Analysis and Control of Cyber-Physical Systems

Homework 5 — 2 May 2024

Problem 1. Consider a (non-switched) spring-mass-damper system whose state equation is:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

where x_1 is the position of the mass ($x_1 = 0$ is the equilibrium position), x_2 is the velocity of the mass, $k > 0$ is the spring constant, $b > 0$ is the friction constant of the damper.

(a) The total energy of the system is $E = E_p + E_k$ where the potential energy (stored in the spring) is $E_p = 1/2 \cdot kx_1^2$ and the kinetic energy is $E_k = 1/2 \cdot mx_2^2$. Verify that E is a Lyapunov function satisfying:

$$(1)\ E(0) = 0; \qquad (2)\ E(x) > 0 \text{ for all } x \neq 0; \qquad (3)\ \frac{d}{dt} E(x(t)) \leq 0 \text{ for all } x \neq 0.$$

Note that the last condition of the Lyapunov function E is slightly different from the ones we discussed in class: $d/dt\ E(x(t)) = 0$ for some $x \neq 0$. In fact, $d/dt\ E(x(t)) = 0$ for the line $x_2 = 0$. However, since the system cannot stay on the line $x_2 = 0$ with $x_1 \neq 0$, the energy will be continuously decreasing so that system is indeed asymptotically stable.

(b) Assume $m = k = b = 1$. Determine a Lyapunov function $V(x) = x^T P x$ by solving the Lyapunov equation: $A^T P + PA = -Q$ where Q is an arbitrary positive-definite matrix you choose. If you solve the Lyapunov equation manually, you may choose $Q = I$.

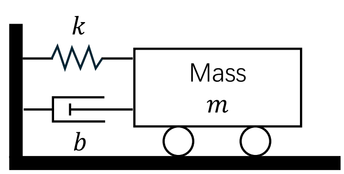

Figure 1: A spring-mass-damper system.

Problem 2. Consider a switched system $H = \{A_1, A_2\}$ with

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -0.5 & -5 \end{bmatrix}$$

(a) Simulate the evolution of each mode $i = 1, 2$ and see if each mode is stable.

(b) Plot[1] $\sigma_{\max}(e^{A_i t})$ versus t for each mode $i = 1, 2$. Determine by the plots if the system H is stable under arbitrary switching law $\ell(t)$.

[1] Here $\sigma_{\max}(M)$ is the maximal *singular value* of matrix M.

Problem 3. Consider a linear switched system $H = \{A_1, \ldots, A_n\}$ in which all matrices A_i are stable and *normal*, i.e., $A_i^T A_i = A_i A_i^T$. Prove that such a system is stable under arbitrary switching law $\ell(t)$. To prove this you should find a suitable positive-definite matrix P such that $V(x) = x^T P x$ is a common quadratic Lyapunov function for all A_i's.

(a) **[Hint]** There is a Lemma for *normal* matrices. "If a continuous-time system $\dot{x} = Ax$ is normal and stable, then $A^T + A \prec 0$, i.e., $Z = A^T + A$ is negative-definite."

(b) If you find the problem difficult, try $n = 2$, i.e., $H = \{A_1, A_2\}$.

Problem 4. Consider an LSS that consists of two unstable modes:

$$A_1 = \begin{bmatrix} 15 & -6 \\ 25 & -6 \end{bmatrix} \quad A_2 = \begin{bmatrix} -8 & -5 \\ -5 & 2 \end{bmatrix}.$$

1. Show that the system is quadratically stabilizable finding a stable linear combination of matrices A_1, A_2. Design the corresponding state feedback switching law $\ell(x(t))$ that stabilizes the system.

2. Draw the regions of the state space where each mode is active according to ℓ. Show the system evolution for two different initial conditions (chattering is allowed).

Problem 5. Consider an LSS that consists of two stable modes:

$$A_1 = \begin{bmatrix} -1 & -10 \\ 1.5 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 5 \\ -1 & -0.5 \end{bmatrix}.$$

1. Determine a minimal dwell time τ_d using singular value analysis such that the system is stable for any switching law $\ell(t)$ that satisfies the minimal dwell time.

2. Determine a minimal dwell time τ_d' using multiple Lyapunov functions (solve Lyapunov functions for P_1, P_2, compute their eigenvalues, etc.). Compare the rate of convergence of two switching laws $\ell(t)$ and $\ell'(t)$ that satisfy the dwell time τ_d and τ_d' above, respectively.

* Use MATLAB to determine the stability of a matrix and to solve the Lyapunov equations.