$ starting from the initial condition $x(0) = 0.5$. What happens when the dynamics switches?

(d) Determine, starting from the same initial condition, the evolution of the system for $t \geq 0$ using the Filippov solution and plot the corresponding graph.

(e) Determine, if it exists, a hybrid automaton with continuous activity functions that describes this system.