UNIVERSITÀ DEGLI STUDI DI CAGLIARI

Dottorato di ricerca in

INGEGNERIA ELETTRONICA ED INFORMATICA

XV Ciclo

Implementation in the ANSYS finite element code of the electric vector potential T-Ω,Ω formulation and its validation with the magnetic vector potential A-V,A formulation.

Relatori

Prof. Ing. Alessandra Fanni

Tesi di dottorato di

Dott. Ing. Pietro Testoni

Implementation in the ANSYS finite element code of the electric vector potential T-Ω,Ω formulation and its validation with the magnetic vector potential A-V,A formulation.

Pietro Testoni

Gennaio 2003

To my mother,

my sister and

my brothers

Acknowledgements

I wish to express my deepest gratitude to Professor Piergiorgio Sonato for proposing and supervising this work and for his appreciated suggestions.

I would also warmly thank Dr. John Swanson for his very helpful e-mail correspondence about the ANSYS programmable capabilities.

Last, but not least, I would like to thank my family for always supporting me.

Contents

Introduction

According to Faraday's law a time varying magnetic field produces an induced electric field which causes an electric current to flow in a conducting material. If the conducting material is not a wire filament, but rather a massive conducting body this current is distributed inside the conductor and is referred to as eddy current.

Eddy currents occur in a wide range of electromagnetic apparatus and devices (synchronous machines, transformers, fusion machines) and have several applications (magnetic levitation, non destructive testing, biomedicine, induction heating…).

The effect of the eddy currents can be beneficial or dangerous. For example, they are necessary for the operation of the electric machines, such as transformers, but eddy currents also cause harmful effects in them: the circulating currents dissipate energy through ohmic losses and they generate a reaction magnetic field in the opposite direction of the inducting field leading to a decrease in the machine efficiency. Also, eddy currents generate a non-uniform current distribution in the conductors cross section causing additional losses.

On the other hand, eddy currents can have beneficial effects: they can be used in metallurgy for induction heating by using the heat generation for melting metal objects or to control the motion of molten metals. They have application in the magnetically levitated trains and can be used on non destructive testing to get information about the materials homogeneity and dimension they flow through.

A good knowledge of space and time distribution of both electric current and magnetic flux is necessary in order to obtain efficient and economical designs of all the apparatus where eddy currents are involved.

The interaction between electrical and magnetic field is described by the Maxwell's equations. Basically, there are two ways to solve this system of equations: analytical and numerical methods. Unfortunately, analytical solutions are possible only for problems involving simple two-dimensional geometries, linear media and steady state problems. Three dimensional, nonlinear and transient problems are very difficult to be solved analytically and algorithms exist only for specific problems [10].

The limitations of analytical methods can be overcome by using numerical techniques, which have been developed in the last 30-40 years with the digital computing advent. Numerical methods can handle any geometrical configuration, time-depending problems and both linear and non-linear cases. The numerical approach consists of dividing the field

region in discrete elements and re-formulating the problem in each element. There are several numerical methods: the finite difference method, the boundary element method, the moment method, and so on. The most powerful is the finite element method because of its major flexibility.

Basically, the solution of the electromagnetic problem can be found re-formulating the Maxwell's equations in terms of potentials. Potentials are auxiliary functions that express various physical properties in ways leading a more workable solution to a problem involving partial differential equations. The potentials can be chosen in a wide variety of ways, and this choice will affect the computational procedure of the field problem leading advantages and disadvantages. Numerical analysts, mathematicians and engineers are involved in the study of various formulations in order to construct efficient and economical numerical algorithms since the system of equation to be solved is usually very large.

Three dimensional eddy current problems can be mathematically formulated in various ways. The variable to be solved may be a vector potential, a scalar potential or a combination of those.

Several finite element commercial codes are available on the market. The formulations they implement are not ever generally applicable to all kind of problems and not ever are the most convenient in terms of availability and power of computing resources.

One of them is ANSYS, which is maybe the most powerful and wide used finite element commercial package.

For the eddy current study, ANSYS implements the magnetic vector potential formulation which uses in the non-conducting regions three degree of freedom, the magnetic vector potential components, and adds an extra degree of freedom, the time-integrated electric voltage, in the conducting regions. This is the most widespread formulation in all the electromagnetic finite element commercial packages.

This work deals with the implementation of a new formulation (the T-Ω,Ω) in ANSYS by using its customization capabilities for creating new element types and adding them in the ANSYS library. The goal of this work has been to implement in ANSYS a simple and economical method for calculating 3-D eddy currents reducing the number of degrees of freedom, from three to one, in the non-conducting regions. On the other hand, the implementation of the new formulation in ANSYS allows to take advantage of the excellent and several capabilities of the commercial code itself: mesh generation, post-processing, graphics window, optimization, coupling and so on.

1. Formulation for the eddy current problem

1.1 Problem definition

The interaction between magnetic fields and electrical phenomena is described by the following subset of Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} \qquad [1.1.1]$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad [1.1.2]$$

$$\nabla \cdot \vec{B} = 0 \qquad [1.1.3]$$

In eq. [1.1.1] the displacement current term is neglected; in fact, if the dimension of the eddy current regions are small compared with the wavelength of the prescribed fields, the $\partial \vec{D} / \partial t$ term can be neglected because it is sufficiently small. The first equation is the Ampere's law, where $\vec{H}$ is the magnetic intensity of the field produced by a current density $\vec{J}$, the second is the Faraday's law, where $\vec{E}$ is the electric field produced by a changing magnetic flux density $\vec{B}$ and the third is the Gauss's law for magnetism, which reflects the fact that there are no point source of magnetic flux.

The field vectors are not independent since they are further related by the material constitutive relationship:

$$\vec{B} = \mu \cdot \vec{H} \qquad [1.1.4]$$

$$\vec{J} = \sigma \cdot \vec{E} \qquad [1.1.5]$$

where μ and σ are the material permeability and conductivity. They may be field dependent and may vary in space, hysteresis and anisotropy are here neglected.

For the solution of the problem we consider also the continuity condition:

$$\nabla \cdot \vec{J} = 0 \qquad [1.1.6]$$

which states the current density solenoidality.

A typical eddy current problem consists (Fig. 1.1.1) of an eddy current region Ω_1 with non zero conductivity σ_1 and magnetic permeability μ_1 bounded by a surface S_{12} and a surrounding region Ω_2 free of eddy currents, which may contain source currents $\vec{J}_s$, bounded by a surface S_o which may be extended to infinity. The whole problem domain is done by the sum of Ω_1 and Ω_2 and will be denoted by Ω_0. The S_o surface can be divided into two parts in accordance of the two types of boundary condition of practical importance: on S_B the normal component of the flux density is prescribed, on S_H the tangential component of the

magnetic field intensity is prescribed.

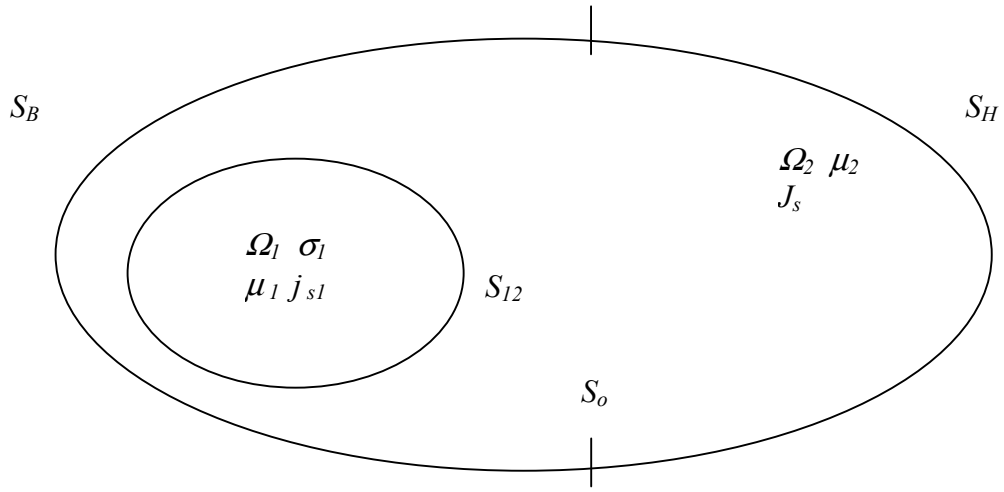


Fig.1.1.1 Electromagnetic field regions

The Ω_0 boundary conditions are related to the field values of the components normal or tangential to the boundaries:

$$\vec{B} \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.1.7]$$

$$\vec{H} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.1.8]$$

Of course also the interface conditions between the conductive and non conductive media must be satisfied:

$$\vec{B}_1 \cdot \hat{n}_1 + \vec{B}_2 \cdot \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.1.9]$$

$$\vec{H}_1 \times \hat{n}_1 + \vec{H}_2 \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.1.10]$$

$$\vec{J} \cdot \hat{n}_1 = 0 \qquad \text{on } S_{12} \quad [1.1.11]$$

where $\hat{n}$ is the outer normal on the corresponding surface. These interface conditions can be easily found expressing eq. [1.1.1], [1.1.3] and [1.1.6]:

$$\nabla \times \vec{H} = \vec{J} \qquad [1.1.1]$$

$$\nabla \cdot \vec{B} = 0 \qquad [1.1.3]$$

$$\nabla \cdot \vec{J} = 0 \qquad [1.1.6]$$

in integral form respectively:

$$\oint_C \vec{H} \cdot dl = I \qquad [1.1.12]$$

$$\oint_S \vec{B} \cdot ds = 0 \qquad [1.1.13]$$

$$\oint_S \vec{J} \cdot ds = 0 \qquad [1.1.14]$$

The first one states that the integral of $\vec{H}$ along a closed path C equals the current I enclosed in it, the other equations state that the flux of $\vec{B}$ and $\vec{J}$ over any closed surface S is zero. We consider the interface between two materials 1 and 2 with different properties and consider the rectangular closed path in Fig. 1.1.2. By applying eq. [1.1.12], as the height of the rectangle goes to zero there will be no contribution from the vertical sides and eq. [1.1.10] must be verified if the surface current density is neglected.

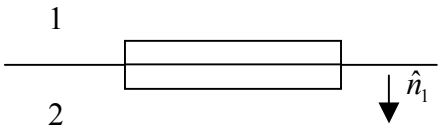


Fig. 1.1.2 Magnetic field at a boundary

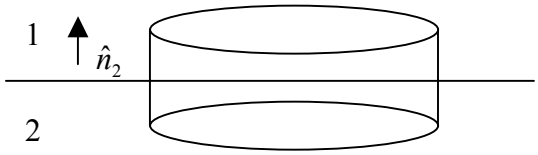


Fig. 1.1.3 Magnetic field and current density flux at a boundary

Also, if we consider the disk in Fig. 1.1.3, by applying eq. [1.1.13] and [1.1.14] as its height goes to zero no flux leaves the volume through the cylinder side and eq. [1.1.9] and [1.1.11] must be verified.

Particular care must be taken in problems involving multiply connected regions. A region Λ is defined as simple connected if any line l belonging to Λ can be reduced in a point by a continuous deformation of l itself. In other words, if any closed line l in Λ has an open surface which belongs to Λ and has l as contour. If it is not, the region Λ is called as multiply connected. The cylinder in Fig. 1.1.4 represents a simple connected region, while the torus in Fig. 1.1.5 is a multiply connected region.

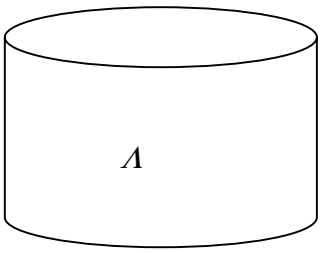


Fig. 1.1.4 Simply connected region

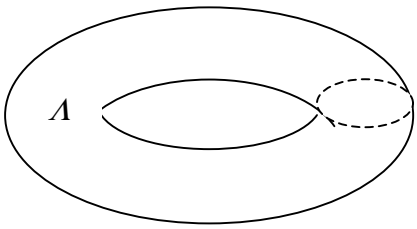


Fig. 1.1.5 Multiply connected region

1.2 The *T-Ω, Ω* formulation

Maxwell's equations are first order coupled differential equations which can be very difficult to solve in boundary values problems. The way for reducing mathematical complexity is therefore to formulate problems in terms of potentials. Potentials are auxiliary functions, which are used frequently for calculating electromagnetic fields. They, in fact, permit the construction of efficient and economical numerical algorithms. One of the widespread formulations is the *A,V–A* formulation, which uses the magnetic vector potential $\vec{A}$ both in Ω_1 and Ω_2 as well the electric scalar potential in Ω_1. It is defined expressing the flux density in terms of an auxiliary vector $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A} \qquad [1.2.1]$$

The *A,V–A* formulation together with other useful formulations will be described in detail in the next paragraphs.

This work deals with the *T-Ω, Ω* formulation, which has the advantage to permit a reduction of computing cost by decreasing the degrees of freedom from three to one in all the non-conducting region.

Starting from [1.1.6]

$$\nabla \cdot \vec{J} = 0 \qquad [1.1.6]$$

we can express the current density in terms of an auxiliary vector $\vec{T}$, which is called the electric vector potential:

$$\vec{J} = \nabla \times \vec{T} \,. \qquad [1.2.2]$$

by using the well known vector identity $\nabla \cdot \nabla \times \vec{T} = 0$, where $\vec{T}$ is any sufficiently differentiable vectorial function.

We note that from eq. [1.1.1] $\nabla \times \vec{H} = \vec{J}$ as well, so $\vec{T}$ and $\vec{H}$ differ by the gradient of a scalar and have the same units:

$$\vec{H} = \vec{T} - \nabla\Omega \qquad \text{in } \Omega_1 \quad [1.2.3]$$

where Ω is a magnetic scalar potential.

Using [1.1.2], [1.1.4] and [1.1.5]:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad [1.1.2]$$

$$\vec{B} = \mu \cdot \vec{H} \qquad [1.1.4]$$

$$\vec{J} = \sigma \cdot \vec{E} \qquad [1.1.5]$$

we obtain:

$$\nabla \times \frac{1}{\sigma} \nabla \times \vec{T} + \mu \frac{\partial}{\partial t} \left(\vec{T} - \nabla \Omega \right) = 0 \qquad \text{in } \Omega_1 \ [1.2.4]$$

and from [1.1.3]:

$$\nabla \cdot \vec{B} = 0 \qquad [1.1.3]$$

we obtain:

$$\nabla \cdot \mu \left(\vec{T} - \nabla \Omega \right) = 0 \qquad \text{in } \Omega_1 \ [1.2.5]$$

It should be noted that by taking the divergence of both sides of [1.2.4] the solenoidality of the magnetic flux density [1.2.5] is satisfied and would be at this point superfluous.

In current-free regions the magnetic field can be found from the scalar potential:

$$\vec{H} = -\nabla \Omega \qquad \text{in } \Omega_2 \ [1.2.6]$$

where Ω results from [1.2.5]:

$$-\nabla \cdot \mu \nabla \Omega = 0 \qquad \text{in } \Omega_2 \ [1.2.7]$$

The potential $\vec{T}$ is not yet fully defined because a vector field can be uniquely defined only defining its divergence and its curl, and by determining suitable boundary conditions on the interface between conducting and non conducting regions. The divergence of $\vec{T}$ is not yet defined and consequently $\vec{T}$ and Ω remain ambiguous. Defining the divergence of $\vec{T}$ in addition to its curl is referred to as a choice of gauge. Any of this value may be chosen without affecting the physical problem. On the other hand this choice will affect the computation ease. The best known gauge condition used in electromagnetics is the Coulomb gauge:

$$\nabla \cdot \vec{T} = 0 \qquad [1.2.8]$$

This condition permits to append [2] the left-side of [1.2.4]

$$\nabla \times \frac{1}{\sigma} \nabla \times \vec{T} + \mu \frac{\partial}{\partial t} \left(\vec{T} - \nabla \Omega \right) = 0 \qquad \text{in } \Omega_1 \ [1.2.4]$$

by a term $\nabla \frac{1}{\sigma} \nabla \cdot \vec{T}$:

$$\nabla \times \frac{1}{\sigma} \nabla \times \vec{T} - \nabla \frac{1}{\sigma} \nabla \cdot \vec{T} + \mu \frac{\partial}{\partial t} \left(\vec{T} - \nabla \Omega \right) = 0 \qquad \text{in } \Omega_1 \ [1.2.9]$$

Taking the divergence of [1.2.9]:

$$\nabla\cdot\left(\nabla\times\frac{1}{\sigma}\nabla\times\vec{T}\right)-\nabla^2\frac{1}{\sigma}\nabla\cdot\vec{T}+\frac{\partial}{\partial t}\nabla\cdot\left(\mu\left(\vec{T}-\nabla\Omega\right)\right)=0 \qquad \text{in } \Omega_1 \quad [1.2.10]$$

The first term of [1.2.10]is zero for any vector, the third is zero as well considering [1.1.3],

$$\nabla\cdot\vec{B}=0 \qquad [1.1.3]$$

so the scalar $\frac{1}{\sigma}\nabla\cdot\vec{T}$ satisfies Laplace's equation

$$\nabla^2\frac{1}{\sigma}\nabla\cdot\vec{T}=0 \qquad \text{in } \Omega_1 \quad [1.2.11]$$

If we set the boundary condition $\frac{1}{\sigma}\nabla\cdot\vec{T}=0$ on the interface between the conducting and non-conducting media, then from [1.2.11] we conclude that $\frac{1}{\sigma}\nabla\cdot\vec{T}=0$ in Ω1. This satisfies the gauge condition in the conductor.

The boundary condition of the electric vector potential [7] is determined by [1.1.11]

$$\vec{J}\cdot\hat{n}_1=0 \qquad [1.1.11]$$

and can be expressed as:

$$\hat{n}_1\times\vec{T}=0 \qquad [1.2.12]$$

By taking the divergence of both sides of [1.2.9] the solenoidality of the magnetic flux density is no more satisfied and it must be enforced.

The interface conditions [1.1.9], [1.1.10]

$$\vec{B}_1\cdot\hat{n}_1+\vec{B}_2\cdot\hat{n}_2=0 \qquad [1.1.9]$$

$$\vec{H}_1\times\hat{n}_1+\vec{H}_2\times\hat{n}_2=0 \qquad [1.1.10]$$

must be satisfied in the interface surface between the conducting and non conducting media.

Substituting equations [1.1.4], [1.2.3], [1.2.6]

$$\vec{B}=\mu\cdot\vec{H} \qquad [1.1.4]$$

$$\vec{H}=\vec{T}-\nabla\Omega \qquad [1.2.3]$$

$$\vec{H}=-\nabla\Omega \qquad [1.2.6]$$

in [1.1.9] and [1.1.10] gives:

$$\mu\cdot\left(\vec{T}_1-\nabla\Omega_1\right)\cdot\hat{n}_1+\mu\cdot\left(-\nabla\Omega_1\right)\cdot\hat{n}_2=0 \qquad \text{on } S_{12} \quad [1.2.13]$$

$$\left(\vec{T}_1-\nabla\Omega_1\right)\times\hat{n}_1+\left(-\nabla\Omega_1\right)\times\hat{n}_2=0 \qquad \text{on } S_{12} \quad [1.2.14]$$

Also the boundary conditions [1.1.7] and [1.1.8]

$$\vec{B}\cdot\hat{n}=0 \qquad \text{on } S_B \quad [1.1.7]$$

$$\vec{H} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.1.8]$$

must be satisfied, by using [1.2.3], [1.2.6]

$$\vec{H} = \vec{T} - \nabla\Omega \qquad \text{in } \Omega_1 \quad [1.2.3]$$

$$\vec{H} = -\nabla\Omega \qquad \text{in } \Omega_2 \quad [1.2.6]$$

the first one can be written as:

$$\mu \cdot \left(\vec{T} - \nabla\Omega\right) \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.2.15]$$

$$\mu \cdot \left(-\nabla\Omega\right) \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.2.16]$$

for conducting and non-conducting regions respectively, the second one is:

$$\left(\vec{T} - \nabla\Omega\right) \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.2.17]$$

$$\left(-\nabla\Omega\right) \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.2.18]$$

for conducting and non-conducting regions respectively.

In the discretization by the finite element method [1.2.13] is satisfied implicitly, the continuity of the scalar potential together with [1.2.12] ensures that [1.2.14]is satisfied too.

The differential equations [1.2.5], [1.2.7] and [1.2.9],

$$\nabla \cdot \mu\left(\vec{T} - \nabla\Omega\right) = 0 \qquad \text{in } \Omega_2 \quad [1.2.5]$$

$$-\nabla \cdot \mu\nabla\Omega = 0 \qquad \text{in } \Omega_2 \quad [1.2.7]$$

$$\nabla \times \frac{1}{\sigma}\nabla \times \vec{T} - \nabla\frac{1}{\sigma}\nabla \cdot \vec{T} + \mu\frac{\partial}{\partial t}\left(\vec{T} - \nabla\Omega\right) = 0 \qquad \text{in } \Omega_1 \quad [1.2.9]$$

the interface conditions[1.2.13] and [1.2.14],

$$\mu \cdot \left(\vec{T}_1 - \nabla\Omega_1\right) \cdot \hat{n}_1 + \mu \cdot \left(-\nabla\Omega_1\right) \cdot \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.2.13]$$

$$\left(\vec{T}_1 - \nabla\Omega_1\right) \times \hat{n}_1 + \left(-\nabla\Omega_1\right) \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.2.14]$$

the boundary conditions [1.2.15], [1.2.16], [1.2.17] and [1.2.18]

$$\mu \cdot \left(\vec{T} - \nabla\Omega\right) \cdot \hat{n} = 0 \qquad \text{on } S_B \text{ and } \sigma \neq 0 \quad [1.2.15]$$

$$\mu \cdot \left(-\nabla\Omega\right) \cdot \hat{n} = 0 \qquad \text{on } S_B \text{ and } \sigma = 0 \quad [1.2.16]$$

$$\left(\vec{T} - \nabla\Omega\right) \times \hat{n} = 0 \qquad \text{on } S_H \text{ and } \sigma \neq 0 \quad [1.2.17]$$

$$\left(-\nabla\Omega\right) \times \hat{n} = 0 \qquad \text{on } S_H \text{ and } \sigma = 0 \quad [1.2.18]$$

are the *T-Ω, Ω* formulation equations.

1.3 The *A,V-A* formulation

The *A,V–A* formulation uses as degrees of freedom the magnetic vector potential $\vec{A}$ both in Ω_1 and Ω_2 as well the electric scalar V potential in Ω_1.

Starting from [1.1.3]

$$\nabla \cdot \vec{B} = 0 \qquad [1.1.3]$$

we can express the flux density in terms of an auxiliary vector $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A} \qquad [1.3.1]$$

by using the well known vector identity $\nabla \cdot \nabla \times \vec{A} = 0$.

With this substitution equation [1.1.2]

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad [1.1.2]$$

becomes:

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \qquad [1.3.2]$$

and using the second vector useful identity $\nabla \times \nabla V = 0$, where V is any sufficiently differentiable scalar function, we can write:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V \qquad [1.3.3]$$

The variables $\vec{A}$ and V are usually referred to as the magnetic vector potential and the electric scalar potential. Substituting [1.3.1] and [1.3.3] in [1.1.1], [1.1.4] and [1.1.5]

$$\nabla \times \vec{H} = \vec{J} \qquad [1.1.1]$$

$$\vec{B} = \mu \cdot \vec{H} \qquad [1.1.4]$$

$$\vec{J} = \sigma \cdot \vec{E} \qquad [1.1.5]$$

results:

$$\nabla \times \nu \nabla \times \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.4]$$

$$\nabla \times \nu \nabla \times \vec{A} = \vec{J}_s \qquad \text{in } \Omega_2 \quad [1.3.5]$$

The equation [1.3.4] automatically satisfies the divergence-free property of the current density in Ω_1 and would be superfluous to specify it.

The boundary conditions [1.1.7], [1.1.8], [1.1.9], [1.1.10]

$$\vec{B}\cdot\hat{n}=0 \qquad \text{on } S_B \quad [1.1.7]$$

$$\vec{H}\times\hat{n}=0 \qquad \text{on } S_H \quad [1.1.8]$$

$$\vec{B}_1\cdot\hat{n}_1+\vec{B}_2\cdot\hat{n}_2=0 \qquad \text{on } S_{12} \quad [1.1.9]$$

$$\vec{H}_1\times\hat{n}_1+\vec{H}_2\times\hat{n}_2=0 \qquad \text{on } S_{12} \quad [1.1.10]$$

are now formulated in terms of the magnetic vector potential as:

$$\hat{n}\cdot\nabla\times\vec{A}=0 \qquad \text{on } S_B \quad [1.3.6]$$

$$\nu\nabla\times\vec{A}\times\hat{n}=0 \qquad \text{on } S_H \quad [1.3.7]$$

$$\hat{n}_1\cdot\nabla\times\vec{A}_1+\hat{n}_2\cdot\nabla\times\vec{A}_2=0 \qquad \text{on } S_{12} \quad [1.3.8]$$

$$\nu\nabla\times\vec{A}_1\times\hat{n}_1+\nu\nabla\times\vec{A}_2\times\hat{n}_2=0 \qquad \text{on } S_{12} \quad [1.3.9]$$

Having $\vec{A}$ itself continuous between the two regions ensures the automatic satisfaction of the interface conditions [1.3.8] and [1.3.9].

To fully define the potential $\vec{A}$ we have to define the divergence of $\vec{A}$ in addition to its curl. Also in the *A,V–A* formulation [3] the Coulomb gauge is used :

$$\nabla\cdot\vec{A}=0\,.$$

This condition permits to append the left-side of [1.3.4] and [1.3.4]

$$\nabla\times\nu\nabla\times\vec{A}+\sigma\left\{\frac{\partial\vec{A}}{\partial t}+\nabla V\right\}=0 \qquad \text{in } \Omega_1 \quad [1.3.4]$$

$$\nabla\times\nu\nabla\times\vec{A}=\vec{J}_s \qquad \text{in } \Omega_2 \quad [1.3.5]$$

by a term $-\nabla\nu\nabla\cdot\vec{A}$:

$$\nabla\times\nu\nabla\times\vec{A}-\nabla\nu\nabla\cdot\vec{A}+\sigma\left\{\frac{\partial\vec{A}}{\partial t}+\nabla V\right\}=0 \qquad \text{in } \Omega_1 \quad [1.3.10]$$

$$\nabla\times\nu\nabla\times\vec{A}-\nabla\nu\nabla\cdot\vec{A}=\vec{J}_s \qquad \text{in } \Omega_2 \quad [1.3.11]$$

The divergence-free property of the current density is not automatically satisfied, and must be enforced. Substituting [1.3.3]

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\nabla V \qquad [1.3.3]$$

in [1.1.5]

$$\vec{J}=\sigma\cdot\vec{E} \qquad [1.1.5]$$

and taking the divergence we obtain:

$$\nabla \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.12]$$

Moreover, also the fact that the normal component of the current density along the interface surface is zero must be enforced, substituting [1.3.3] and [1.1.5]in[1.1.11]

$$\vec{J} \cdot \hat{n}_1 = 0 \qquad [1.1.11]$$

gives:

$$n \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.13]$$

Taking the divergence of [1.3.10] and [1.3.11], considering [1.3.12] and the fact that $\vec{J}_s$ is divergence-free we have:

$$\nabla^2 \left(\nu \nabla \cdot \vec{A} \right) = 0 \qquad \text{in } \Omega \quad [1.3.14]$$

To ensure the Coulomb gauge satisfaction in the whole region Ω_o proper boundary conditions must be enforced on $\nu \nabla \cdot \vec{A}$ along the boundary S_o. O. Biro and K. Preis have shown in [3] that the satisfaction of the Coulomb gauge can be ensured by imposing:

$$\frac{\partial}{\partial n} \left(\nu \nabla \cdot \vec{A} \right) = 0 \qquad \text{on } S_H \quad [1.3.15]$$

and:

$$\nu \nabla \cdot \vec{A} = 0 \qquad \text{on } S_B \quad [1.3.16]$$

The uniqueness of the solution depends also by the imposition of suitable boundary conditions on $\vec{A}$ itself along S_o. The boundary condition [1.3.6]

$$\hat{n} \cdot \nabla \times \vec{A} = 0 \qquad \text{on } S_B \quad [1.3.6]$$

can be replaced by the:

$$\hat{n} \times \vec{A} = 0 \qquad \text{on } S_B \quad [1.3.17]$$

The boundary condition[1.3.7] on S_H

$$\nu \nabla \times \vec{A} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.3.7]$$

is then replaced by:

$$\hat{n} \cdot \vec{A} = 0 \qquad \text{on } S_H \quad [1.3.18]$$

On summary, the differential equations [1.3.10], [1.3.11] and [1.3.12],

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.10]}$$

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} = \vec{J}_s \qquad \text{in } \Omega_2 \text{ [1.3.11]}$$

$$\nabla \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.12]}$$

the boundary conditions [1.3.15], [1.3.16] , [1.3.17] and [1.3.18]

$$\nu \nabla \cdot \vec{A} = 0 \qquad \text{on } S_B \text{ [1.3.16]}$$

$$\hat{n} \times \vec{A} = 0 \qquad \text{on } S_B \text{ [1.3.17]}$$

$$\frac{\partial}{\partial n} \left(\nu \nabla \cdot \vec{A} \right) = 0 \qquad \text{on } S_H \text{ [1.3.15]}$$

$$\hat{n} \cdot \vec{A} = 0 \qquad \text{on } S_H \text{ [1.3.18]}$$

the interface condition [1.3.8], [1.3.9] and [1.3.13]

$$\hat{n}_1 \cdot \nabla \times \vec{A}_1 + \hat{n}_2 \cdot \nabla \times \vec{A}_2 = 0 \qquad \text{on } S_{12} \text{ [1.3.8]}$$

$$\nu \nabla \times \vec{A}_1 \times \hat{n}_1 + \nu \nabla \times \vec{A}_2 \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \text{ [1.3.9]}$$

$$n \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } S_{12} \text{ [1.3.13]}$$

are the *A,V-A* formulation equations.

1.4 The $A,V\text{-}\Psi$ formulation

The $A,V\text{-}\Psi$ formulation permits to reduce the computational cost of the problem solution by using a scalar magnetic potential in the free of eddy currents region Ω_2. In this region, in fact, it is possible to split the magnetic field intensity vector $\vec{H}$ into two parts:

$$\vec{H} = \vec{H}_s + \vec{H}_m \qquad [1.4.1]$$

where $\vec{H}_s$ is the field generated by the sources current in Ω_2 and satisfies [1.1.1]

$$\nabla \times \vec{H} = \vec{J}_s \qquad [1.1.1]$$

thus $\vec{H}_m$ is irrotational

$$\nabla \times \vec{H}_m = 0 \qquad [1.4.2]$$

and can be expressed as the gradient of the reduced magnetic scalar potential ϕ:

$$\vec{H}_m = -\nabla\phi \qquad [1.4.3]$$

and so [1.4.1] becomes:

$$\vec{H} = \vec{H}_s - \nabla\phi \qquad [1.4.4]$$

The field $\vec{H}_s$ can be computed in any point by using the Biot-Savart law:

$$\vec{H}_s = \frac{1}{4\pi}\int_{vols} \frac{\vec{J}_s \times \vec{r}}{|r^3|}\, dvol_s \qquad [1.4.5]$$

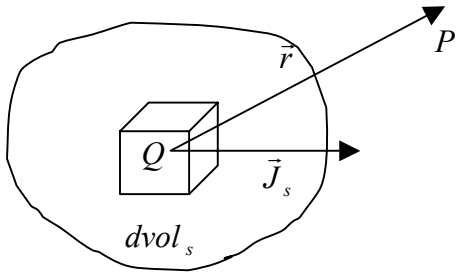


Fig.1.4.1 Biot Savart law application

Where (Fig. 1.4.1):

- $\vec{r}$ position vector from the current source point Q to the node point P
- $\vec{J}_s$ current source density vector in $d(vol_s)$
- vol_s volume where the current source is defined.

Substituting [1.4.5] in [1.1.4]

$$\vec{B} = \mu \cdot \vec{H} \qquad [1.1.4]$$

and taking the divergence gives:

$$\nabla \cdot \mu \nabla \phi = \nabla \cdot \mu \vec{H}_s \qquad [1.4.6]$$

Eq. [1.4.6] is a generalized form of the Poisson equation. In magnetic materials the two parts of the field $\vec{H}_s$ and $\vec{H}_m$ tend to be of similar magnitude but opposite in direction, so that cancellation occurs in computing the field intensity $\vec{H}$, giving a loss in accuracy, especially when the magnetic permeability μ is large [4]. This problem can be overcome if the ferromagnetic region is free of source currents. In this case, in fact, the total field can be represented by a scalar potential called as the total magnetic scalar potential Ψ since [1.1.1]:

$$\nabla \times \vec{H} = \vec{J}_s \qquad [1.1.1]$$

can be written as:

$$\nabla \times \vec{H} = 0 \qquad [1.4.7]$$

and thus:

$$\vec{H} = -\nabla \Psi \qquad [1.4.8]$$

Substituting [1.4.8] in [1.1.4]

$$\vec{B} = \mu \cdot \vec{H} \qquad [1.1.4]$$

and taking the divergence gives:

$$\nabla \cdot \mu \nabla \Psi = 0 \qquad \text{in } \Omega_2 \quad [1.4.9]$$

Eq. [1.4.9] is a generalized form of the Laplace equation.

In the region Ω_1 which is described by the magnetic vector potential [1.3.4] is still valid

$$\nabla \times \nu \nabla \times \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.4]$$

Again, the eq. [1.3.4] automatically satisfies the divergence-free property of the current density in Ω_1 and would be superfluous to specify it.

The boundary conditions [1.1.7], [1.1.8]

$$\vec{B} \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.1.7]$$

$$\vec{H} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.1.8]$$

can now be specified in term of the scalar potential:

$$\mu(\nabla \Psi) \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.4.10]$$

$$\left(-\nabla \Psi\right)\times \hat{n} = 0 \qquad \text{on } S_H \quad [1.4.11]$$

which can also be expressed as:

$$\Psi = 0 \qquad \text{on } S_H \quad [1.4.12]$$

The interface conditions [1.1.9] and[1.1.10]

$$\vec{B}_1 \cdot \hat{n}_1 + \vec{B}_2 \cdot \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.1.9]$$

$$\vec{H}_1 \times \hat{n}_1 + \vec{H}_2 \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.1.10]$$

are now formulated in terms of the magnetic vector and the magnetic scalar potentials as:

$$\hat{n}_1 \cdot \nabla \times \vec{A}_1 - \hat{n}_2 \cdot \mu_2 \left(\nabla \Psi\right) = 0 \qquad \text{on } S_{12} \quad [1.4.13]$$

$$\nu \nabla \times \vec{A}_1 \times \hat{n}_1 - \nabla \Psi \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.4.14]$$

As for the *A,V-A* formulation the uniqueness of the solution depends by the imposition of suitable boundary conditions on $\vec{A}$ itself along the boundary and by a choice of gauge. The same boundary conditions that have been found for the *A,V-A* formulation can be here applied, however it has to be taken into account that $\vec{A}$ is defined only in region Ω_1 and its boundary S_{12} is the interface surface between the conducting and non conducting media.

It can be noted that from [1.4.14] it is possible to define the tangential component of $\vec{H}$ and [1.4.13] can be treated as a boundary condition for the magnetic scalar potential in Ω_2. This means that the interface surface must be treated as S_H in the *A,V-A* formulation where [1.1.8]

$$\vec{H} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.1.8]$$

must be applied. Therefore, to ensure the uniqueness of $\vec{A}$, the normal component of $\vec{A}$ itself must be prescribed on the interface surface and the divergence of $\vec{A}$ must be enforced to be zero in Ω_1 (Coulomb gauge).

By rewriting [1.3.10] and [1.3.12]

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.10]$$

$$\nabla \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.12]$$

Taking the divergence of [1.3.10], considering [1.3.12] we have:

$$\nabla^2 \left(\nu \nabla \cdot \vec{A}\right) = 0 \qquad \text{in } \Omega_1 \quad [1.3.14]$$

Taking the normal component of [1.3.10] along the interface surface yields to:

$$n \cdot \nabla \times \nu \nabla \times \vec{A} - \frac{\partial}{\partial n} \nu \nabla \cdot \vec{A} + n \cdot \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{on } S_{12} \text{ [1.4.15]}$$

the first term is zero by using [1.3.7]

$$\nu \nabla \times \vec{A} \times \hat{n} = 0 \qquad \text{on } S_H \quad [1.3.7]$$

and because it can be rearranged [3] as

$$n \cdot \nabla \times \nu \nabla \times \vec{A} = -\nabla \cdot \left(n \times \nabla \times \nu \nabla \times \vec{A} \right) \qquad \text{on } S_{12} \text{ [1.4.16]}$$

the third terms is zero from [1.3.13]

$$n \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \quad [1.3.13]$$

and so[1.4.15]can be rewritten as:

$$\frac{\partial}{\partial n} \nu \nabla \cdot \vec{A} = 0 \qquad \text{on } S_{12} \text{ [1.4.17]}$$

Eq. [1.3.14] with [1.4.17] ensures that $\nu \nabla \cdot \vec{A} = cost$ in Ω_1 , since the normal component of $\vec{A}$ is supposed to be zero on S_{12}, this constant can only be zero. This ensure the Coulomb gauge satisfaction.

On summary, the differential equations [1.3.10], [1.4.9] and [1.3.12],

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.10]}$$

$$\nabla \cdot \mu \nabla \Psi = 0 \qquad \text{in } \Omega_2 \quad [1.4.9]$$

$$\nabla \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.12]}$$

the boundary conditions [1.4.10], [1.4.12] ,

$$\mu (\nabla \Psi) \cdot \hat{n} = 0 \qquad \text{on } S_B \quad [1.4.10]$$

$$\Psi = 0 \qquad \text{on } S_H \quad [1.4.12]$$

the interface conditions [1.3.13], [1.3.18] , [1.3.17] and [1.3.18]

$$\hat{n}_1 \cdot \nabla \times \vec{A}_1 - \hat{n}_2 \cdot \mu_2 (\nabla \Psi) = 0 \qquad \text{in } S_{12} \quad [1.4.13]$$

$$\nu \nabla \times \vec{A}_1 \times \hat{n}_1 - \nabla \Psi \times \hat{n}_2 = 0 \qquad \text{in } S_{12} \quad [1.4.14]$$

$$n \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } S_{12} \quad [1.3.13]$$

$$\hat{n} \cdot \vec{A} = 0 \qquad \text{on } S_{12} \quad [1.3.18]$$

are the A,V-Ψ formulation equations.

1.5 The *A,V-A-Ψ* formulation

The magnetic scalar potential can not be used in the non conducting region if the conducting region is multiply connected. This problem can be overcome if in the non conducting holes the magnetic vector potential formulation is used.

In Fig. 1.5.1 the conducting region Ω_1 is multiply connected and region Ω_3 , where the magnetic vector potential formulation hold, is selected to have in conjunction with Ω_1 a simple connected region.

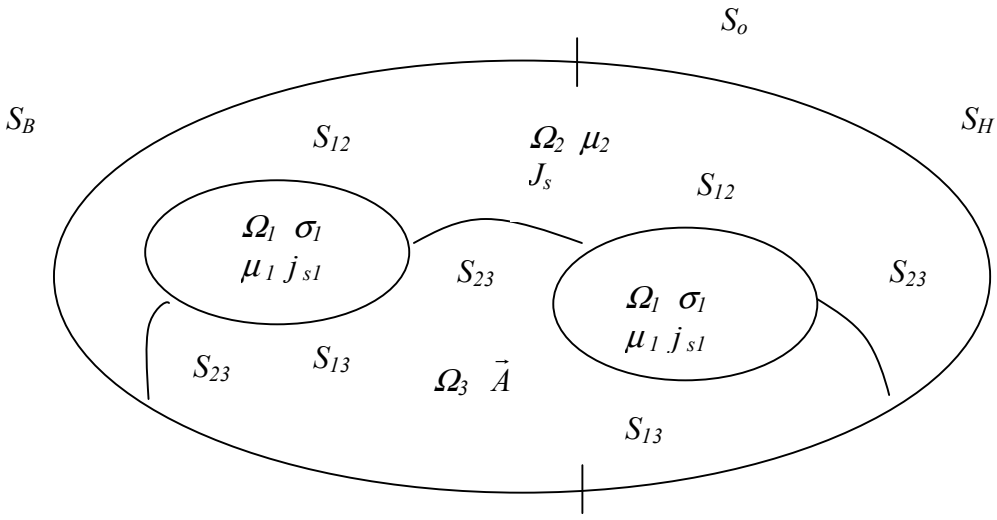


Fig.1.5.1 Electromagnetic field regions with multiply connected conductor

So, the magnetic scalar formulation can be used in the non conducting region Ω_2, which now surrounds a simple connected region done by the sum of Ω_1 and Ω_3 .

The above considerations yield in a mixed formulation combining the *A,V-A* and the *A,V-Ψ* formulations and the equation to be solved came from them. This equations are here summarized.

The differential equations [1.3.10], [1.3.11], [1.4.9] and [1.3.12],

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} + \sigma \left\{ \frac{\partial \vec{A}}{\partial t} + \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.10]}$$

$$\nabla \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{in } \Omega_1 \text{ [1.3.12]}$$

$$\nabla \cdot \mu \nabla \Psi = 0 \qquad \text{in } \Omega_2 \text{ [1.4.9]}$$

$$\nabla \times \nu \nabla \times \vec{A} - \nabla \nu \nabla \cdot \vec{A} = 0 \qquad \text{in } \Omega_3 \text{ [1.3.11]}$$

the boundary conditions [1.3.15], [1.3.16], [1.3.17], [1.3.18], [1.4.10] and [1.4.12] ,

$$\nu \nabla \cdot \vec{A} = 0 \qquad \text{on } S_{B3} \text{ [1.3.16]}$$

$$\hat{n} \times \vec{A} = 0 \qquad \text{on } S_{B3} \text{ [1.3.17]}$$

$$\frac{\partial}{\partial n} \left(\nu \nabla \cdot \vec{A} \right) = 0 \qquad \text{on } S_{H3} \text{ [1.3.15]}$$

$$\hat{n} \cdot \vec{A} = 0 \qquad \text{on } S_{H3} \text{ [1.3.18]}$$

$$\mu \left(\nabla \Psi \right) \cdot \hat{n} = 0 \qquad \text{on } S_{B2} \text{ [1.4.10]}$$

$$\Psi = 0 \qquad \text{on } S_{H2} \text{ [1.4.12]}$$

the interface conditions [1.3.13], [1.3.18] , [1.3.17] and [1.3.18]

$$\hat{n}_1 \cdot \nabla \times \vec{A}_1 + \hat{n}_2 \cdot \nabla \times \vec{A}_2 = 0 \qquad \text{on } S_{12} \quad [1.3.8]$$

$$\nu \nabla \times \vec{A}_1 \times \hat{n}_1 + \nu \nabla \times \vec{A}_2 \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \quad [1.3.9]$$

$$\hat{n}_{13} \cdot \nabla \times \vec{A}_1 - \hat{n}_2 \cdot \mu_2 (\nabla \Psi) = 0 \qquad \text{on } S_{12} \text{ and } S_{23} \quad [1.4.13]$$

$$\nu \nabla \times \vec{A}_1 \times \hat{n}_{13} - \nabla \Psi \times \hat{n}_2 = 0 \qquad \text{on } S_{12} \text{ and } S_{23} \quad [1.4.14]$$

$$n \cdot \left\{ -\sigma \frac{\partial \vec{A}}{\partial t} - \sigma \nabla V \right\} = 0 \qquad \text{on } S_{12} \text{ and } S_{13} \quad [1.3.13]$$

$$\hat{n} \cdot \vec{A} = 0 \qquad \text{on } S_{12} \text{ and } S_{23} \quad [1.3.18]$$

are the *A,V-A-*Ψ formulation equations.

2. The *T-Ω, Ω* finite element formulation

The finite element method is a numerical approach by which general differential equations can be solved in an approximate manner. It's a characteristic feature of the finite element method to divide the field region where the differential hold into a number of finite elements which have the same dimension (one, two or three-dimensional) of the problem to be solved (Fig. 2.1.1). Each element has a certain number of nodes which are often located at its boundaries. The collection of all elements and nodes is called as the finite element mesh.

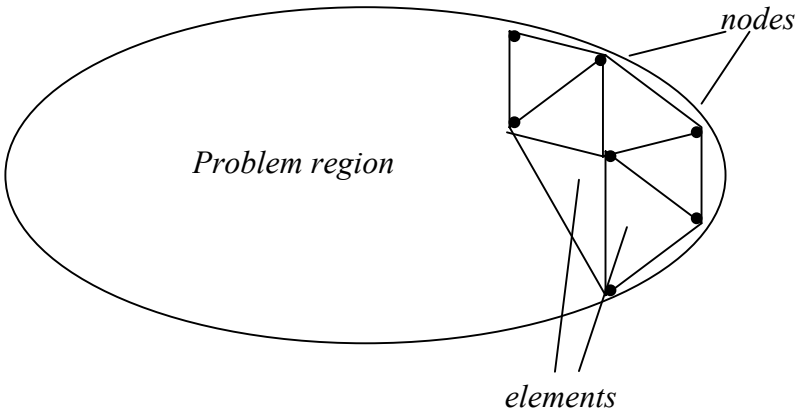


Fig. 2.1.1 Problem region discretization

Instead of seeking the approximations that hold directly the whole region, the approximation is carried out over each element. The unknown field quantity is represented within each element as a combination of interpolatory functions known as shape functions. Such functions are chosen in a relative simple form by using polynomial expansions. In fact, polynomial functions give an excellent description of the field under investigation, simultaneously permitting the process of integration and differentiation. A relationship involving the unknown field quantity at the nodal points is then obtained for the problem formulation for a typical element. The number of nodes for each element define the variation law of the shape functions: linear, quadratic, cubic. The next step is the assembly of the element equations to obtain the equations for the overall system. The imposition of the boundary conditions leads to the final system of equations to be solved. The solution can be achieved by minimizing a functional with respect to each of the nodal potentials or applying directly the Galerkin [3] procedure. Once the problem of finding the potential in each

element node is solved it is possible to compute other desired quantities and represent them in tabular or graphical form.

2.1 Finite element representation of the scalar Ω and vector T potentials

The finite element implementation of the *T-Ω, Ω* formulation has been carried out by using four-noded, first-order, tetrahedral elements. A tetrahedral element in the global *x, y, z* system is shown in Fig. 2.1.1. The numbers on the element indicate the local numeration of the nodes. Inside each element the magnetic scalar potential Ω and the electric vector potential $\vec{T}$ can be expressed by a linear combination of the shape functions associated with the nodes.

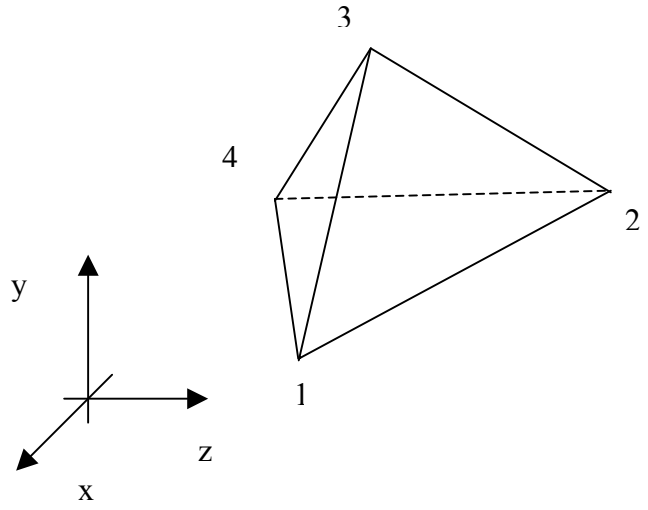


Fig. 2.1.1 Finite tetrahedral element

Within an element the scalar potential Ω is approximated as:

$$\Omega = \sum_{i=1}^{m} \Omega i \cdot Ni \qquad [2.1.1]$$

where *Ni* is the nodal shape function corresponding to node *i*. The index *m* is the number of the element nodes and *m = 4* for tetrahedral elements. The coefficient Ωi is the degree of freedom and it is the value of the magnetic scalar potential Ω on the node *i*.

The electric scalar potential $\vec{T}$ is treated as three scalar components, $\vec{T}x$, $\vec{T}y$, $\vec{T}z$ in the Cartesian co-ordinate system. Each node then has three degrees of freedom instead of one. In each element the electric vector potential $\vec{T}$ can be approximated as:

$$\vec{T} = \sum_{i=1}^{m} \vec{Ti} \cdot Ni = \sum_{i=1}^{m} \left(Txi \cdot \vec{x} + Tyi \cdot \vec{y} + Tzi \cdot \vec{z} \right) \cdot Ni \qquad [2.1.2]$$

where the coefficient $\vec{Ti}$ is the value of the electric vector potential $\vec{T}$ on the node i, Txi, Tyi, Tzi are the components of $\vec{Ti}$.

For a tetrahedral element the shapes functions are defined as:

$$Ni = \frac{ai + bi \cdot x + ci \cdot y + di \cdot z}{6 \cdot vol} \qquad [2.1.3]$$

where all the parameters depend on the element node co-ordinates.

The element volume multiplied by six [5] is given by the determinant of the coefficient matrix:

$$6 \cdot vol = \begin{vmatrix} 1 & x1 & y1 & z1 \\ 1 & x2 & y2 & z2 \\ 1 & x3 & y3 & z3 \\ 1 & x4 & y4 & z4 \end{vmatrix}$$

while the $a1$, $b1$, $c1$, $d1$ constants can be obtained calculating the cofactors of the coefficient matrix:

$$a1 = \begin{vmatrix} x2 & y2 & z2 \\ x3 & y3 & z3 \\ x4 & y4 & z4 \end{vmatrix} \quad b1 = -\begin{vmatrix} 1 & y2 & z2 \\ 1 & y3 & z3 \\ 1 & y4 & z4 \end{vmatrix} \quad c1 = \begin{vmatrix} 1 & x2 & z2 \\ 1 & x3 & z3 \\ 1 & x4 & z4 \end{vmatrix} \quad d1 = -\begin{vmatrix} 1 & x2 & y2 \\ 1 & x3 & y3 \\ 1 & x4 & y4 \end{vmatrix}$$

the other coefficients are derived by cycling interchange of the subscripts.

2.2 The Galerkin's method

The finite element method uses variational formulations or weighted residual methods to solve boundary-value problems.

In the variational method the partial differential equations of the field problem is formulated in terms of an equivalent energy related expression called a functional, which in some applications may represent the stored energy or the dissipated power in the system.

This functional has the property of being stationary with the correct set of function representing the required solution of the problem. The solution of the field problem is then obtained by minimizing the functional with respect to a set of trial solutions.

The weighted residual method is a more general and a more universally applicable method than the variational approach because there are a wide number of cases in which the variational expression does not exist and the weighted residual method can be applied. The Galerkin's method is a special case of the weighted residual methods; in such kind of methods a weighted error on the solution domain has to be minimized.

Let us assume that the governing partial differential equation of the electromagnetic system we are dealing with is of the form:

$$Lu = f \qquad [2.2.1]$$

where L is a differential operator, f is a known function (the excitation) and u is the unknown function. When L is chosen, it specifies the actual form of the differential equation given by [2.2.1]. In the weighted residual methods the unknown function u is approximated in terms of a linear combination of n functions called as basis or trial functions.

$$u \cong u^{app} = \sum_{i=1}^{n} \Phi i \cdot qi \qquad [2.2.2]$$

Here $q1 \ldots qn$ are unknown parameters and $\Phi 1 \ldots \Phi n$ are the trial functions, each defined in the domain of L. The equation [2.2.1] may be taken as an element-wise approximation, in this case $q1 \ldots qn$ would be the values of u at the nodal points and the trial functios would become the shape functions: this is the application of the weighted residual method to the finite element method. Substituting [2.2.2] in [2.2.1] we obtain a residual:

$$R = Lu^{app} - f \qquad [2.2.3]$$

since $u \neq u^{app}$. To determine the coefficients $q1 \ldots qn$ so that u^{app} is a approximation of u the residual is forced to be zero, in an average sense, by setting weighted integrals of the residual equal to zero, i.e:

$$\langle w_m, R \rangle = 0 \qquad [2.2.4]$$

where w_m ($m = 1 \ldots n$) is a set of weighting functions and the notation $\langle w_m, R \rangle$ indicates an inner product defined as $\int w_m R dv = 0$ in the domain solution. Assuming L is linear and substituting [2.2.2] and [2.2.3] in [2.2.4] we obtain:

$$\left\langle w_m, \sum_{i=1}^{n} qiL\Phi i - f \right\rangle = 0 \quad \text{or}$$

$$\sum_{i=1}^{n} qi \left\langle w_m, \sum_{i=1}^{n} L\Phi i \right\rangle = \langle w_m, f \rangle \qquad [2.2.5]$$

The set of equations given by [2.2.5] can be cast into a matrix form as:

$$[K]\,[q] = [C] \qquad [2.2.6]$$

where:

$$[K] = \begin{bmatrix} \int w_1 L\Phi_1 dv & \int w_1 L\Phi_2 dv & ..... & \int w_1 L\Phi_n dv \\ \int w_2 L\Phi_1 dv & \int w_2 L\Phi_2 dv & ..... & \int w_2 L\Phi_n dv \\ . & . & ..... & . \\ . & . & & . \\ \int w_n L\Phi_1 dv & \int w_n L\Phi_2 dv & .... & \int w_n L\Phi_n dv \end{bmatrix} \qquad [C] = -\begin{bmatrix} \int w_1 f dv \\ \int w_2 f dv \\ . \\ . \\ \int w_n f dv \end{bmatrix}$$

and [q] is the column vector of the coefficients *q1…qn.* The equation [2.2.6] consists of n linear equations from which the vector [q] can be determined, once [q] has been obtained eq. [2.2.2] provides the required approximate solution.

A special case of the above described weighted residual method is the Galerkin's method in which the weighting functions are chosen to be identical to the basis functions.

2.3 Discretization of the *T-Ω, Ω* formulation equations

The Galerkin's form of equations [1.2.9], [1.2.5] and [1.2.7]

$$\nabla\times\frac{1}{\sigma}\nabla\times\vec{T}-\nabla\frac{1}{\sigma}\nabla\cdot\vec{T}+\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)=0 \qquad [1.2.9]$$

$$\nabla\cdot\mu\left(\vec{T}-\nabla\Omega\right)=0 \qquad [1.2.5]$$

$$-\nabla\cdot\mu\nabla\Omega=0 \qquad [1.2.7]$$

is:

$$\int_{vol}\boldsymbol{N}_i\cdot\left(\nabla\times\frac{1}{\sigma}\nabla\times\vec{T}-\nabla\frac{1}{\sigma}\nabla\cdot\vec{T}+\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)\right)dvol=0 \qquad [2.3.1]$$

$$\int_{vol}N_i\left(\nabla\cdot\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)\right)dvol=0 \qquad [2.3.2]$$

$$\int_{vol}N_i\left(\nabla\cdot\mu\frac{\partial}{\partial t}\left(-\nabla\Omega\right)\right)dvol=0 \qquad [2.3.3]$$

By using the divergence of cross product rule:

$$\nabla\cdot\left(\vec{P}\times\vec{F}\right)=\left(\nabla\times\vec{P}\right)\cdot\vec{F}-\vec{P}\cdot\left(\nabla\times\vec{F}\right) \qquad [2.3.4]$$

where $\vec{P}$ and $\vec{F}$ are vector functions of position, integrating over the volume and applying the divergence theorem to the left-hand side and rearranging, gives:

$$\int_{vol}\vec{P}\cdot\left(\nabla\times\vec{F}\right)dvol=\int_{vol}\left(\nabla\times\vec{P}\right)\cdot\vec{F}dvol-\oint_{S}\left(\vec{P}\times\vec{F}\right)\cdot\hat{n}ds \qquad [2.3.5]$$

where S is a closed surface with outward normal $\hat{n}$. Making the substitution $\vec{F}=\nabla\times\vec{Q}$ eq. [2.3.1] can be written as:

$$\int_{vol}\vec{P}\cdot\left(\nabla\times\nabla\times\vec{Q}\right)dvol=\int_{vol}\left(\nabla\times\vec{P}\right)\cdot\left(\nabla\times\vec{Q}\right)dvol-\oint_{S}\left(\vec{P}\times\nabla\times\vec{Q}\right)\cdot\hat{n}ds \qquad [2.3.6]$$

The first term on the left side of eq. [2.3.1] can be rewritten from [2.3.6] by substituting $\vec{P}$ with the shape functions $\boldsymbol{N}_i$ and $\nabla\times\vec{Q}$ with $\frac{1}{\sigma}\nabla\times\vec{T}$:

$$\int_{vol}\boldsymbol{N}_i\cdot\left(\nabla\times\frac{1}{\sigma}\nabla\times\vec{T}\right)dvol=\int_{vol}\left(\nabla\times\boldsymbol{N}_i\right)\cdot\left(\frac{1}{\sigma}\nabla\times\vec{T}\right)dvol-\oint_{S12}\boldsymbol{N}_i\times\left(\frac{1}{\sigma}\nabla\times\vec{T}\right)\cdot\hat{n}_{12}ds \qquad [2.3.7]$$

The second term on the left side of equation can be rewritten by using the Green Gauss theorem:

$$\int_{vol} u\nabla\cdot\vec{Q}dvol = -\int_{vol}(\nabla u)\cdot\vec{Q}dvol + \oint_{S} u\vec{Q}\cdot\hat{n}ds \qquad [2.3.8]$$

where u and $\vec{Q}$ are sufficiently differentiable function of position. Substituting $\vec{Q}$ with $\boldsymbol{N}_i$ and u with $\frac{1}{\sigma}\nabla\cdot\vec{T}$ gives:

$$\int_{vol}\left(\frac{1}{\sigma}\nabla\cdot\vec{T}\right)\nabla\cdot N_i dvol = -\int_{vol}\left(\nabla\frac{1}{\sigma}\nabla\cdot\vec{T}\right)\cdot N_i dvol + \oint_{S}\left(\frac{1}{\sigma}\nabla\cdot\vec{T}\right)N_i\cdot\hat{n}ds \qquad [2.3.9]$$

and rearranging:

$$-\int_{vol} N_i\cdot\left(\nabla\frac{1}{\sigma}\nabla\cdot\vec{T}\right)dvol = \int_{vol}\nabla\cdot N_i\left(\frac{1}{\sigma}\nabla\cdot\vec{T}\right)dvol + \oint_{S12} N_i\cdot\hat{n}\frac{1}{\sigma}\nabla\cdot\vec{T}ds \qquad [2.3.10]$$

Eq. [2.3.1]can now be rewritten as:

$$\begin{aligned}&-\int_{vol}\left((\nabla\times\boldsymbol{N}_i)\cdot\left(\frac{1}{\sigma}\nabla\times\vec{T}\right)+(\nabla\cdot N_i)\left(\frac{1}{\sigma}\nabla\cdot\vec{T}\right)+\boldsymbol{N}_i\cdot\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)\right)dvol\\&-\oint_{S12}\left(N_i\cdot\hat{n}\frac{1}{\sigma}\nabla\cdot\vec{T}\right)ds+\oint_{S12}\left(\boldsymbol{N}_i\times\left(\frac{1}{\sigma}\nabla\times\vec{T}\right)\cdot\hat{n}_{12}\right)ds\end{aligned} \qquad [2.3.11]$$

Equations [2.3.2] and [2.3.3] can be rewritten by using the vector integration by parts rule:

$$-\int_{vol}\nabla N_i\cdot\left(\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)\right)dvol + \oint\mu\frac{\partial}{\partial t}\left(\vec{T}-\nabla\Omega\right)\cdot\hat{n} = 0 \qquad [2.3.12]$$

$$-\int_{vol}\nabla N_i\cdot\left(\mu\frac{\partial}{\partial t}\left(-\nabla\Omega\right)\right)dvol + \oint\mu\frac{\partial}{\partial t}\left(-\nabla\Omega\right)\cdot\hat{n} = 0 \qquad [2.3.13]$$

by imposing the interface conditions on the normal components of $\vec{B}$ and $\vec{J}$ and on the tangential $\vec{H}$ and the condition $\hat{n}_1\times\vec{T}=0$ on the interface surface between the conducting and non-conducting media all the surface integrals disappear and Eq. [2.3.11] , [2.3.12] and [2.3.13] can be rewritten as:

$$\boldsymbol{K}\vec{u} + \boldsymbol{D}\frac{d\vec{u}}{dt} = 0 \qquad [2.3.14]$$

if the terms multiplying the potentials are collected in a matrix $\boldsymbol{K}$ (the stiffness matrix) and the terms multiplying the time derivatives of the potentials in a matrix $\boldsymbol{D}$ (the damping matrix); $\vec{u}$ is the vector of the nodal values potentials. The name of these matrix has its origin in the structural and mechanical field, where the finite element method was applied firstly. The method was not applied in electromagnetism until 1968, in fact such kind of

electromagnetic problems required not yet existing powerful calculus systems due to the fact that also the air surrounding the conductors has to be modelled. The [2.3.14] matrix equation describe the behaviour of each element independently of the others.

2.4 Assembling element equations

A equation system like [2.3.14] valid for the whole region where the problem is defined can be found by assembling together all the element equations and enforcing the conditions that adjacent elements are connected ensuring the nodal potential continuity.
To do that, we can rewrite [2.3.14] for the whole region where the problem is defined:

$$\boldsymbol{K}_g \vec{u}_g + \boldsymbol{D}_g \frac{d\vec{u}_g}{dt} = 0 \qquad [2.4.1]$$

where the subscript *g* indicates global parameters.
The assembling matrix process is best illustrated by considering a simple example (Fig. 2.4.1) of a three triangular finite elements in a two dimensional region.

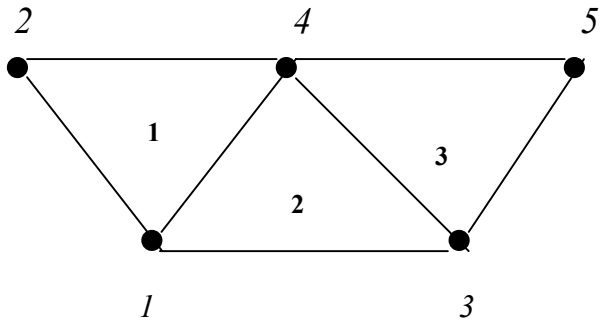


Fig. 2.4.1 Assembly of three element

The numbering of node *1, 2, 3, 4, 5* is called global numbering. It is also possible to define, for each element, a local element numbering (Fig. 2.4.2) in counterclockwise sequence starting from any node.

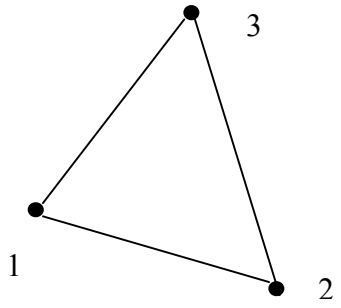


Fig. 2.4.2 Local numbering of the element

For example, for element 2 in Fig. 2.4.1, the global numbering *1, 3, 4* correspond to the local numbering 1, 2, 3 of the element in Fig. 2.4.2. So, a helpful table can be built:

Local Numbering	1	2	3
Global Numbering of element 1	*1*	*4*	*2*
Global Numbering of element 2	*1*	*3*	*4*
Global Numbering of element 3	*3*	*5*	*4*

Table 2.4.1 Local and global numbering correspondence

In this case, the dimension of the global vector of the nodal values potentials $\vec{u}_g$ is 5 and the global stiffness and damping matrices have dimension 5x5 since five nodes are involved.
The global stiffness matrix K_g can be written as:

$$K_g = \begin{bmatrix} K_{11}^1 + K_{11}^2 & K_{13}^1 & K_{12}^2 & K_{12}^1 + K_{13}^2 & 0 \\ K_{31}^1 & K_{33}^1 & 0 & K_{32}^1 & 0 \\ K_{21}^2 & 0 & K_{22}^2 + K_{11}^3 & K_{23}^2 + K_{13}^3 & K_{12}^3 \\ K_{21}^1 + K_{31}^2 & K_{23}^1 & K_{32}^2 + K_{31}^3 & K_{22}^1 + K_{33}^2 + K_{33}^3 & K_{32}^3 \\ 0 & 0 & K_{21}^3 & K_{23}^3 & K_{22}^3 \end{bmatrix} \qquad [2.4.2]$$

The global matrix coefficients can be found by using the fact that the potential distribution must be continuous across the element boundaries. For example, elements 1 and 2 have node *1* in common; hence:

$$K_{g11} = K_{11}^1 + K_{11}^2$$

Nodes *1* and *4* belong simultaneously to elements 1 and 2; hence:

$$K_{g14} = K_{12}^1 + K_{13}^2$$

Since there is no coupling between nodes *2* and *3*:

$$K_{g23} = 0$$

The subscript in the first side of each matrix term indicates the nodal global numeration. In the second side, the superscript indicates the involved element, while the subscript indicates the local numeration. The global damping matrix can be assembled in the same way.

It can be shown that:

- the global stiffness and damping matrices are symmetric
- the global stiffness and damping matrices are sparse and banded.

2.5 Transient and harmonic analysis

A transient dynamic analysis is used to determine the dynamic response of the system under the action of any general time-dependent loads. It is possible to use this type of analysis to determine the time-varying quantities such as magnetic field intensity, the magnetic flux density and the current density in the problem definition domain as they respond to any combination of static, transient, and harmonic loads. The basic equation solved by a transient dynamic analysis is eq. [2.4.1]

$$\boldsymbol{K}_g \vec{u}_g + \boldsymbol{D}_g \frac{d\vec{u}_g}{dt} = 0 \qquad [2.4.1]$$

At any given time, *t*, this equation can be thought of as a "static" equilibrium equation. The ANSYS program uses a time integration method to solve this equation at discrete time-points. The time increment between successive time-points is called the “integration time step”.

To specify loads and boundary conditions, it is possible to divide the load-versus-time curve into suitable load steps. Each "corner" on the load-time curve is one load step, as shown in Fig. 2.5.1

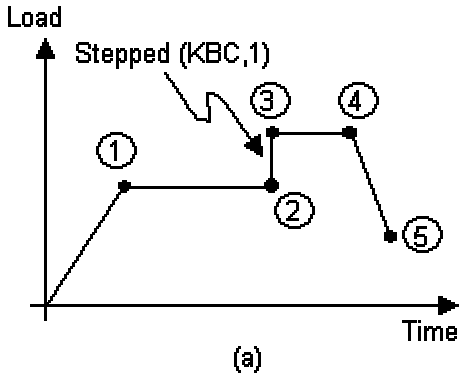


Fig. 2.5.1 Load-time curve

By specifying more than one load step, it is possible to step or ramp the load itself :

- If a load is stepped, then its full value is applied instantaneously
- If a load is ramped, then its value increases gradually with the full value occurring at the end of the load step

In the case of harmonic analysis the potentials can be considered varying sinusoidally with time and can be represented by phasors. The vector $\vec{u}_g$ can be divided into a real and an imaginary part:

$$\vec{u}_g = \vec{u}_{gRE} + j\vec{u}_{gIM} \qquad [2.5.1]$$

and eq. [2.4.1] can be written as:

$$\boldsymbol{K}\vec{u}_g + j\omega\boldsymbol{D}\vec{u}_g = 0 \qquad [2.5.2]$$

where ω is the frequency of the potential field variation. Substituting [2.5.1] in [2.5.2] we can see that a harmonic analysis provides two sets of solution: the real and imaginary part of a complex solution. The measurable quantity $\vec{u}_g(x,y,z,t)$ can be described as the real part of the complex function:

$$\vec{u}_g(x,y,z,t) = RE\left\{\left(\vec{u}_{gRE} + j\vec{u}_{gIM}\right)\cdot e^{j\omega t}\right\} \qquad [2.5.3]$$

2.6 Boundary conditions

Electromagnetic fields are rarely truly bounded, but in most cases they exist in an infinitely extending space. However, in the solution of electromagnetic problems boundary conditions can be imposed by using symmetry conditions and interface properties between different materials.

For example, an iron body in air does not bound the magnetic field. But since the permeability of the iron is much higher than the one of the air, the flux lines in air are perpendicular to the interface surface between iron and air. So, a boundary condition can be imposed in this interface surface if only the field distribution in air is of interest.

In other cases, symmetry lines or planes permit to impose boundary conditions to have the magnetic field perpendicular or parallel to them.

For example in a fusion machine (Fig. 2.6.1) the magnetic field is generated by a central solenoid and by some coils which surround the vacuum chamber. Since all the geometry is toroidal, an axis-symmetric model can be built. In the whole region of the problem the fields

distribution can be described by the magnetic vector potential $\vec{A}$.

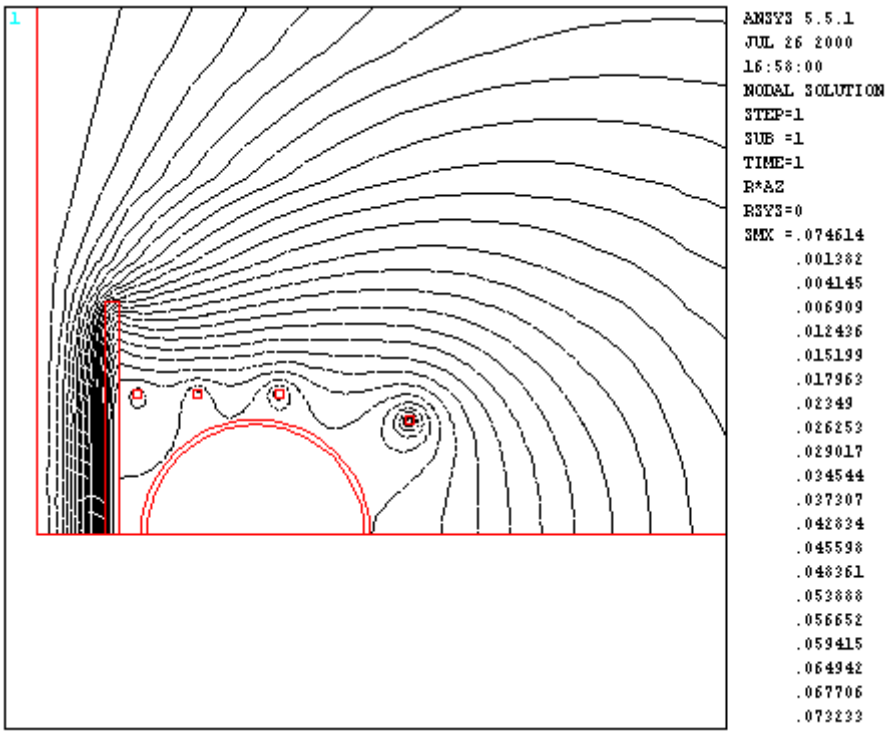


Fig. 2.6.1 Magnetic field distribution in a fusion machine

Since the current density posses only the component perpendicular to the plane of the model, the magnetic vector potential has only the component directed in the same direction of the current density. The magnetic vector potential formulation reduces in this case to a Poisson equation:

$$\nabla \cdot \nu \nabla A_z = -J_z \qquad [2.6.1]$$

where A_z and J_z are scalar and are the component of $\vec{A}$ and $\vec{J}$ in the toroidal direction.

The vertical line bisects the coils and the central solenoid, with exactly similar but oppositely directed current on its left and right sides. It must therefore be the separatrix curve which separates the flux lines belonging to the right side from the flux line belonging to the left side, so the magnetic field must be parallel to this line. A parallel magnetic field boundary condition must be enforced in it, enforcing the fact that this line must be a flux line along which the vector potential A_z has a fixed value. For convenience, this value may be taken to zero, so that:

$$A_z = 0 \qquad [2.6.2]$$

Boundary conditions in which the potential is prescribed are often referred to as Dirichlet conditions [6].

The bottom edge of the model is also a symmetry line: the currents in the coils are directed in the same sense above and below this line, so the flux lines must be perpendicular to this lines. A perpendicular magnetic field boundary condition must be enforced in it by imposing:

$$\frac{\partial A_z}{\partial n} = 0 \qquad [2.6.3]$$

where *n* is the normal to the surface.

Boundary conditions in which the normal derivative of the potential is prescribed are often referred to as Neumann conditions [6].

In the finite element solution and in this particular case, this type of boundary condition is called as natural because it does not need to be specified by the user.

Considering Fig. 2.6.2, it is possible to summarize how to apply Dirichlet and Neumann boundary conditions in three-dimensional problem regions involving the magnetic vector or scalar potential formulations.

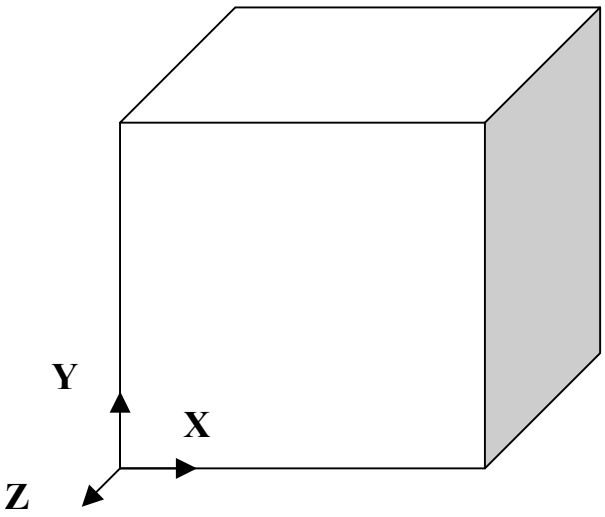


Fig. 2.6.2 Boundary conditions on a three dimensional region

Dirichlet boundary condition.

This is a perpendicular magnetic field condition. It consists in imposing the following condition:

- in the case of the magnetic scalar potential formulation: $\Omega = cost$ on the plane where the magnetic field has to be perpendicular
- in the case of the vector scalar potential formulation: $A_x = 0$ or $A_y = 0$ or $A_z = 0$ if the plane where the condition has to be imposed is perpendicular to the X, Y or Z axis.

Neumann boundary condition.

This is a parallel magnetic field condition. It consists in imposing the following condition:

- in the case of the magnetic scalar potential formulation this is a natural boundary condition because it will be imposed by the finite element solution process itself.
- in the case of the vector scalar potential formulation: $A_x = 0$ and $A_y = 0$ if Z = *cost* is the plane where the magnetic field has to be perpendicular. The conditions on the others planes can be easily found by rotation of the X, Y and Z indexes.

2.6 Derived fields from the degree of freedom solution

The derived electromagnetic field results are the magnetic field intensity $\vec{H}$, the magnetic flux density $\vec{B}$ and the current density $\vec{J}$.

The magnetic field intensity is defined as eq. [1.2.3] and [1.2.6]

$$\vec{H} = \vec{T} - \nabla\Omega \qquad [1.2.3]$$

$$\vec{H} = -\nabla\Omega \qquad [1.2.6]$$

respectively for conducting and non conducting regions.

The gradient of the magnetic scalar potential $-\nabla\Omega$ is computed using the element shape function as:

$$\vec{H}_\Omega = \sum_{i=1}^{m} Ni \cdot \nabla\Omega i \qquad [2.5.1]$$

the part which depends from the vector electric potential is computed as:

$$\vec{H}_T = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n} Ni\left(x_j, y_j, z_j\right)\cdot \vec{T}i \qquad [2.5.2]$$

where *m* and *n* equals the number of element nodes. In the non conducting regions we have $\vec{H} = \vec{H}_\Omega$, while in the conducting regions $\vec{H} = \vec{H}_T + \vec{H}_\Omega$.

The magnetic flux density can be easily computed from the equation $\vec{B} = \mu \cdot \vec{H}$.

The current density is defined in eq. [1.2.2]

$$\vec{J} = \nabla \times \vec{T} \qquad [1.2.2]$$

as the curl of the electric vector potential and can be computed using the element shape functions as:

$$\begin{aligned}
\vec{J}_x &= \left(\sum_{i=1}^{m}\frac{\partial Ni}{\partial y}\cdot\vec{T}_{zi} - \frac{\partial Ni}{\partial z}\cdot\vec{T}_{yi}\right)\\
\vec{J}_y &= \left(\sum_{i=1}^{m}\frac{\partial Ni}{\partial z}\cdot\vec{T}_{xi} - \frac{\partial Ni}{\partial x}\cdot\vec{T}_{zi}\right) \qquad [2.5.3]\\
\vec{J}_z &= \left(\sum_{i=1}^{m}\frac{\partial Ni}{\partial x}\cdot\vec{T}_{yi} - \frac{\partial Ni}{\partial y}\cdot\vec{T}_{xi}\right)
\end{aligned}$$

3 Implementation in ANSYS of the *T-Ω, Ω* formulation

The ANSYS code allows to write routines and subroutines in FORTRAN 77 or in C and either to compile and link them into the program, creating a custom version of the ANSYS program. This capability is called as User Programmable Features (UPFs).

UPFs allow to create new elements, add them to the ANSYS element library and use them as regular elements. The customisation of the new formulation has been done by modifying the **uel101**, **uel102**, **uec101** and **uec102** subroutines which reside in the subdirectory /ansys56/customize/user. The **uec** routines allow to describe the element characteristics such us: 2-D or 3-D geometry, numbers of nodes, degree of freedom set, and so on. The routine **elccmt**, which reside in the subdirectory /ansys56/customize/include, describes the input for these routines in detail. The **uel** routines permit to calculate the element stiffness and damping matrices and the element load vector. The element printout also is generated, and the variables to be saved are calculated and stored in the results file. The ANSYS distribution medium has also "include-decks" which can be included in the subroutines. This "include-decks" also called "commons" contain important amounts of data, such us: solution options, output control informations, element characteristics and so on. The "include-decks" permit the communication between the option defined by the user during the model creation and analysis solution and the costumized routines. The modified routines and also the **anscust.bat**, **makefile**, **ansysex.def**, **ansysb.dll** and **mnflib.dll** files (which are on the distribution medium) have to be moved in a working directory where the FORTRAN files will be compiled and a custom ANSYS executable version will be created by running the **anscust.bat** file. In this way, two elements have been created: the **USER101** for non-conducting regions and the **USER102** for conducting regions. To run the ANSYS executable version the following string:

ansys56cust.exe -custom <path of the working dir>\ansys.exe -p ansysuh

has to be typed in the commands prompt of the operative system. It is also possible to run the executable ANSYS version in interactive mode selecting the ansys.exe file from the "Execute a costumized ANSYS executable" command of the run window of ANSYS . The ANSYS output of the user-linked version will include the following note: "This ANSYS version was linked by licence".

3.1 The Uec routines

The Uec routines permit to define the element characteristics by flagging the positions of an array which is called "ielc" and which is an argument of the routine itself. The flags description can be found in the elccmt routine. The meaning of the most important Uec subroutines flags for the USER101 and USER102 are reported below:

c GEOMETRY
ielc(KDIM)=3 define a three-dimensional element
ielc(ISHAP)=JTET define a four-noded element
ielc(IDEGEN)=1 no degenerations is allowed

c ELEMENT USAGE
ielc(KELSTO)=1 orientation of the quantities global coordinates
ielc(MATRQD)=200 a not well defined material property must be input for this element

c ELEMENT NODES
ielc(NMNDMX)=4 maximum number of nodes per element
ielc(NMNDMN)=4 minimum number of nodes per element
ielc(NMNDAC)=4 active number of nodes per elements (nodes that have dofs)
ielc(NMDFPN)=1 maximum number of degrees of freedom per node for this element
ielc(MATRXS)=10 matrices possible for this element type
ielc(KDOFS)= MAG degree of freedom set for this element

c ELEMENT LOADS & SURFACES
ielc(NMPTSF)=3 load varying linearly over triangular surface

c POSDATA FILE
ielc(NMNDNO)=4 number of nodes that have nodal output stored on postdata file
ielc(KCONIT)=1 contourable nodal items are present
ielc(KMAGC)=1 element supporting eddy currents

ielc(NMSMIS)=5 maximum number of items on the record for save variables

ielc(NMNMIS)=20
ielc(NMNMUP)=24
ielc(NCPTM1)=299
ielc(NCPTM2)=354
ielc(JSTPR)=1

c NUMBER OF SAVED VARIABLES
ielc(NMMSVR)=181 number of saved variables for linear magnetic effects

3.2 The Uel routines

The Uel routines have the function to compute element matrices, load vectors and results and to maintain the element solution data.
The Uel routines input arguments are:

elem: element label (number)
ielc: array of element type characteristics
elmdat: array of element data
eomask: bit pattern for element output
nodes : array of element node numbers
locsvr: location of the saved variables
kelreq: matrix and load vector form requests
kelfil: keys indicating incoming matrices and load vectors
nr: matrix and load vector size
xyz: nodal coordinates
u: element nodal solution values

The Uel routines output arguments are:
kelout: keys indicating created matrices and load vectors
zs: stiffness matrix
damp: damping matrix
zsc: applied load vector

zscnr: applied load imaginary vector

elvol: element volume

elmass: element mass

center: centroid location

elener: element energies

edindx: element result data file indexes

lcerst: position on result file

The Uel routines call other subroutines, the most important are:

PROPEV subroutine permit to get the element material properties,
their input arguments are:

iel: the element number

mtr: the material number

lp: keys for which specific value is requested

tem: temperature at which to evaluate material

n: number of material properties to be evaluate

the output argument is:

prop: values of material properties

SVGIDX subroutine permit to get the index for saved variables,
its input argument is:

locsvr: pointer to location of index

its output argument is:

svindx: the index of the saved variables

SVRGET subroutine permit to fetch the saved variables for an element,
their input argument are:

svindx: the index of the saved variables

nset: the set number in the index

nsvr: number of dp words expected in the set

its output argument is:

svr: data in the set

ELDWRT subroutine permit to output the element data to the result file,
their input arguments are:
ielem: the element number
edtype: the element type
lcrest: pointer to results file position
edindx: the index to results file data
nval: total number of values
the output argument is:
value: the output value

SVRPUT subroutine permit to write an element's saved variable set,
their input arguments are:
svindx: the index of the saved variables
nset: the set number in the index
leng: number of dp words in the set
svr: data in the set
its output argument is:
svindx: the update index

SVPIDX subroutine permit to write out the saved variables index vector,
their input arguments are:
locsvr: pointer to start of the saved variable for element
svindx: the index of the saved variables for this element
its output argument is:
locsvr: pointer to start of the saved variable for the next element

4. Validation of the implementation of the *T-Ω,Ω* formulation

To validate the implementation in ANSYS of the new *T-Ω,Ω* formulation, three benchmark problems have been considered.

Results obtained are always compared with the results from the commercial formulations in ANSYS.

The first problem is the analysis of a conducting cube in homogeneus magnetic field the second benchmark is the hollow sphere in uniform magnetic field, the last is the FELIX brick experiment.

The first one has been proposed by O. Birò and K. Preis in a published paper on eddy current numerical computation [3], the last two benchmark are from some International Workshops for the comparison of eddy current codes.

In particular, the hollow sphere analysis is the problem number 6, while the FELIX brick experiment is the problem number 4.

The idea to develop certain benchmark problems to validate 3-D eddy current computer codes was born in 1985 in the Office of Fusion Energy, U.S. Department of fusion.

The goal of the benchmarks is: "to show the effectiveness of numerical techniques and associated computer codes in solving electromagnetic field problems, and to gain confidence in their predictions…." [15]

Results obtained with the new *T-Ω,Ω* formulation are in very good agreement with those from other codes and with analytical solutions and experimental solution where them are available.

4.1 Conducting cube in homogeneous magnetic field

A conducting cube is immersed in a uniform magnetic field of 1T with harmonic time variation of 50 Hz. Symmetry conditions permit to discretize only 1/8 of the entire cube. The model used for the analysis is shown in Fig. 4.1.1

This problem is studied by O. Birò and K. Preis in [3]. Each side of the conducting region is 1 cm long, while the boundaries are 5 cm farthest from the axes origin. The model includes 5872 elements and 1202 nodes. The cube conductivity is $\sigma = 5{,}7 \cdot 10^7$ S/m.

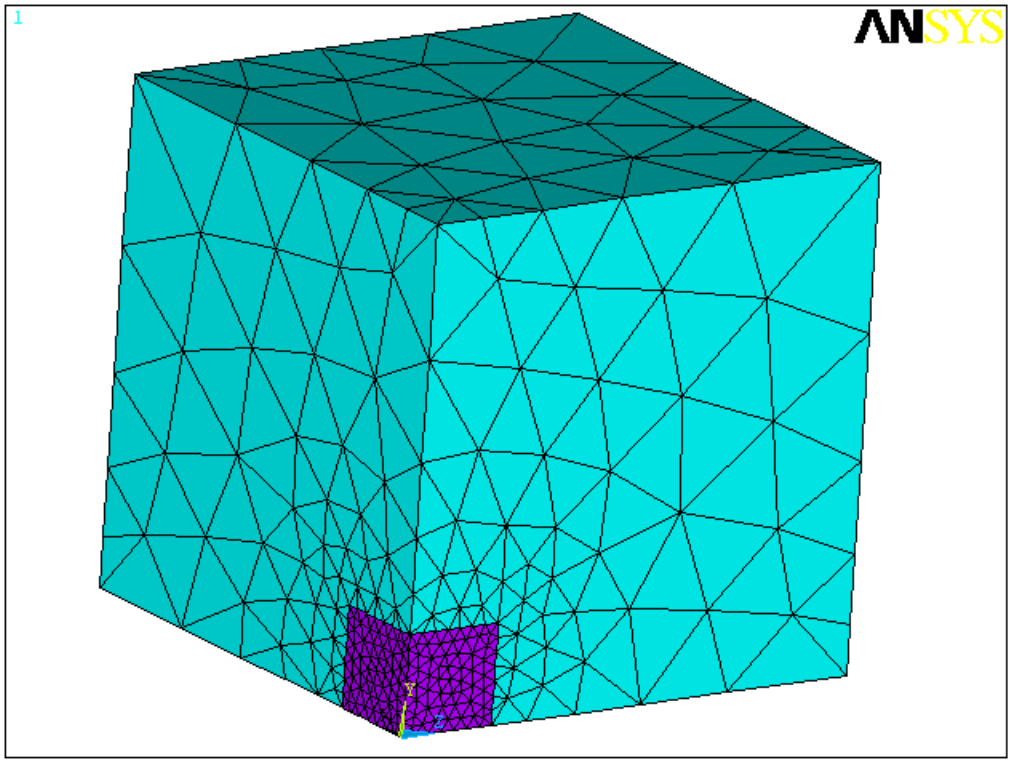


Fig. 4.1.1 Conducting Cube Model

The magnetic field and the eddy currents distributions in the conducting cube (real and imaginary parts) obtained, by using the same mesh, with the implemented *T-Ω,Ω* formulation and with the ANSYS commercial *A-V,A* formulation are shown in the following pictures. On the left we have results from the *T-Ω,Ω* formulation, on the right from the *A-V,A* formulation.

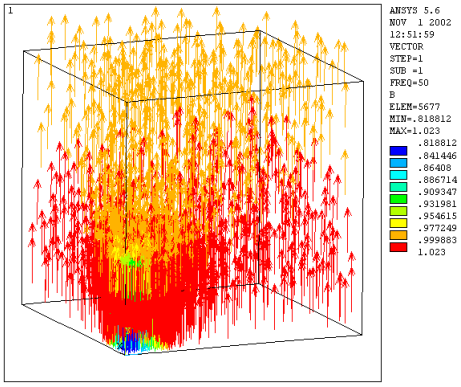


Fig. 4.1.2 B Re part T-Ω,Ω formulation

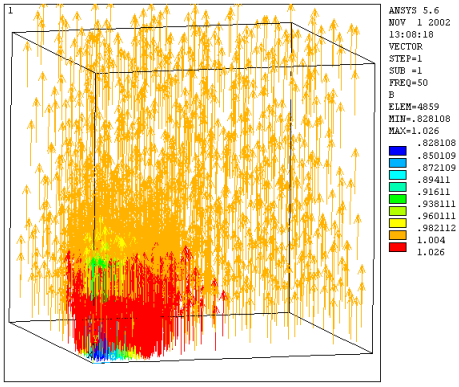


Fig. 4.1.3 B Re part A-V,V formulation

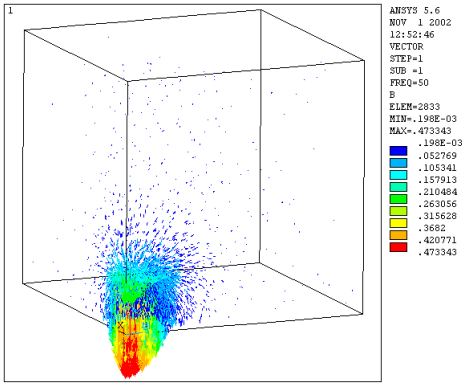


Fig. 4.1,4 B Imag part T-Ω,Ω formulation

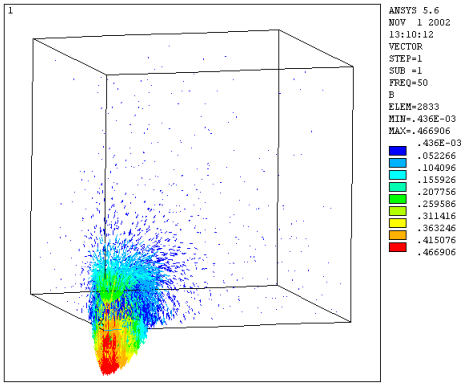


Fig. 4.1.5 B Imag part A-V,V formulation

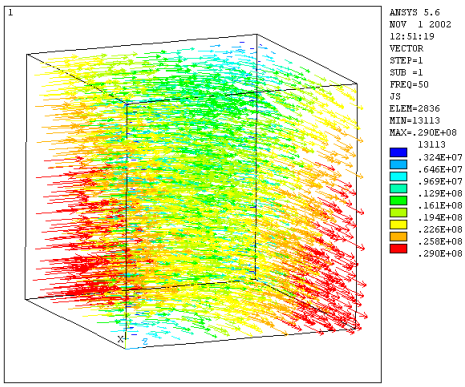


Fig. 4.1.6 J Re part T-Ω,Ω formulation

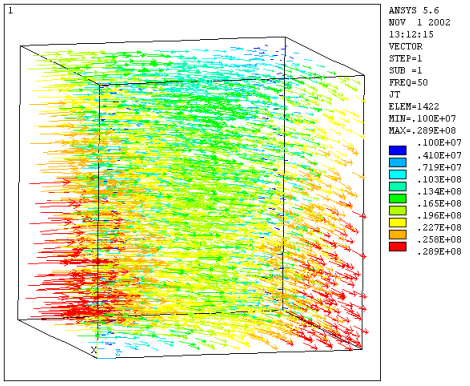


Fig. 4.1.7 J Re part A-V,V formulation

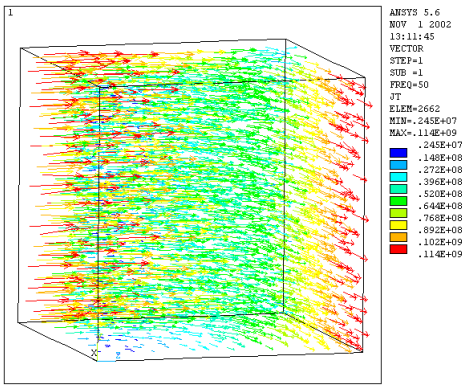


Fig. 4.1.8 J Imag part T-Ω,Ω formulation

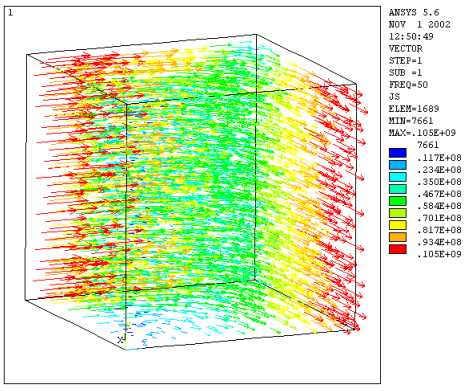


Fig. 4.1.9 J Imag part A-V,V formulation

As can be noted results obtained with the two different formulations are in fairly

correspondence. The real and imaginary parts of the X component of the current density along the conducting cube Z axis is reported in Fig. 4.1.10.

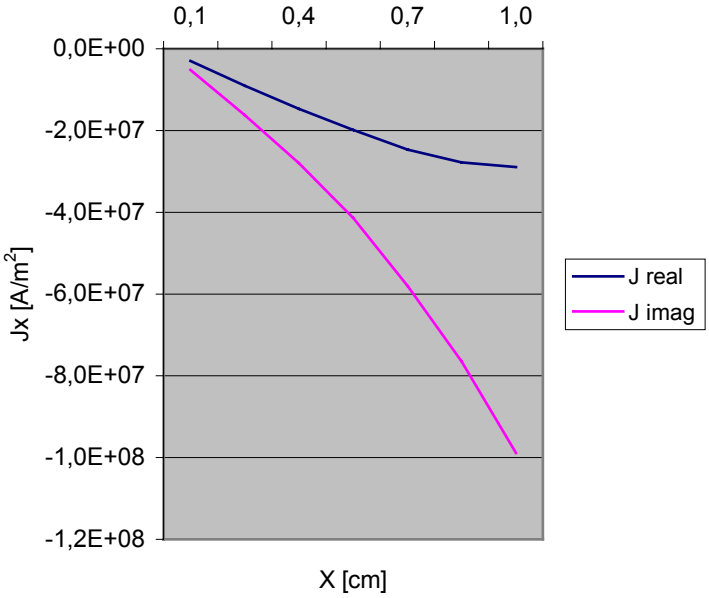


Fig. 4.1.10 Current density along the x axis cube

The current densities distributions depicted in the plot are in excellent agreement with those (Fig. 4.1.11) reported by O. Birò in [3].

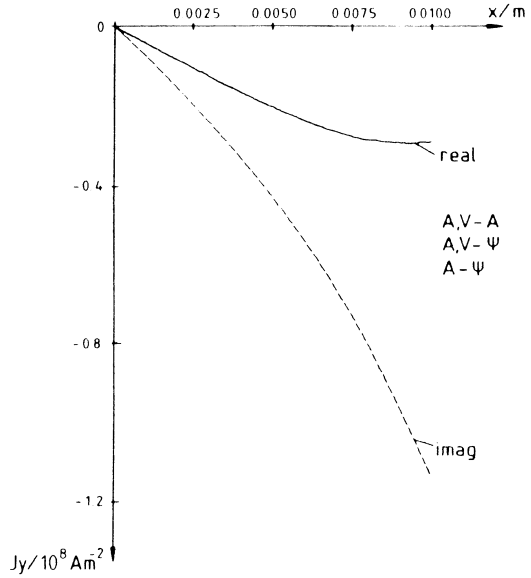


Fig. 4.1.11 Current density along the x axis cube as reported in [3]

The computer storage and the CPU time (s) as computed by ANSYS in obtaining the

solution for both the *T-Ω,Ω* formulation and the *A-V,A* formulation are reported in Table 4.1.1:

	Solution file size (MB)	CPU time processing	CPU time solution	CPU time results	CPU time total
T-Ω,Ω formulation	39.6	18.9	12.2	39.4	83.9
A-V,A formulation	134.8	21.9	22.1	51.9	113.7

Table. 4.1.1 Runtime statistics and computer storage

In Table 4.1.1 the total CPU time and also the CPU time for the element processing phase, the CPU time for solving the equation system and the CPU time for calculating the derived quantities have been considered.
Moreover, also the computer storage size in MB of the files which ANSYS uses to obtain the solution and to write results has been reported in table 4.1.1.
In Table 4.1.2 the degrees of freedom number in air and in the conductive material is shown for both the formulations:

	DOFs in air	DOFs in conductor	Total number of DOFs
T-Ω,Ω formulation	691	2828	3519
A-V,A formulation	2073	2828	4901

Table. 4.1.2 Degrees of freedom

The *A-V,A* formulation requires an higher CPU time because of the element nodes in air require three degrees of freedom instead of one of the *T-Ω,Ω* formulation.

4.2 Hollow sphere in uniform field

A hollow conducting sphere is immersed in a uniform field of 1 T oscillating at 50 Hz. Symmetry conditions permit to discretize only one arbitrary circumferential slice of 20 degrees of the entire sphere. The model used for the analysis is shown in Fig. 4.2.1 and Fig. 4.2.2

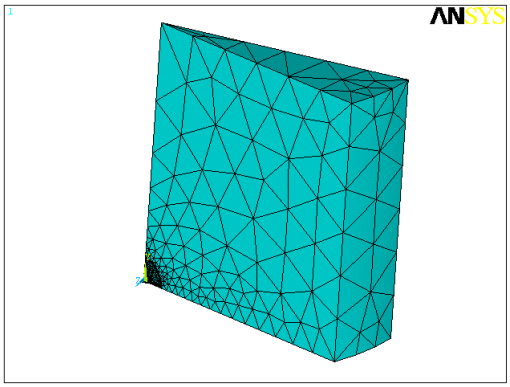


Fig. 4.2.1 Hollow Sphere Model

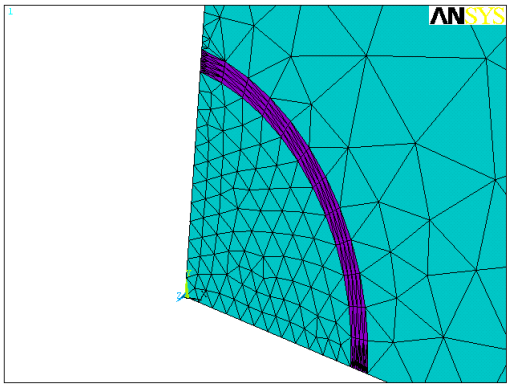


Fig. 4.2.2 Hollow Sphere Model detail

The sphere has an inner radius of 0.05 m and an outer radius of 0.055 m. A rectangular exterior boundary is located at a distance of 0.6 m where the external field load is applied. The conducting sphere has a conductivity of $5 \cdot 10^8$ S/m. The model includes 2506 elements and 792 nodes. This is a typical case of multiply connected regions; the region inside the conductor is treated by introducing an artificial region of low conductivity (one thousand smaller than the sphere region) rendering the problem simply connected.

This is the benchmark problem 6 defined in the International Workshops for Eddy Current Code Comparison and has been proposed by C. R. I. Emson [15].

The problem consists in determining the magnetic field and eddy current distribution, in determining the peak flux density at the centre of the sphere and the average power loss within the sphere. Results can be compared with results form other codes and also with the analytical solution. ANSYS solves this problem by using the mixed formulation. The conducting region is modelled with the vector potential formulation . The region of air inside and outside of the sphere is modelled with the Reduced Scalar Potential. The two regions are interfaced with 100 interface elements at the sphere-air interface boundary.

The results of the magnetic field and the eddy currents distributions in the conducting cube (real and imaginary parts) obtained with the implemented *T-Ω,Ω* formulation and with the

ANSYS commercial mixed formulation are showed in the following pictures. On the left we have results from the *T-Ω,Ω* formulation, on the right from the mixed formulation.

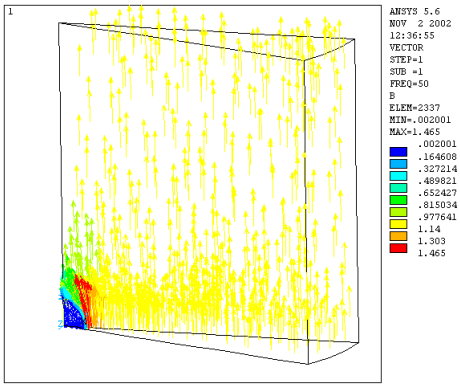


Fig. 4.2.3 B Re part T-Ω,Ω formulation

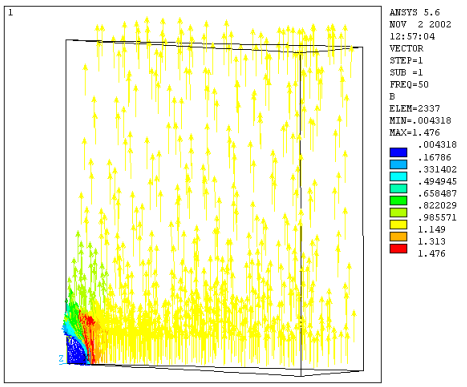


Fig. 4.2.4 B Re part mixed formulation

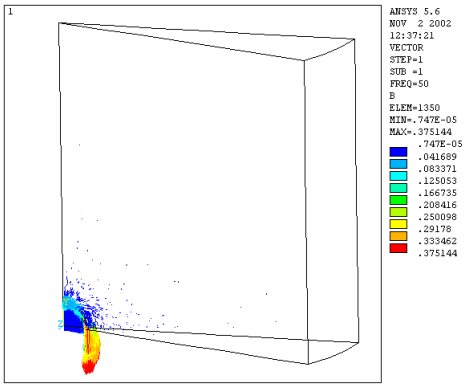


Fig. 4.2.5 B Imag part T-Ω,Ω formulation

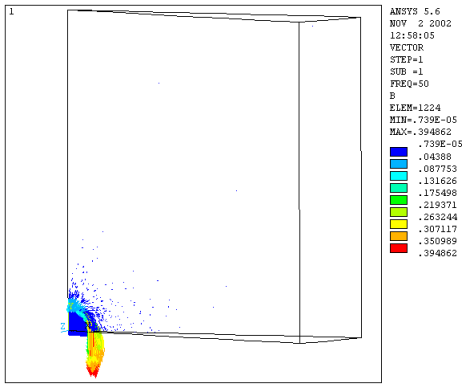


Fig. 4.2.6 B Imag part mixed formulation

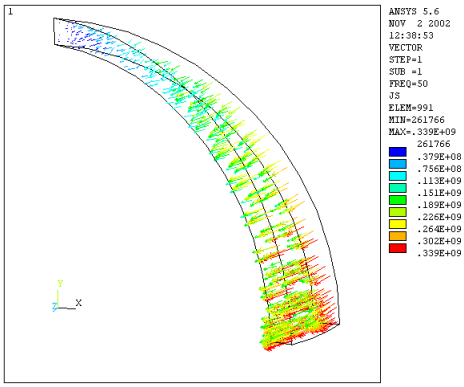


Fig. 4.2.7 J Re part T-Ω,Ω formulation

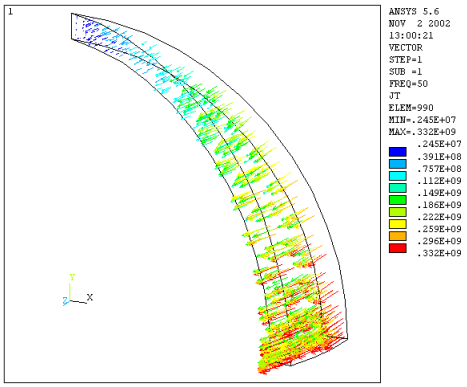
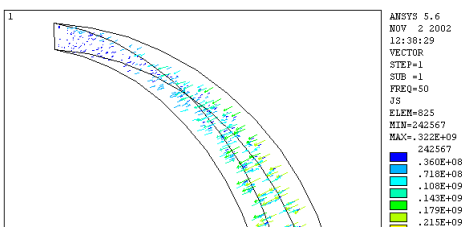


Fig. 4.2.8 J Re part mixed formulation



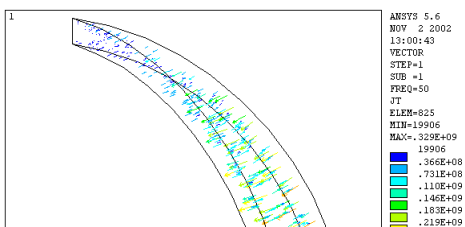


Fig. 4.2.9 J Imag part T-Ω,Ω formulation

Fig. 4.2.10 J Imag part mixed formulation

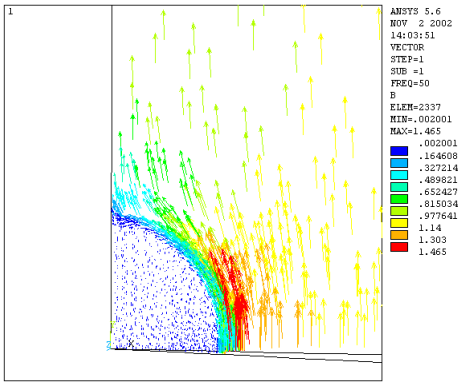


Fig. 4.2.10 B Real part detail T-Ω,Ω formulation

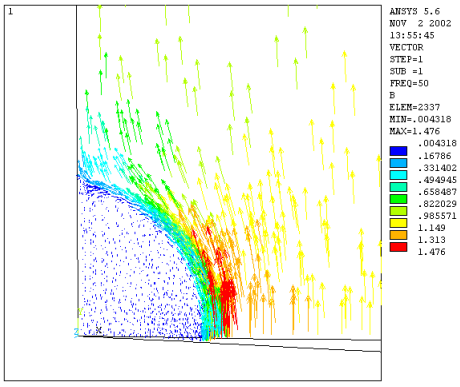


Fig. 4.2.11 B Real part detail mixed formulation

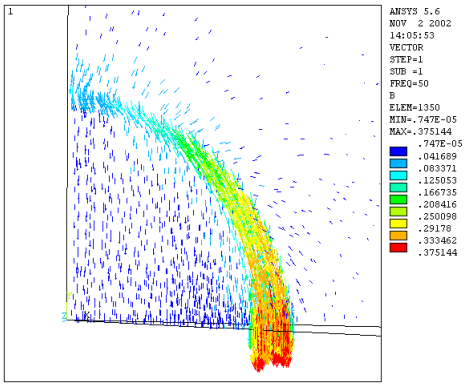


Fig. 4.2.12 B Imag part detail T-Ω,Ω formulation

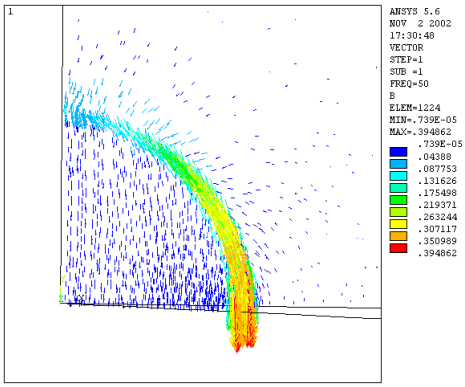


Fig. 4.2.12 B Imag part detail mixed formulation

As can be noted results obtained with the two different formulations are in fairly correspondence. In table 4.2.1 the peak flux density at the centre of the sphere and the average power loss within the sphere as computed from the two formulations are compared with those computed analytically.

	T-Ω,Ω formulation	*Mixed formulation*	*Analytical solution*
B_y (0,0), T	5.431E-02	5.443E-02	5.424E-02
Power loss, W	9401.02	10108.1691	10062

Table 4.2.1 Results comparison

The computer storage and the CPU time (s) as computed by ANSYS in obtaining the solution for both the *T-Ω,Ω* formulation and the A-V,A formulation are reported in Table 4.2.2:

	Solution file size (MB)	CPU time processing	CPU time solution	CPU time results	CPU time total
T-Ω,Ω formulation	18.4	8.4	2	18.9	34
A-V,A formulation	30.1	10.7	Not reported	22.2	35.8

Table. 4.2.2 Runtime statistics and computer storage

In table 4.2.3 the degrees of freedom number in air and in the conductive material is shown for both the formulations:

	DOFs in air	DOFs in conductor	DOFs in interface elements	Total number of DOFs
T-Ω,Ω formulation	336	8568		8904
A-V,A formulation	610	2140	316	3066

Table. 4.2.3 Degrees of freedom

The CPU time in obtaining the solution for the *T-Ω,Ω* formulation is quite the same than the one for the *A-V,A* formulation even if the number of degrees of freedom is bigger, this is because in the mixed formulation there are the interface elements that increase the total element number and also because the user element subroutine for the *T-Ω,Ω* formulation is more parsimonious.

4.3 The FELIX brick

A rectangular aluminium brick with a rectangular hole is immersed in a uniform field which decays exponentially with time. Symmetry conditions permit to build the model of only one eighth of the brick. The model used for the analysis is shown in figure 4.3.1 and 4.3.2

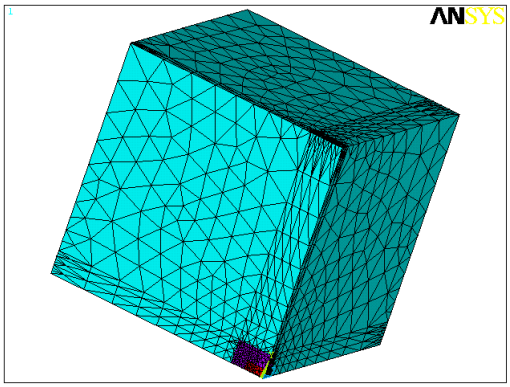


Fig. 4.3.1 Whole FELIX Brick Model

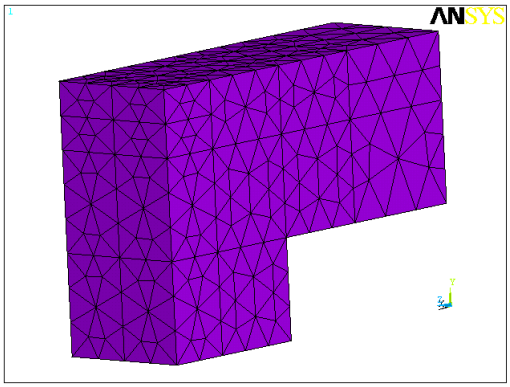


Fig. 4.3.2 FELIX Brick Model

The brick is made of aluminium alloy 6001, of resistivity $3.94 \cdot 10^{-8}$ Ohm m. Dimensions in the X, Y and Z directions are respectively 0.1524 m, 0.1016 m and 0.0508 m. A rectangular hole 0.0889 m x 0.0381 m penetrates the brick through the centre of the large faces. The model includes 16668 elements and 3614 nodes. The region inside the conductor is treated by introducing an artificial region of low conductivity to render the problem simply connected. This is the benchmark problem 4 defined in the International Workshops for Eddy Current Code Comparison and has been proposed by A. Kameary [15]. The problem has been solved by 11 different computer codes.

The problem consists in determining the time evaluations of the magnetic, the average power loss field and eddy current at different locations. The applied magnetic field is directed in the Z direction and it varies as:

$$Bz = Bo \quad \text{for } t<0$$

$$Bz = Bo\, e^{-t/\tau} \quad \text{for } t>0$$

where Bo = 0.1 T and τ = 0.0119 s.

The transient analysis problem solution has been obtained considering a 20 ms time range and by dividing it in 20 load steps, so using 20 iterations.

The time evolution of the total power loss in the brick as calculated by using the *T-Ω,Ω*

formulation and the one from the other codes [15] are shown in Fig. 4.3.3 and Fig. 4.3.4.

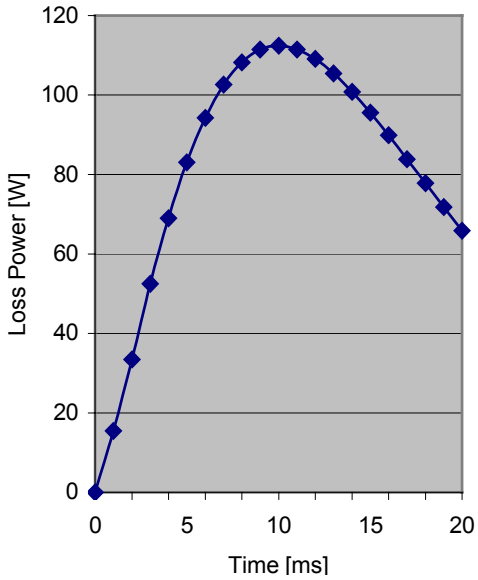


Fig. 4.3.3 Total power loss time evolution

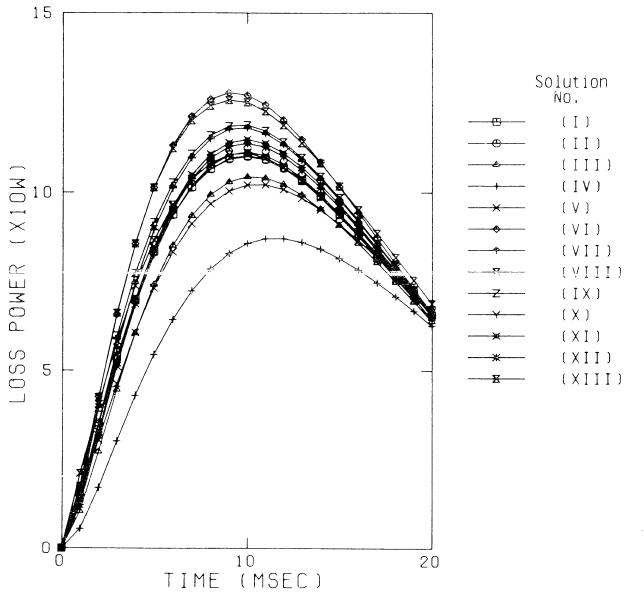


Fig. 4.3.4 Total power loss time evolution as reported in [15]

The total magnetic flux density at t = 4, 8, 12 and 20 ms as calculated by using the *T-Ω,Ω* formulation and the one from the other codes [15] are shown in Fig. 4.3.5 and Fig. 4.3.6.

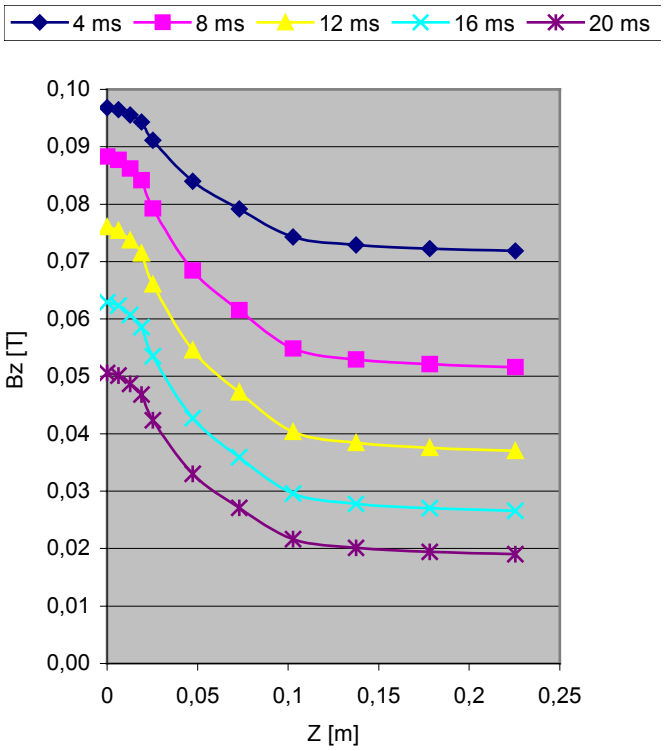


Fig. 4.3.5 Magnetic field on Z axis

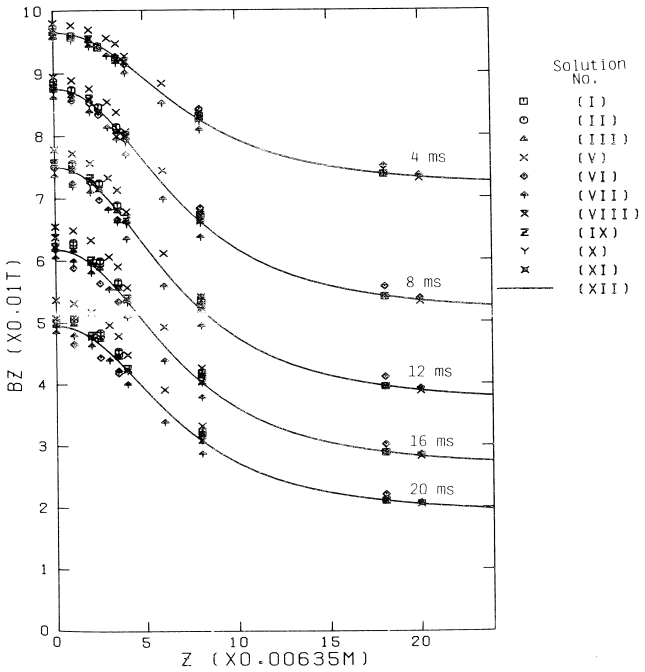


Fig. 4.3.6 Magnetic field on Z axis as reported in [15]

In table 4.3.1 the peak of the induced flux density at the centre of the hole and at different locations, the average power loss within the brick as computed from the two formulations are computed and compared with the mean values from the other codes.

	T-Ω,Ω formulation	*Mean*
B_z (Z=0.0), T	3.95E-02 (12)[1]	3.88 E-02
B_z (Z=0.0127), T	3.72E-02 (11)	3.63 E-02
B_z (Z=0.0254), T	2.97E-02 (11)	2.96 E-02
Power loss, W	112.4 (10)	110.6

Table. 4.3.1 Results comparison

The time evolution of the magnetic field and the eddy currents distributions in the conducting brick obtained with the implemented *T-Ω,Ω* formulation and with the *A-V,A* ANSYS commercial formulation are shown in the following pictures. On the left we have results from the *T-Ω,Ω* formulation, on the right from the commercial formulation.

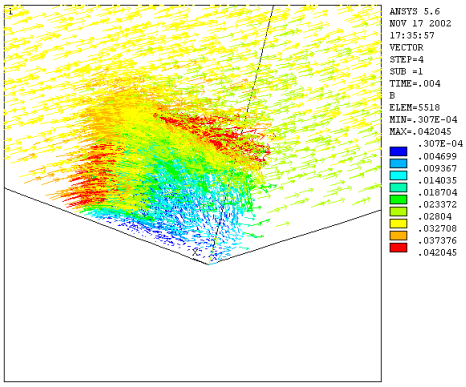


Fig. 4.3.7 B 4 ms T-Ω,Ω formulation

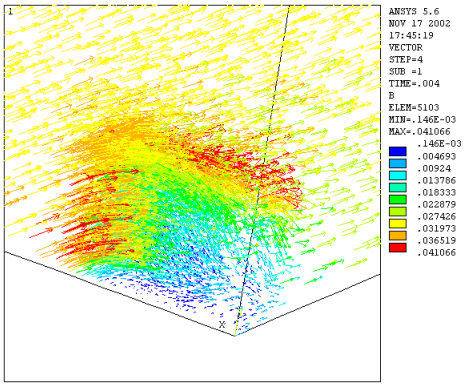


Fig. 4.3.8 B 4 ms A-V, A formulation

[1] Iteration number

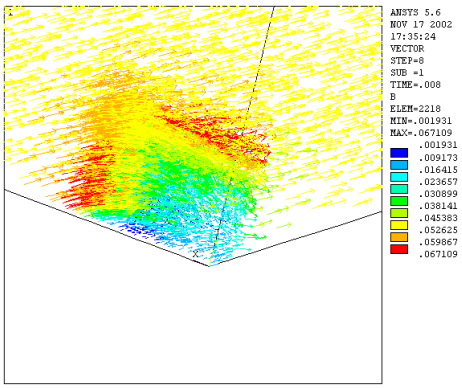


Fig. 4.3.9 B 8 ms T-Ω,Ω formulation

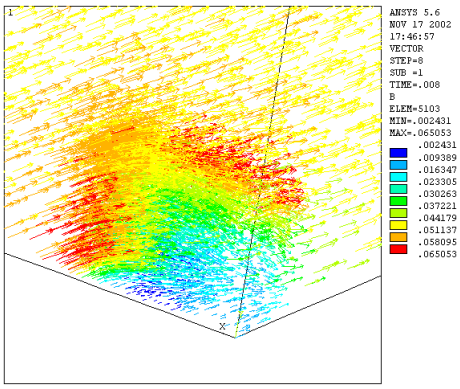


Fig. 4.3.10 B 8 ms A-V, V formulation

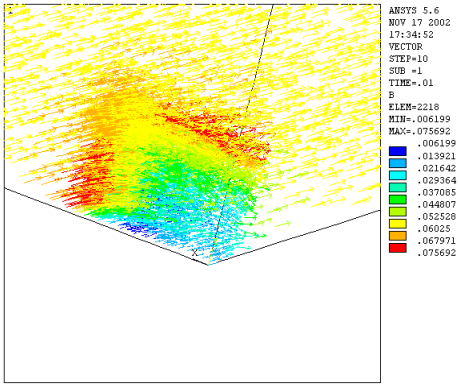


Fig. 4.3.11 B 10 ms T-Ω,Ω formulation

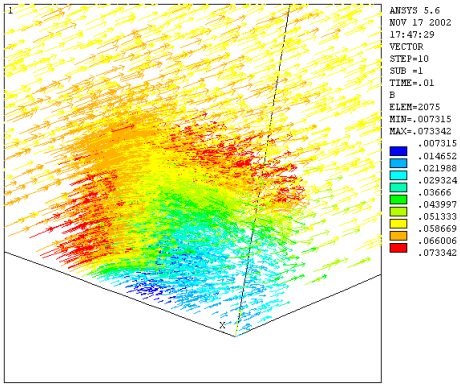


Fig. 4.3.12 B 10 ms A-V, V formulation

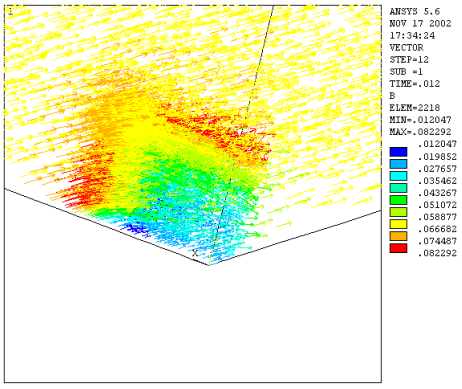


Fig. 4.3.13 B 12 ms T-Ω,Ω formulation

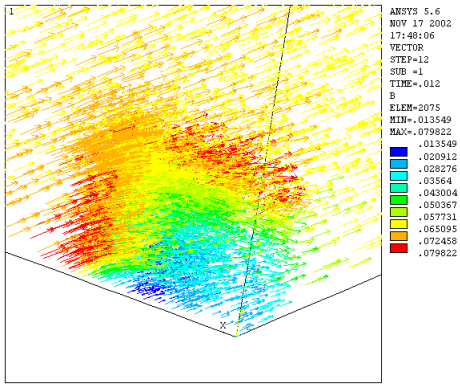


Fig. 4.3.14 B 12 ms A-V, V formulation

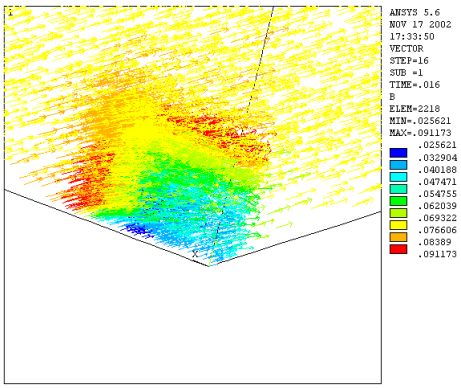


Fig. 4.3.15 B 16 ms T-Ω,Ω formulation

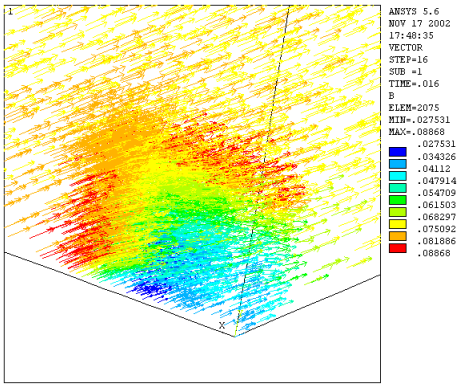


Fig. 4.3.16 B 16 ms A-V, V formulation

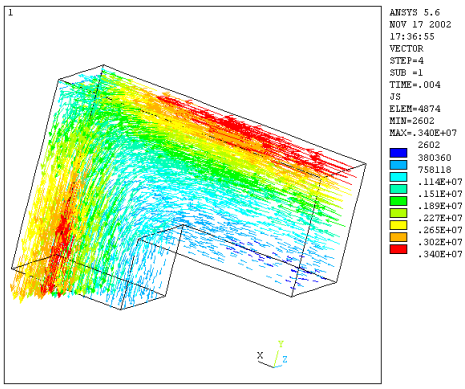


Fig. 4.3.17 J 4 ms T-Ω,Ω formulation

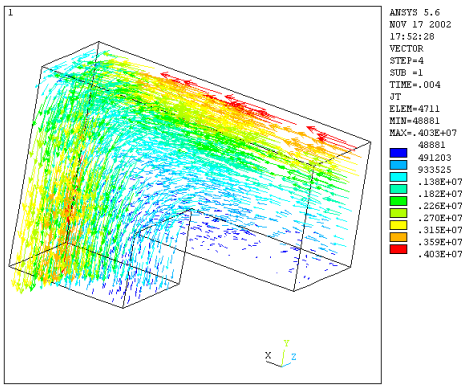


Fig. 4.3.18 J 4 ms A-V, A formulation

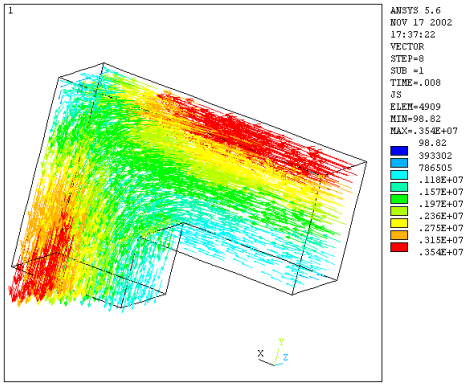


Fig. 4.3.19 J 8 ms T-Ω,Ω formulation

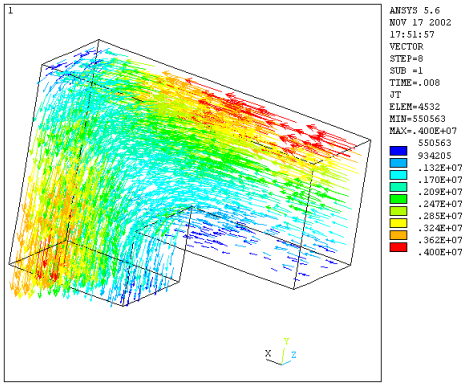


Fig. 4.3.20 J 8 ms A-V, V formulation

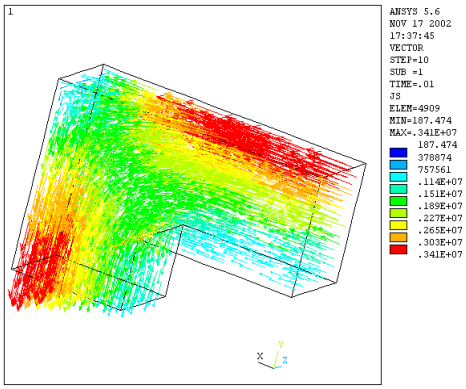


Fig. 4.3.21 J 10 ms T-Ω,Ω formulation

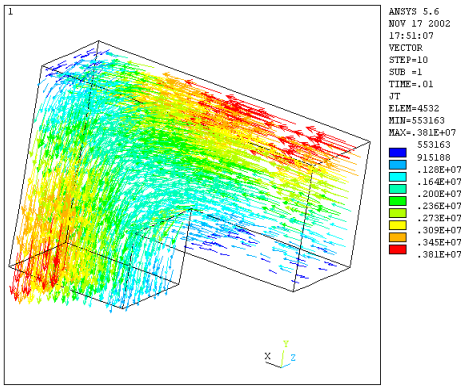


Fig. 4.3.22 J 10 ms A-V, V formulation

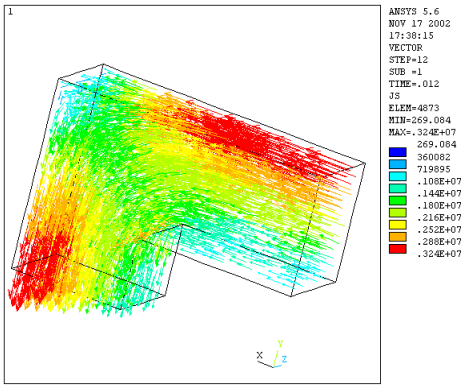


Fig. 4.3.23 J12 ms T-Ω,Ω formulation

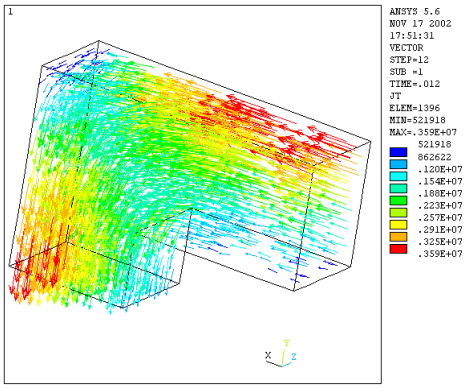


Fig. 4.3.24 J 12 ms A-V, V formulation

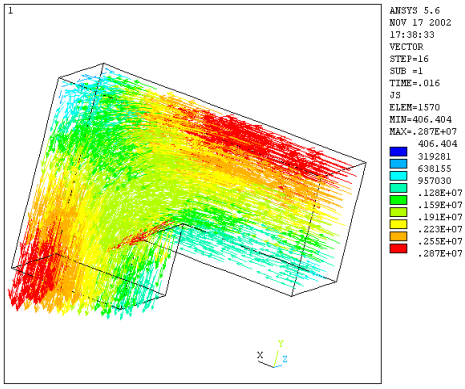


Fig. 4.3.25 J 16 ms T-Ω,Ω formulation

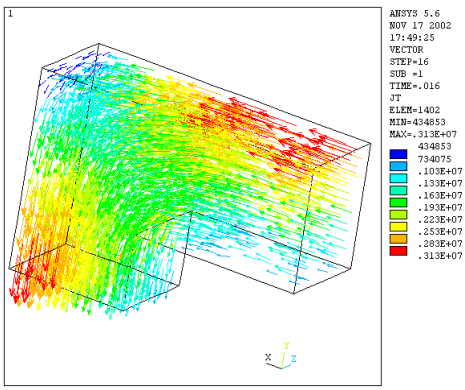


Fig. 4.3.26 J 16 ms A-V, V formulation

The computer storage and the CPU time for the first iteration as computed by ANSYS in obtaining the solution for both the *T-Ω,Ω* formulation and the *A-V,A* formulation are reported in Table 4.3.2:

	Solution file size (MB)	CPU time processing	CPU time solution	CPU time results	CPU time total
T-Ω,Ω formulation	23.4	9.7	2.7	11.5	27.6
A-V,A formulation	283.1	61.41	124.8	72.7	162.2

Table. 4.3.2 Runtime statistics and computer storage

In table 4.3.3 the degrees of freedom number in air and in the conductive material is shown for both the formulations:

	DOFs in air	DOFs in conductor	Total number of DOFs
T-Ω,Ω formulation	3027	3112	6139
A-V,A formulation	9495	2688	12183

Table. 4.3.3 Degrees of freedom

The *A-V,A* formulation requires much more CPU than the *T-Ω,Ω* formulation because most part of the element model are in air.

5. Conclusions

The aim of this work has been to implement a new formulation (the *T-Ω,Ω)* in the ANSYS commercial package by using its customization capabilities for creating new element types and adding them in the ANSYS library.

The goal has been to implement in ANSYS a simple and economical method for calculating 3-D eddy currents and magnetic field distributions.

For the eddy current study, ANSYS implements the magnetic vector potential formulation which uses in the non-conducting regions three degree of freedom, the magnetic vector potential components, and adds an extra degree of freedom, the time-integrated electric voltage, in the conducting regions.

The *T-Ω,Ω* formulation permit to reduce the number of unknown variables in the non-conducting regions from three to one.

Moreover, the implementations in ANSYS allows to take advantage of the excellent and several capabilities of the commercial code itself: mesh generation, post-processing, graphics window, optimization coupling and so on.

The implemented formulation has been applied to three bench-mark problems: the cube in an homogeneous sinusoidal magnetic field [3], the hollow sphere and the FELIX brick from from some International Workshops for the comparison of eddy current codes [15].

Results obtained are also always compared with the results from the commercial formulations in ANSYS.

Results obtained with the new *T-Ω,Ω* formulation are in very good agreement with those from other codes and with analytical solutions and experimental solution where them are available.

It has been shown that the problem of the multiply connected regions can be overcome by introducing an artificial region of low conductivity rendering the problem simply connected.

The uniqueness of the problem solution can be ensured by imposing proper interface conditions on the electric vector potential in the interface surface between conducting and non-conducting regions.

It has been shown that the *T–Ω,Ω* formulation has the big advantage that the computer storage and CPU time can be considerably reduced especially when most part of the analyzed region is current free.

References[1]

[1] Kriezis E. E., Tisiboukis T.D, Panas S.M. and Tegopoulos J.A.. Eddy currents: Theory and applications. Proceedings of the IEEE Vol. 80 N. 10, October 1992, pp 1556,1589

[2] M.V.K. Chari, S.J. Salon, Numerical methods in electromagnetism. Academic Press 2000

[3] O. Biro and K. Preis. On the use of magnetic vector potential in the finite element analysis of three-dimensional eddy currents. IEEE Transaction on Magnetics Vol. 25 N. 4, July 1989, pp 3145,3159

[4] K.J. Binns, P.J. Lawrenson, C.W. Trowbridge, The Analytical and Numerical Solution of Electric and Magnetic Fields. John Wiley & Sons, Inc.1992

[5] M. N. O. Sadiku, Numerical Techniques in Electromagnetics. CRC Press, 2001

[6] D. A. Lowther, P.P.Silvester, Computer-Aided Design in Magnetics. Springer-Verlag, 1985

[7] M.Touma, Three-dimensional Finite Element Computation of Eddy Currents in Syncronous Machines. Technical Report No 350 1998 Department of Electric Power Engineering, Goteborg

[8] P.P. Silvester, R. L. Ferrari, Finite Element for electrical engineers. Cambridge University Press,1996

[9] N. Ottosen and H. Petersson, Introduction to the Finite Element Method. Prentice Hall.1992

[10] Heinz Knoepfel, Magnetic Fields: A comprehensive theoretical treatise for practical use. John Wiley & Sons, Inc.2000

[11] Finite Element in Electrical and Magnetic Field Problems. Edited by M.V.K. Chari and P.P. Silvester . John Wiley & Sons, 1980

[12] T. Nakata, N. Takahashi, K. Fujiwara and Y. Okada. Improvements of the *T-Ω, Ω* formulation for 3-D eddy current analysis. IEEE Transaction on Magnetics Vol. 24 N. 1, Juanuary 1988, pp 94,97

[13] W. Jaibing and T. Baoqian. Calculation of 3D eddy current problems using a modified

[1] References are reported in importance order

T-Ω method. IEEE Transaction on Magnetics Vol. 24 N. 1, Juanuary 1988, pp 114,117

[14] R. Albanese, R. Martone, G. Miano and G. Rubinacci. A T formulation for 3D finite element eddy current computation. IEEE Transaction on Magnetics Vol. 21 N. 6, November 1985, pp. 2299,2302

[15] COMPEL. The International Journal for Computation and Mathematics in Electrical and Electronic Engineering Vol. 7, Nos 1&2, 3-9 1988 BOOLE PRESS LIMITED

[16] Y. Yingyinng, X. Dexin, W. Jinming and O. Mohammed. A Multi-step Method for 3-D Nonlinear Transient Eddy Current Problems. IEEE Transaction on Magnetics Vol. 37 N. 5, September 2001, pp 3194,3197

[17] Guide to ANSYS User Programmable Features. ANSYS release 5.6. February 2000

[18] ANSYS, Inc. Theory Manual . ANSYS release 5.6, March 2001