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Petri Net Languages and Infinite Subsets of $\mathbb{N}^m$

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Families of Petri net languages are usually defined by varying the type of transition labeling and the class of subsets of $\mathbb{N}^m$ to be used as sets of final markings (m is the number of places). So far three main classes of subsets have been studied: the trivial class containing as single element $\mathbb{N}^m$, the class of finite subsets of $\mathbb{N}^m$, and the class of ideals (or covering subsets) of $\mathbb{N}^m$. In this paper we extend the known hierarchy of Petri net languages by considering the classes of semi-cylindrical, star-free, recognizable, rational (or semi-linear) subsets of $\mathbb{N}^m$. We compare the related Petri net languages. For arbitrarily labeled and for λ-free labeled Petri net languages, the above hierarchy collapses: one does not increase the generality by considering semi-linear accepting sets instead of the usual finite ones. However, for free-labeled and for deterministic Petri net languages, we show that one gets new distinct subclasses of languages, for which several decidability problems become solvable. We establish as intermediate results some properties of star-free subsets of general monoids. © 1999 Academic Press

1. INTRODUCTION

Petri net (PN) languages have received a lot of attention since the late seventies [14–16, 19–21, 27]. Comprehensive surveys on PN languages can be found in the work of Jantzen [16] and Peterson [21]. Different classes of PN languages have been defined, depending on the choice of transition labeling (free, λ-free, arbitrary) and on the choice of the final markings set F. In the literature [21, 16], three choices are common for F. Choosing F as the set of all reachable markings leads to the definition of **P**-type languages, which represent the prefix-closed behaviors of nets. Choosing F as a finite set leads to the definition of **L**-type languages. Choosing

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F as a covering set (i.e., *ideal*, see Section 3 below), leads to the definition of **G**-type languages.

Vidal-Naquet [27] and Pelz [20] studied the classes of deterministic languages. Determinism is a property of the labeling and of the net structure. Deterministic languages were introduced as a trade-off between modelling power and analytical tractability; it has been proved, in fact, that several important properties such as language containment, become decidable when the net is deterministic.

Recently, PN have become a standard model for the study of discrete event systems (DESs) and have been used within many different approaches such as supervisory control [13, 17, 26], logic controller [6], max-plus algebra [2], and stochastic processes [1]. Supervisory control theory [23] considers a DES as a language generator. In particular, two languages are associated with a given DES G: $L(G)$, the *prefix-closed* behavior, i.e. the set of all words generated by the system; and $L_{\mathrm{m}}(G)$, the *marked* behavior, i.e. the set of words that are accepted reaching a final state. Final states are useful to represent desirable terminal properties. E.g., in a manufacturing system one many require that no parts are left partially unprocesses within the system at the end of an operation.

When PN are used as DES models, both **L**-type and **G**-type languages have been used to represent the marked behavior of a net [11]. However, in many real cases, it is useful to consider more general sets of final markings. For instance, terminal states are frequently specified by constraints on a subset of resources (pallets being in a fixed position, machines being idle, etc.). This can be modeled by fixing the value of a *partial marking* (restriction of the marking to a subset of places), i.e. by taking as set of final markings a *cylinder* of $\mathbb{N}^m$, where m is the number of places of the net. More generally, we may take a *semicylindrical* subset, which is a finite union of cylinders. The semicylindrical subsets have appeared in the Petri net literature as incompletely specified sets [21], but until now have received relatively little attention. In some cases, it is desirable to mix partial marking constraints and covering constraints. For instance, the specification "at least two parts must be finished and the pallet must be empty" will be represented by a constraint of the form $M(p) \geqslant 2$ and $M(p') = 0$, with obvious notations. A natural way to handle such constraints is to introduce, *star-free* subsets of $\mathbb{N}^m$, which are the closure of finite subsets by the Boolean operations and addition. Other specifications require more sophisiticated sets. For instance, making lots of size k may be modeled requiring that the final number of tokens in a given place be a multiple of k. The corresponding set of final markings is *recognizable*, but not star-free, unless $k = 1$. Other useful specifications on the terminal behavior may require that the markings of two places p, p' be in a bounded-fairness relation (i.e., $|M(p) - M(p')| \leqslant K$ for some constant K). This arises, for instance, if two different tasks (service of customers, production of parts) have to be performed in almost identical quantities. More generally, one may wish to include ratio specifications (i.e., $|M(p) - rM(p')| \leqslant K$ for some integers r and K). This kind of properties can be expressed by allowing the set of final markings to be a *rational* (=*semilinear*) subset of $\mathbb{N}^m$. Indeed, rational final sets turn out to be natural, since the subsets of $\mathbb{N}^m$ definable by Presburger formulae [15] are precisely the rational subsets: Presburger formulae seem to contain all the practically "reasonable" specifications on final markings.

In this paper, we study the natural hierarchy of subsets of $\mathbb{N}^m$: finite, ideal, (semi)cylindrical, star-free, recognizable, rational (or semilinear). All these classes are standard, except the class of star-free subsets—a commutative analogue of star-free languages [22]—that with a remarkable exception [12] has received little attention in the literature. In the course of the paper we also incidentally derive some general results about star-free subsets of cartesian products of arbitrary monoids.

Considering different classes of final marking sets one obtains different classes of languages. Thus we study the hierarchy of Petri net languages, induced by the above hierarchy of subsets of $\mathbb{N}^m$.

For arbitrarily labeled and for λ-free labeled Petri net languages, this hierarchy collapses; one does not increase the generality by considering semilinear accepting sets instead of the usual finite ones. However, for free-labeled and for deterministic Petri net languages, we show that one gets new distinct subclasses of Petri net languages. We also prove that language containment remains decidable for the new deterministic classes we define.

The paper is structured as follows. Section 2 presents the notation on Petri nets. Section 3 introduces various classes of subsets of $\mathbb{N}^m$ and recalls their basic properties. All these classes are standard, except the class of star-free subsets of $\mathbb{N}^m$, which is characterized in Section 4, where general properties of star-free subsets of groups and of cartesian products of monoids are established. In Section 5 a Petri net language is associated to each of these classes. The properties of these languages for arbitrary and λ-free labeling are also studied. In Section 6 deterministic languages are considered. Part of this work has been presented in [8].

2. NOTATION

We first recall some classical definitions about Petri nets. See [18, 21] for more details. A *Place/Transition net* (P/T net) or a *Petri net* is a 4-tuple $N = (P, T, \text{Pre}, \text{Post})$, where P is a finite set of *places*, T is a finite set of *transitions*, $\text{Pre}: P \times T \to \mathbb{N}$, and $\text{Post}: P \times T \to \mathbb{N}$ are the *input* and *output functions*.

A *marking* is a vector $M: P \to \mathbb{N}$. A *marked net* (N, M_0) is a net N equipped with an initial marking M_0.

A transition $t \in T$ is *enabled* by a marking M if $M \geqslant \text{Pre}(\cdot, t)$. The *firing* of an enabled transition t generates a new marking $M' = M + \text{Post}(\cdot, t) - \text{Pre}(\cdot, t)$. When a marking M' is reached from marking M by executing a *firing sequence* of transitions $\sigma = t_1 \cdots t_k$ we write $M[\sigma\rangle M'$. We write $M[\sigma\rangle$ to indicate that σ may be executed from M. The set of markings reachable on a net N from a marking M is called the *reachability set* of M and is denoted as $R(N, M)$.

Let Σ denote a finite alphabet. A *Σ-labeled Petri net* [16, 21] is a 2-tuple $G = (N, l)$, where $N = (P, T, \text{Pre}, \text{Post})$ is a Petri net, $l: T \to \Sigma$ is a labeling function that assigns to each transition a label from the alphabet of events Σ.

Note that in our definition of labeled nets, we are assuming that l is a λ-free labeling function, according to the terminology of Peterson [21]; i.e., no transition is labeled with the empty string λ, while several transitions may have the same label. The mapping l will be extended to a morphism $T^* \to \Sigma^*$.

A labeled net G with an initial marking $M_0 \in \mathbb{N}^P$, and a (possibly infinite) set of *final* or *accepted* markings $F \subset \mathbb{N}^P$ can be considered as a language generator. The language *accepted* by G is the set of labels of firing sequences leading from the initial marking to a final marking:

$$L(G, M_0, F) = \{l(\sigma) \mid \sigma \in T^*, M_0[\sigma\rangle M, M \in F\}. \tag{1}$$

We denote by $\leqslant$ the prefix order on Σ^* (i.e., $u \leqslant w$ if $w = uz$ for some $z \in \Sigma^*$). For all $a \in \Sigma$, $w \in \Sigma^*$, we denote by $|w|_a$ the number of occurrences of the symbol a in w. We write $\subset$ for the inclusion of sets and $\subsetneq$ for the strict inclusion (i.e., $A \subsetneq B$ iff $A \subset B$ and $A \neq B$), we denote by $\not\leftrightarrow$ the incomparability relation ($A \not\leftrightarrow B$ iff $A \not\subset B$ and $B \not\subset A$).

A *monoid* $(S, \cdot)$ is a set S with an associative law $\cdot$ and a unit element e. A *commutative* monoid will be denoted additively ($+$ instead $\cdot$; 0 instead of e).

3. SOME CLASSICAL CLASSES OF SUBSETS OF $\mathbb{N}^m$

We next introduce various classes of subsets of $\mathbb{N}^m$, and recall their basic closure properties. All these facts are standard, except the characterization of star-free subsets of $\mathbb{N}^m$. These properties will be used intensively in the sequel, when defining the corresponding classes of Petri net languages:

1. We denote by $\mathtt{Triv}(\mathbb{N}^m)$ the "trivial" subset $\{\mathbb{N}^m\}$ of $\mathscr{P}(\mathbb{N}^m)$.[4]

2. We denote by $\mathtt{Fin}(\mathbb{N}^m)$ the set of finite subsets of $\mathbb{N}^m$.

3. Given a subset $I \subset \{1, ..., m\}$ and a vector $v \in \mathbb{N}^m$, the *cylinder* of basis (I, v) is the subset $C(I, v) = \{x \in \mathbb{N}^m \mid \forall i \in I, x_i = v_i\}$. We denote by $\mathtt{SCyl}(\mathbb{N}^m)$ the set of *finite unions* of cylinders, that we call *semicylindrical* subsets.

4. An *ideal* of the additive monoid $\mathbb{N}^m$ is a set $X \subset \mathbb{N}^m$ such that $x \in X$, $y \in \mathbb{N}^m \Rightarrow x + y \in X$. Thus an ideal X is an *upper set* for the usual order $\leqslant$, i.e., $x \in X$, $x \leqslant y \Rightarrow y \in X$. The set of ideals of $\mathbb{N}^m$ will be denoted by $\mathtt{Id}(\mathbb{N}^m)$. A *principal* ideal is a set of the form $\uparrow(x) = \{y \in \mathbb{N}^m \mid x \leqslant y\}$. As is well known [25, Theorem 3.12], an ideal of $\mathbb{N}^m$ is finitely generated (i.e., it is a finite union of principal ideals).

5. A subset $X \subset \mathbb{N}^m$ is *star-free* if it can be written as a finite expression involving finite sets, vector sum of subsets, and the Boolean operations (union, intersection, complement). More formally, the set of star-free subsets $\mathtt{Sf}(\mathbb{N}^m)$ is the least subset $\mathscr{X} \subset \mathscr{P}(\mathbb{N}^m)$ such that $\mathscr{X} \supset \mathtt{Fin}(\mathbb{N}^m)$, and $\forall X, Y \in \mathscr{X}$, $X \cup Y \in \mathscr{X}$, $X \cap Y \in \mathscr{X}$, $\complement X \in \mathscr{X}$, $X + Y \in \mathscr{X}$. A more effective characterization of star-free subsets if provided below.

6. A subset $X \subset \mathbb{N}^m$ is *recognizable* if there exists a finite monoid $(S, \cdot)$, a subset $K \subset S$, and a morphism $\varphi: (\mathbb{N}^m, +) \to (S, \cdot)$ such that $X = \varphi^{-1}(K)$. We denote by $\mathtt{Rec}(\mathbb{N}^m)$ the set of recognizable subsets. A more effective characterization of recognizable subsets is provided below.

[4] Here $\mathscr{P}(X)$ denotes the power set of a set X.

7. We denote by $\mathtt{Rat}(\mathbb{N}^m)$ the set of *rational* subsets of $\mathbb{N}^m$, i.e., the least subset of $\mathbb{N}^m$ containing $\mathtt{Fin}(\mathbb{N}^m)$ and stable by the operations $\cup$, $+$, $*$.[5] As is well known (see, e.g., [7]), a subset X is rational iff it is semilinear, i.e., iff it can be written as $X = \bigcup_{i \in I} (u_i + V_i^*)$ for some finite family $\{(u_i, V_i)\}_{i \in I} \subset \mathbb{N}^m \times \mathtt{Fin}(\mathbb{N}^m)$.

We denote by

$$\mathscr{H}(\mathbb{N}^m) = \{\mathtt{Triv}(\mathbb{N}^m), \mathtt{Fin}(\mathbb{N}^m), ..., \mathtt{Rat}(\mathbb{N}^m)\}$$

the set of above classes. When the specification of m will be irrelevant or clear from the context, we will write more simply $\mathscr{H}$, $\mathtt{Triv}$, $\mathtt{Fin}$, etc., instead of $\mathscr{H}(\mathbb{N}^m)$, $\mathtt{Triv}(\mathbb{N}^m)$, $\mathtt{Fin}(\mathbb{N}^m)$, etc.

To obtain more effective characterizations of $\mathtt{Rec}$ and $\mathtt{Sf}$, we observe that these classes can be defined in a general (possibly noncommutative) monoid $(S, \cdot)$, and not only in $(\mathbb{N}^m, +)$, by merely replacing $\mathbb{N}^m$ by S and $+$ by $\cdot$ in the above definitions. Since $(\mathbb{N}^m, +)$ is the m-fold cartesian product of $(\mathbb{N}, +)$, this raises the question of relating recognizable (resp. star-free) subsets of the cartesian product monoid $S \times S'$ with recognizable (resp. star-free) subsets of S and S' for arbitrary monoids $(S, \cdot)$ and $(S', \cdot)$. In the case of recognizable subsets, the answer is given by the following classical result. Given two subsets $\mathscr{X} \subset \mathscr{P}(S)$, $\mathscr{X}' \subset \mathscr{P}(S')$, we denote by $\mathscr{X} \otimes \mathscr{X}'$ the subset of $\mathscr{P}(S \times S')$ with as elements all finite unions of sets of the form $X \times X'$ with $X \in \mathscr{X}$, $X' \in \mathscr{X}'$.

Lemma 1 [3, Theorem 1.5]. *For arbitrary monoids $(S, \cdot)$ and $(S', \cdot)$, we have*

$$\mathtt{Rec}(S \times S') = \mathtt{Rec}(S) \otimes \mathtt{Rec}(S').$$

Hence we have the following elementary characterization of recognizable subsets of $\mathbb{N}^m$ which will be used later on.

Proposition 2. *Let X be a subset of $\mathbb{N}^m$. Three assertions are equivalent:*

1. *X is recognizable;*
2. *X is a finite union of sets of the form*

$$D(v, a) = \{x \in \mathbb{N}^m \mid (\forall i \in \{1, ..., m\})(\exists k \in \mathbb{N})\ x_i = ka_i + v_i\}, \tag{2}$$

where v, $a \in \mathbb{N}^m$;

3. *X is a finite union of sets of the form*

$$A_1 \times \cdots \times A_m, \tag{3}$$

where each A_j can be written as $A_j = v_j + a_j\mathbb{N}$, with v_j, $a_j \in \mathbb{N}$.

[5] Recall that for $X \subset \mathbb{N}^m$, $X^* = \{0\} \cup X \cup (X + X) \cup (X + X + X) \cup \cdots$.

Proof. We first prove $2 \Leftrightarrow 3$. This result is immediate, observing that $A_1 \times \cdots \times A_m = D(v, a)$ with $a = (a_j)_{1 \leqslant j \leqslant m}$ and $v = (v_j)_{1 \leqslant j \leqslant m}$.

We finally prove $1 \Leftrightarrow 3$. By Lemma 1 it is sufficient to prove $1 \Leftrightarrow 3$ when $m = 1$. Recall that in the one-dimensional case, rational and recognizable subsets coincide; i.e., $\mathtt{Rat}(\mathbb{N}) = \mathtt{Rec}(\mathbb{N})$. This is a special case of the Kleene–Shützenberger theorem (see, e.g., [4]). Since rational and semilinear subsets of $\mathbb{N}$ coincide, recognizable subsets are exactly the finite unions of subsets of the form $u + \{v_1, ..., v_k\}^*$ with $u, v_1, ..., v_k \in \mathbb{N}$. By using the identities $(Y \cup Z)^* = Y^* + Z^*$ for all $Y, Z \subset \mathbb{N}$ and $\{b\}^* + \{c\}^* = F_{b,c} \cup (k_{b,c} + \{\gcd(b, c)\}^*)$ for all $b, c \in \mathbb{N}$, where $F_{b,c}$ (resp. $k_{b,c}$) is a finite subset (resp. an element) of $\mathbb{N}$, depending on b, c (the first identity is a classical commutative rational identity [5]; the second identity follows readily from Bezout's theorem), we can rewrite $u + \{v_1, ..., v_k\}^*$ as a finite union of sets of the form $v + a\mathbb{N}$, with $v, a \in \mathbb{N}$. Thus $1 \Leftrightarrow 3$. ∎

4. STAR-FREE SUBSETS OF $\mathbb{N}^m$

The definition of star-free subsets extends that of star-free languages, seen as subsets of free (noncommutative) monoids. Schützenberger's characterization (see, e.g., [22]) of star-free languages in terms of aperiodic syntactic monoids and its extension to trace monoids [12] is a deep result. For star-free subsets of $\mathbb{N}^m$, we next give a much more elementary characterization, based on the following star-free analogue of Lemma 1.

LEMMA 3. *For arbitrary monoids* $(S, \cdot)$ *and* $(S', \cdot)$, *we have*

$$\mathtt{Sf}(S \times S') \subset \mathtt{Sf}(S) \otimes \mathtt{Sf}(S').$$

Moreover, the equality holds if S *and* S' *admit finite sets of generators* Σ *and* Σ', *respectively, such that* $e = \complement(\Sigma S)$ *and* $e' = \complement(\Sigma' S')$, *where* e, e' *denote the unit elements of* S, S', *respectively.*

Proof. Clearly,

(i) $\mathtt{Sf}(S) \otimes \mathtt{Sf}(S') \supset \mathtt{Fin}(S \times S')$, and

(ii) $\mathtt{Sf}(S) \otimes \mathtt{Sf}(S')$ is stable by union.

Let $H, K \in \mathtt{Sf}(S)$, $H', K' \in \mathtt{Sf}(S')$. Since $(H \times H')(K \times K') = HK \times H'K'$, we get that

(iii) $\mathtt{Sf}(S) \otimes \mathtt{Sf}(S')$ is stable by product.

Since $\complement(H \times H') = \complement H \times S' \cup S \times \complement H'$, $S = \complement \varnothing \in \mathtt{Sf}(S)$, and similarly, $S' \in \mathtt{Sf}(S')$, we get that

(iv) $\mathtt{Sf}(S) \otimes \mathtt{Sf}(S')$ is stable by complement.

Since $\mathtt{Sf}(S \times S')$ is the least subset of $\mathscr{P}(S \times S')$ that satisfies (i)–(iv), we get that $\mathtt{Sf}(S \times S') \subset \mathtt{Sf}(S) \otimes \mathtt{Sf}(S')$, as announced.

To show that the equality holds under the assumption of the lemma, we have to check that if $H \in \mathtt{Sf}(S)$ and $H' \in \mathtt{Sf}(S')$, then $H \times H' \in \mathtt{Sf}(S \times S')$. Since $H \times H' = (H \times e')(e \times H')$, it is enough to check that $H \times e' \in \mathtt{Sf}(S \times S')$.

Since $e' = \complement(\Sigma' S')$, we have for all $K \subset S$, $\complement K \times e' = \complement K \times \complement(\Sigma' S') = \complement(K \times S' \cup S \times \Sigma' S')$, i.e.,

$$\complement K \times e' = \complement \left(K \times S' \cup \bigcup_{a' \in \Sigma'} (e \times a')(S \times S') \right). \tag{4}$$

We will also use the elementary identities, valid for all K, $L \subset S$, $K' \subset S'$,

$$(K \cup L) \times K' = K \times K' \cup L \times K', \tag{5}$$

$$KL \times K' = (K \times K')(L \times e'). \tag{6}$$

Let $H \in \mathtt{Sf}(S)$ be given by a finite expression involving the operations $\bigcup$, $\complement$, $\cdot$ and empty or one-element subsets of S. Properties (4)–(6) allow us to rewrite $H \times e'$ as a finite expression involving the operators $\bigcup$, $\complement$, $\cdot$, and subsets of the form $R \times K'$, where R (resp., K') is an empty, one element, or full—i.e., $R = S$ (resp., $K' = S'$)—subset of S (resp., S'). It remains to prove that for any such R, K', we have $R \times K' \in \mathtt{Sf}(S \times S')$. If either $R = \emptyset$, or $K' = \emptyset$, or both R and K' are one-element subsets, or $R = S$ and $K' = S'$, this is clear. It remains to consider the case $R = S$ and $K' = \{m'\}$, with $m' \in S'$ (the dual case follows by symmetry). We have $R \times K' = S \times m' = (e \times m')(S \times e')$. Applying (4) again, we get $S \times e' = \complement \emptyset \times e' = \complement(\bigcup_{a' \in \Sigma'} (e \times a')(S \times S'))$, which shows that $R \times K'$ is star-free. ∎

Let $\mathtt{co}-\mathtt{Fin}(S)$ denote the class of subsets of S with finite complement. The fact that the inclusion in Lemma 3 can be strict will be derived from the following general observation.

LEMMA 4. *If G is a group, then*

$$\mathtt{Sf}(G) = \mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G). \tag{7}$$

Proof. The inclusion $\mathtt{Sf}(G) \supset \mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G)$ is trivial. To show that the equality holds, we have to check that $\mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G)$ is stable by the Boolean operations, which is immediate, and also by the product. Let X, $Y \in \mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G)$. We distinguish the following cases.

— If X, $Y \in \mathtt{Fin}(G)$, $XY \in \mathtt{Fin}(G) \subset \mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G)$.

— If $X \neq \emptyset$ and $Y \in \mathtt{co}-\mathtt{Fin}(G)$, we have $XY \supset mY$, where m denotes any element of X. The following assertions are equivalent:

$$z \in \complement(mY),$$

$$\forall y \in Y, \qquad z \neq my$$

$$\forall y \in Y, \qquad m^{-1}z \neq y$$

$$m^{-1}z \in \complement Y;$$

thus, $\complement(mY) = m \complement Y \in \mathtt{Fin}(G)$, i.e., $mY \in \mathtt{co}-\mathtt{Fin}(G)$. Since XY contains $mY \in \mathtt{co}-\mathtt{Fin}(G)$, $XY \in \mathtt{co}-\mathtt{Fin}(G) \subset \mathtt{Fin}(G) \cup \mathtt{co}-\mathtt{Fin}(G)$.

— The remaining case $X \in \mathtt{co-Fin}(G)$ and $Y \neq \emptyset$ follows by symmetry. ∎

The following counterexample shows that the inclusion in Lemma 3 can be strict.

EXAMPLE 5. Consider the group $G = \mathbb{Z}^2$ and $X = 0 \times \mathbb{Z}$. By definition, $X \in \mathtt{Sf}(\mathbb{Z}) \otimes \mathtt{Sf}(\mathbb{Z})$, but $X \notin \mathtt{Fin}(\mathbb{Z}^2) \cup \mathtt{co-Fin}(\mathbb{Z}^2)$, and $\mathtt{Fin}(\mathbb{Z}^2) \cup \mathtt{co-Fin}(\mathbb{Z}^2) = \mathtt{Sf}(\mathbb{Z}^2)$ by Lemma 4.

The following proposition characterizes star-free subsets of $\mathbb{N}^m$.

PROPOSITION 6. *Let X be a subset of* $\mathbb{N}^m$. *Three assertions are equivalent:*

1. *X is star-free;*
2. *X is a finite union of sets of the form*

$$K(I, v) = \{x \in \mathbb{N}^m \mid x \geqslant v,\ \forall i \in I,\ x_i = v_i\} = \uparrow(v) \cap C(I, v), \tag{8}$$

where $v \in \mathbb{N}^m$, $I \subset \{1, ..., m\}$;

3. *X is a finite union of sets of the form*

$$A_1 \times \cdots \times A_m, \tag{9}$$

where each A_j *can be written as* $A_j = v_j + a_j\mathbb{N}$, *with* $v_j \in \mathbb{N}$ *and* $a_j \in \{0, 1\}$.

Proof. We first prove $2 \Leftrightarrow 3$. This result is immediate, observing that $A_1 \times \cdots \times A_m = K(I, v)$ with $I = \{j \in \{1, ..., m\} \mid a_j = 0\}$ and $v = (v_j)_{1 \leqslant j \leqslant m}$.

We finally prove $1 \Leftrightarrow 3$. We note that $\mathbb{N}^m$ has a finite set of generators $\Sigma = \{(1, 0, ..., 0),\ (0, 1, ..., 0), ...,\ (0, 0, ..., 1)\}$, with $\{0\} = \complement(\Sigma + \mathbb{N}^m)$. Thus, by Lemma 3 $\mathtt{Sf}(\mathbb{N}^m) = \mathtt{Sf}(\mathbb{N}) \otimes \cdots \otimes \mathtt{Sf}(\mathbb{N})$, and it is sufficient to prove $1 \Leftrightarrow 3$ when $m = 1$. Let $\mathscr{X}$ denote the set of finite unions of sets of the form $v + a\mathbb{N}$, with $v \in \mathbb{N}$, $a \in \{0, 1\}$. Clearly, $v + 0\mathbb{N} = v$ and $v + 1\mathbb{N} = v + \mathbb{N}$ are star-free for $\mathbb{N} = \complement\emptyset$ and $\emptyset \in \mathtt{Fin}(\mathbb{N})$. Thus, $\mathscr{X} \subset \mathtt{Sf}(\mathbb{N})$. Since $\mathscr{X}$ is clearly closed under the Boolean operations and under $+$, and $\mathtt{Fin}(\mathbb{N}) \subset \mathscr{X}$, we get $\mathtt{Sf}(\mathbb{N}) \subset \mathscr{X}$. Thus, $\mathtt{Sf}(\mathbb{N}) = \mathscr{X}$ which shows $1 \Leftrightarrow 3$. ∎

We conclude this preliminary part by comparing all the classes of $\mathscr{H}$.

PROPOSITION 7. *The classes* $\mathscr{X} \in \mathscr{H}$ *are ordered as shown, where an arrow* $X \to Y$ *means that* $X \subset Y$. (*All the inclusions are strict for* $m \geqslant 1$, *except the inclusion* $\mathtt{Rec}(\mathbb{N}^m) \subset \mathtt{Rat}(\mathbb{N}^m)$ *which is strict for* $m \geqslant 2$. *Classes that are not connected by a directed path are incomparable.*)

$$\begin{array}{ccccc} \mathtt{Fin}(\mathbb{N}^m) & \to & \mathtt{SCyl}(\mathbb{N}^m) & \searrow & \\ & \nearrow & & & \mathtt{Sf}(\mathbb{N}^m) \to \mathtt{Rec}(\mathbb{N}^m) \to \mathtt{Rat}(\mathbb{N}^m) \\ \mathtt{Triv}(\mathbb{N}^m) & \to & \mathtt{Id}(\mathbb{N}^m) & \nearrow & \end{array}$$

Proof. The inclusions $\mathtt{Fin} \subsetneq \mathtt{SCyl}$, $\mathtt{Triv} \subsetneq \mathtt{Id}$, $\mathtt{Triv} \subsetneq \mathtt{SCyl}$, are obvious. We note that $\mathtt{SCyl}$ and $\mathtt{Id}$ are incomparable. Since $\mathtt{SCyl} \cup \mathtt{Id} \subset \mathtt{Sf}$ by Proposition 6, this implies that the inclusions $\mathtt{SCyl} \subset \mathtt{Sf}$, $\mathtt{Id} \subset \mathtt{Sf}$ are strict.

To show that $\mathtt{Sf} \subset \mathtt{Rec}$, it is enough to note that, trivially, $\mathtt{Fin} \subset \mathtt{Rec}$ and that $\mathtt{Rec}$ is closed by the Boolean operations and vector sum. The first closure property is a classical result [3, Chap. III, Proposition 1.1], which holds in an arbitrary monoid. The second closure property follows from the characterization of recognizable subsets of $\mathbb{N}^m$ given in Proposition 2, point 2.

The inclusion $\mathtt{Sf} \subset \mathtt{Rec}$ is strict; e.g., $2\mathbb{N}^m$ is a recognizable subset which is not star-free (by the characterization of Eq. (8)).

Classically, the inclusion $\mathtt{Rec} \subset \mathtt{Rat}$ holds in an arbitrary finitely generated monoid (see e.g., [3, Chap. 3]). The strict inclusion for $\mathbb{N}^m$, $m \geqslant 2$, is well known (e.g., this follows from characterization (3); consider the diagonal $D = (1, 1, ..., 1)^* \subset \mathbb{N}^m$ which is rational but not recognizable). ∎

Proposition 8. *All the classes of* $\mathscr{H}$ *are stable by union and intersection.* $\mathtt{Sf}$, $\mathtt{Rec}$, $\mathtt{Rat}$ *are stable by complement.* $\mathtt{Fin}$, $\mathtt{Id}$, $\mathtt{Sf}$, $\mathtt{Rec}$, $\mathtt{Rat}$ *are stable by sum.*

Proof. The closure properties for $\mathtt{Fin}$, $\mathtt{SCyl}$, $\mathtt{Id}$, $\mathtt{Sf}$ are automatic. The closure of recognizable subsets by $\cap$, $\cup$, $\complement$ is universal and elementary; it holds for an arbitrary monoid and not only for $\mathbb{N}^m$ (see, e.g., [3]). The closure of rational subsets of $\mathbb{N}^m$ by complement and intersection is classical (see, e.g., Eilenberg and Schützenberger [7]). The other assertions are clear. ∎

5. PETRI NET LANGUAGES

With each of the above classes of subsets of $\mathbb{N}^m$, we associate a class of Petri net languages.

Definition 9. Let $\mathscr{X} \in \mathscr{H}$. We say that a language L is a $\mathscr{X}$-type Petri net language if there exists a λ-free labeled PN $G = (N, l)$ with initial marking M_0 and set of accepting states $F \in \mathscr{X}$ such that $L = L(G, M_0, F)$. We denote by $\mathscr{L}\mathscr{X}$ the set of $\mathscr{X}$-type languages.

Remark 10. Some of these classes are well known in the literature:

- For $\mathscr{X} = \mathtt{Fin}$, we obtain the class $\mathscr{L}\mathtt{Fin}$ usually denoted **L**, following Peterson.
- For $\mathscr{X} = \mathtt{Triv}$, all the reachable markings are accepted; thus, the associated class $\mathscr{L}\mathtt{Triv}$ coincides with the class **P** of Peterson (composed of prefix closed languages).
- For $\mathscr{X} = \mathtt{Id}$, we obtain the class of weak languages, usually denoted **G** (in which all the markings *covering* a finite set of markings are accepted).
- For $\mathscr{X} = \mathscr{L}\mathtt{SCyl}$, we obtain a class of languages that was first considered by Peterson [21], who called the final marking sets in $\mathtt{SCyl}$ *incompletely specified.*

It is also possible to consider classes of labeling functions other than the λ-free:

- We define the subclasses of *free* PN languages $\mathscr{L}_f\mathscr{X}$ by requiring the labeling $l\colon T \to \Sigma$ to be injective.

• A Σ-labeled Petri net $G=(N, l)$ with initial marking M_0 is *deterministic* if for any $w \in \Sigma^*$ there exists at most one marking reachable in G from M_0 while generating w. The corresponding classes of deterministic PN languages will be denoted $\mathscr{L}_d \mathscr{X}$.

• When we allow l to be erasing (i.e., when l is a map $T \to \{\lambda\} \cup \Sigma$), we obtain the new class of *arbitrary* PN languages $\mathscr{L}_\lambda \mathscr{X}$.

It is clear that for all $\mathscr{X} \in \mathscr{H}$, $\mathscr{L}_f \mathscr{X} \subset \mathscr{L}_d \mathscr{X} \subset \mathscr{L} \mathscr{X} \subset \mathscr{L}_\lambda \mathscr{X}$. We will see in Corollary 18 below that all these inclusions are strict.

The main result of this section consists in showing that for λ-free and arbitrary PN languages, the use of infinite sets of final markings (following the hierarchy outlined in the previous section) does not extend the corresponding classes of PN languages. This result follows from the lemma below, which shows that λ-labeled transitions without output places do not increase the language-defining power of λ-free PN generators.

LEMMA 11. *Let G be an arbitrary labeled Petri net generator with a finite set of final markings* $F \in \mathtt{Fin}$. *Assume that for all transitions* $t \in T$ *labeled by the empty string it holds that* $\mathrm{Post}(\cdot, t) = (0 \cdots 0)$. *Then* $L(G, M_0, F) \in \mathscr{L}\mathtt{Fin}$.

Proof. We will show that there exists a λ-free labeled generator G' and a finite set F' of final markings such that $L(G', M_0, F') = L(G, M_0, F)$. Let $T^\lambda = \{t_1^\lambda, ..., t_r^\lambda\}$ be the set of transitions of G labeled by the empty string and let $T^\Sigma = T \backslash T^\lambda$. Without loss of generality we may assume that for all $t^\lambda \in T^\lambda$, $\mathrm{Pre}(\cdot, t^\lambda) \neq (0 \cdots 0)$ since otherwise t^λ could be removed without changing the language of the net.

A firing sequence σ of G can always be written as

$$\sigma = \sigma_0^\lambda t_1 \sigma_1^\lambda \cdots t_k \sigma_k^\lambda,$$

where $\sigma_i^\lambda = t_{i,1}^\lambda \cdots t_{i,r_i}^\lambda \in (T^\lambda)^*$ and $t_i \in T^\Sigma$. We say that such a sequence is *minimal* if the λ transitions are fired "as soon as possible," formally, if for all i and for all $j: 1 \leqslant j \leqslant r_i$, the sequence $\sigma' = \sigma_0^\lambda t_1 \cdots \sigma_{i-1}^\lambda t_{i,j}^\lambda t_i$ is not firable. Clearly, possibly after a finite number of moves of λ-transitions (which does not modify the label), we may assume that σ is minimal.

We claim that the length of the σ_i^λ factors in a *minimal* sequence is bounded by a fixed integer q (depending on the net and the initial marking). Indeed, since $\mathrm{Post}(\cdot, t^\lambda) = (0 \cdots 0)$, for all λ-transitions t^λ, each firing of t^λ reduces the total number of tokens by at least one unit, and therefore, $|\sigma_0^\lambda| \leqslant q' \overset{\text{def}}{=} \sum_p M_0(p)$. Next, for $i \geqslant 1$ consider $\sigma'' = \sigma_0^\lambda t_1 \cdots t_{i-1} \sigma_{i-1}^\lambda$, with $M_0[\sigma''\rangle M''$. Since $t_{i,j}^\lambda t_i$ is not firable at M'', there exists a place p_j such that $M''(p_j) < \mathrm{Pre}(p_j, t_{i,j}^\lambda) + \mathrm{Pre}(p_j, t_i)$. Let $K_p = \sup_{t \in T^\Sigma, t^\lambda \in T^\lambda} \{\mathrm{Pre}(p, t^\lambda) + \mathrm{Post}(p, t)\}$. We have $M''[t_i\rangle M'$ with $M'(p_j) < K_{p_j}$. Since every firing of a λ-transition consumes at least one token in such a p_j place, we have $|\sigma_i^\lambda| \leqslant q'' \overset{\text{def}}{=} \sum_p (K_p - 1)$. Setting $q = \max(q', q'')$, we obtain $|\sigma_i^\lambda| \leqslant q$ for all factors σ_i^λ of a minimal sequence.

We introduce a new generator G' with the same places and a new alphabet of transitions: $T' = T^\Sigma \cup T^\lambda T^\Sigma \cup \cdots \cup (T^\lambda)^q T^\Sigma$. For each new transition $t' = t_1^\lambda \cdots t_r^\lambda t$, with $t_i^\lambda \in T^\lambda$, $t \in T^\Sigma$, we set $\mathrm{Pre}(\cdot, t') = \sum_i \mathrm{Pre}(\cdot, t_i^\lambda) + \mathrm{Pre}(\cdot, t)$, $\mathrm{Post}(\cdot, t') =$

Post$(\cdot, t)$, $l(t') = l(t)$, so that the firing of t' has the same precondition, effect, and label as the consecutive firing of the sequence $t_1^\lambda \cdots t_r^\lambda t$ in the original net.

Clearly, all the firable sequences of transitions of G' correspond to firable sequences of transitions of the original net which, in addition, lead to the same marking. Conversely, to each minimal sequence of the original net ending with a transition $t \in T^\Sigma$ corresponds a firable sequence of G', which also leads to the same marking. Moreover, this correspondence preserves labels.

We still have to take into account, however, the possibility that some minimal sequence accepted by G may end with a sequence σ_k^λ of λ-transitions. Since the length of σ_k^λ is bounded by q we let

$$F' = F + \bigcup_{j \leqslant q} V^j, \qquad \text{where} \quad V = \{\text{Pre}(\cdot, t^\lambda) \mid t^\lambda \in T^\lambda\}. \quad \blacksquare$$

Theorem 12. *The classes of λ-free PN languages are ordered as shown. (All the inclusions are strict.)*

$$\mathscr{L}\texttt{Triv}(=\mathbf{P}) \to \mathscr{L}\texttt{Id}(=\mathbf{G}) \to \mathscr{L}\texttt{Fin}(=\mathbf{L}) = \mathscr{L}\texttt{CSyl} = \mathscr{L}\texttt{Sf} = \mathscr{L}\texttt{Rec} = \mathscr{L}\texttt{Rat}.$$

Proof. The first two inclusions are well known [16, 19]. Peterson [21] has shown that $\mathscr{L}\texttt{SCyl} = \mathscr{L}\texttt{Fin}$. To prove the other relations, note first that $\mathscr{X} \subset \mathscr{X}'$ implies $\mathscr{L}\mathscr{X} \subset \mathscr{L}\mathscr{X}'$. Hence, by Proposition 7 it follows that $\mathscr{L}\texttt{Fin} \subset \mathscr{L}\texttt{SCyl} \subset \mathscr{L}\texttt{Sf} \subset \mathscr{L}\texttt{Rec} \subset \mathscr{L}\texttt{Rat}$. We prove that $\mathscr{L}\texttt{Rat} \subset \mathscr{L}\texttt{Fin}$.

Let G be a Petri net generator with initial marking M_0 and a linear set of final markings $F = v + \{u_1, ..., u_r\}^*$ (with $v, u_1, ..., u_r \in \mathbb{N}^m$). Construct a new net G' by adding to G a set of r new λ-transitions t_i^λ with Pre$(\cdot, t_i^\lambda) = u_i$ and Post$(\cdot, t_i^\lambda) = (0 \cdots 0)$. Consider $F' = \{v\}$ as set of final markings for G'. It is easy to check that $L(G, M_0, F) = L(G', M_0, F')$. Even if G' is an arbitrary labeled Petri net, Lemma 11 shows that $L(G', M_0, F') \in \mathscr{L}\texttt{Fin}$. Now, let G be a Petri net generator with a semilinear set of final markings F. The semilinear set can be written as the union of a finite number of linear sets F^i. Now $L(G, M_0, F) = \bigcup_i L(G, M_0, F^i)$. But we have shown that for all i, $L(G, M_0, F^i) \in \mathscr{L}\texttt{Fin}$ and, since $\mathscr{L}\texttt{Fin}$ is closed under union [16], we have proved that $\mathscr{L}\texttt{Rat} \subset \mathscr{L}\texttt{Fin}$. $\blacksquare$

Theorem 13. *The classes of arbitrary PN languages are ordered as shown. (The first inclusion is strict.)*

$$\mathscr{L}_\lambda\,\texttt{Triv} \to \mathscr{L}_\lambda\,\texttt{Id} \to \mathscr{L}_\lambda\,\texttt{Fin} = \mathscr{L}_\lambda\,\texttt{SCyl} = \mathscr{L}_\lambda\,\texttt{Sf} = \mathscr{L}_\lambda\,\texttt{Rec} = \mathscr{L}_\lambda\,\texttt{Rat}.$$

Proof. The first two inclusions are classical [16, 19], but it is not known if the second is strict. The construction of the previous theorem may be used to prove that $\mathscr{L}_\lambda\texttt{Fin} = \mathscr{L}_\lambda\texttt{Rat}$. $\blacksquare$

6. DETERMINISTIC PETRI NET LANGUAGES

In this section we consider deterministic Petri net generators. We show that the different classes of final marking sets create a proper hierarchy of deterministic languages similar to that described in Proposition 7. We also show that the complement of a deterministic language is a PN language.

THEOREM 14. *The classes of deterministic PN languages are ordered as shown. (All inclusions are strict. Classes that are not connected by a direct path are incomparable.)*

$$\begin{array}{lll} \mathscr{L}_d\texttt{Fin} \to \mathscr{L}_d\texttt{SCyl} & \searrow & \\ \qquad\quad \nearrow & & \mathscr{L}_d\texttt{Sf} = \mathscr{L}_d\texttt{Rec} \to \mathscr{L}_d\texttt{Rat} \\ \mathscr{L}_d\texttt{Triv} \to \mathscr{L}_d\texttt{Id} & \nearrow & \end{array}$$

Proof. Note first that $\mathscr{X} \subset \mathscr{X}'$ implies $\mathscr{L}_d\mathscr{X} \subset \mathscr{L}_d\mathscr{X}'$. Hence by Proposition 7, it follows that $\mathscr{L}_d\texttt{Fin} \subset \mathscr{L}_d\texttt{SCyl} \subset \mathscr{L}_d\texttt{Sf} \subset \mathscr{L}_d\texttt{Rec} \subset \mathscr{L}_d\texttt{Rat}$, $\mathscr{L}_d\texttt{Triv} \subset \mathscr{L}_d\texttt{Id} \subset \mathscr{L}_d\texttt{Sf}$. We are left to prove that the inclusions are strict and to prove that the other relations hold.

1. ($\mathscr{L}_d\texttt{Triv} \not\leftrightarrow \mathscr{L}_d\texttt{Fin}$). See [11].

2. ($\mathscr{L}_d\texttt{Id} \not\leftrightarrow \mathscr{L}_d\texttt{Fin}$). See [11]. It also holds ($\mathscr{L}_d\texttt{Id} \cap \mathscr{L}_d\texttt{Fin} = \mathbf{Rat}$), where **Rat** denotes the set of regular languages [9].

3. ($\mathscr{L}_d\texttt{Id} \not\leftrightarrow \mathscr{L}_d\texttt{SCyl}$). Since $\mathscr{L}_d\texttt{Id} \not\leftrightarrow \mathscr{L}_d\texttt{Fin} \subset \mathscr{L}_d\texttt{SCyl}$, it holds that $\mathscr{L}_d\texttt{SCyl} \not\subset \mathscr{L}_d\texttt{Id}$. We just need to prove that $\mathscr{L}_d\texttt{Id} \not\subset \mathscr{L}_d\texttt{SCyl}$.

Consider the language $L = \{w \in \{a, b\}^* \mid (\forall s \leqslant w)\ |s|_a \geqslant |s|_b,\quad |w|_a \geqslant |w|_b + 1\}$. $L \in \mathscr{L}_d\texttt{Id}$ since it is the language of the net in Fig. 1a with $F = \uparrow(1)$. We will prove, by contradiction, that $L \notin \mathscr{L}_d\texttt{SCyl}$.

In fact, assume $L = L(G', M_0, F')$ for a deterministic PN generator G' with m places and for a semicylindrical set $F' = \bigcup_{j=1}^{r} C(I_j, v_j)$. For $i > 0$, let M_i be the unique marking reached in G' by generating the string a^i. It is possible to extract from the sequence $M_1, M_2, \ldots$ a subsequence $M_{\alpha(1)}, M_{\alpha(2)}, \ldots$ such that $M_{\alpha(k)} < M_{\alpha(k+1)}$ and such that there exists $I \subset \{1, \ldots, m\}$ with $(\forall i \in I)\ M_{\alpha(k)}(i) = M_{\alpha(k+1)}(i)$, and $(\forall i \notin I)\ M_{\alpha(k)}(i) < M_{\alpha(k+1)}(i)$. Now let σ be the firing sequence from $M_{\alpha(1)}$ such that $l(\sigma) = b^{\alpha(1)}$. The same sequence σ may be fired from all markings $M_{\alpha(k)}$. Thus we can write $M_{\alpha(k)}[\sigma\rangle M'_{\alpha(k)}$. Clearly $M'_{\alpha(k)} < M'_{\alpha(k+1)}$ and $(\forall i \in I)\ M'_{\alpha(k)}(i) = M'_{\alpha(k+1)}(i)$, while $(\forall i \notin I)\ M'_{\alpha(k)}(i) < M'_{\alpha(k+1)}(i)$. Then $M'_{\alpha(1)} \notin F'$, while for all $k > 1$, $M'_{\alpha(k)} \in F'$. Now for $k > 1$ choose $v_{j_k} \in \{v_1, \ldots, v_r\}$ such that $M'_{\alpha(k)} \in C(I_{j_k}, v_{j_k})$. Clearly, $I_{j_k} \not\subset I$; otherwise $M'_{\alpha(1)}$ would be in $C(I_{j_k}, v_{j_k})$ and, hence, would be final. However, if $I_{j_k} \backslash I \neq \emptyset$, then $M'_{\alpha(k')} \notin C(I_{j_k}, v_{j_k})$ if $k \neq k'$. Hence there must exist an infinite set of vectors v_{j_k} and this contradicts the hypothesis that r is finite.

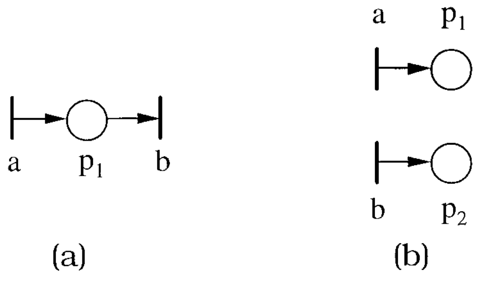

FIG. 1. Two Petri nets.

4. ($\mathscr{L}_d\texttt{Fin} \subsetneq \mathscr{L}_d\texttt{SCyl}$) and ($\mathscr{L}_d\texttt{Triv} \subsetneq \mathscr{L}_d\texttt{SCyl}$). Follow from the fact that $\mathscr{L}_d\texttt{Triv} \subset \mathscr{L}_d\texttt{SCyl}$ and $\mathscr{L}_d\texttt{Fin} \subset \mathscr{L}_d\texttt{SCyl}$, while $\mathscr{L}_d\texttt{Triv} \not\leftrightarrow \mathscr{L}_d\texttt{Fin}$.

5. ($\mathscr{L}_d\texttt{Triv} \subsetneq \mathscr{L}_d\texttt{Id}$). Follows from the fact that $\mathscr{L}_d\texttt{Triv} \subset \mathscr{L}_d\texttt{SCyl}$ and $\mathscr{L}_d\texttt{Id}$ and $\mathscr{L}_d\texttt{SCyl}$ are incomparable.

6. ($\mathscr{L}_d\texttt{SCyl} \subsetneq \mathscr{L}_d\texttt{Sf}$) and ($\mathscr{L}_d\texttt{Id} \subsetneq \mathscr{L}_d\texttt{Sf}$). Follow from the fact that $\mathscr{L}_d\texttt{Id} \subset \mathscr{L}_d\texttt{Sf}$, $\mathscr{L}_d\texttt{SCyl} \subset \mathscr{L}_d\texttt{Sf}$, while $\mathscr{L}_d\texttt{Id} \not\leftrightarrow \mathscr{L}_d\texttt{SCyl}$.

7. ($\mathscr{L}_d\texttt{Sf} = \mathscr{L}_d\texttt{Rec}$). Let $L = L(G, M_0, F)$ be the language generated by a deterministic PN generator G with initial marking M_0 and with recognizable set $F \in \texttt{Rec}(\mathbb{N}^m)$ of final markings. We will show how to construct a new labeled net G' such that $L = L(G', M'_0, F')$ for a suitable choice of initial marking $M'_0 \in \mathbb{N}^{m'}$ and final set $F' \in \texttt{Sf}(\mathbb{N}^{m'})$.

We will first consider the case where F is a single subset of the form (2), i.e., the final periodicity of each place p_i (as in Proposition 2) is given by the integer a_i, while the corresponding maximal integer is v_i. Let $P = \{p_1, ..., p_m\}$ be the set of places of G. The following algorithm may be used to construct the new generator G'.

begin
  $G' = G$.
  for $i = 1$ **to** m **do**
    if $a_i \geqslant 2$ **then**
      * Add to G' a set of a_i new places $P^i_{\text{new}} = \{p^j_i \mid 0 \leqslant j \leqslant a_i - 1\}$.
      * The place p^j_i, with $j = M_0(p_i) \bmod a_i$, will contain one token. All other new places will not be marked, while the places from P will be marked as by M_0.
      * **for** each transition t of G' inputing to or outputing from place p_i **do**
        · Remove t.
        · Add to G' a set of a_i new transitions $T^i_{\text{new}} = \{t^j \mid 0 \leqslant j \leqslant a_i - 1\}$ with the same label as t.
        · The pre arcs of each new transition t^j are as follows. If $p \notin P^i_{\text{new}}$ then $\text{Pre}(p, t^j) = \text{Pre}(p, t)$, else if $p = p^j_i$ then $\text{Pre}(p, t^j) = 1$, else $\text{Pre}(p, t^j) = 0$.
        · The post arcs of each new transition t^j are as follows. If $p \notin P^i_{\text{new}}$ then $\text{Post}(p, t^j) = \text{Post}(p, t)$, else if $p = p^{j'}_i$, with $j' = [\text{Post}(p_i, t) - \text{Pre}(p_i, t) + j] \bmod a_i$, then $\text{Post}(p, t^j) = 1$, else $\text{Post}(p, t^j) = 0$.
      endfor
    endif
  enfor
end

This construction preserves the determinism. To each firing sequences σ of G corresponds one and only one firing sequence σ' of G' with the same label (and vice versa). Furthermore, if $M_0[\sigma\rangle M$ and $M'_0[\sigma'\rangle M'$ we have that $(\forall p_i \in P)$ $M'(p_i) = M(p_i)$, while $(\forall p^j_i \in P^i_{\text{new}})$ $M'(p^j_i) = 1$ if $j = M(p_i) \bmod a_i$, and $M'(p^j_i) = 0$ otherwise.

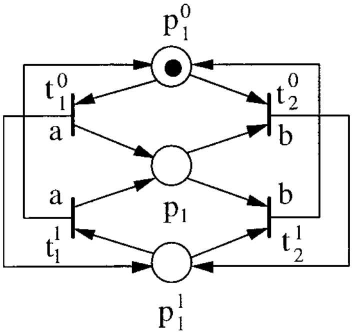

FIG. 2. Petri net obtained by modification of the net in Fig. 1a.

Then given a set $F = D(v, a)$ as given by Eq. (2) we have that $L(G, M_0, F) = L(G', M'_0, F')$, where

$$F' = \{M' \mid (\forall i \in I)\ M'(p_i) = v_i \text{ and } (\forall i \notin I)\ M'(p_i) \geqslant v_i$$
$$\text{and if } a_i \geqslant 2 \text{ then } M'(p_i^{j_i}) = 1 \text{ and } M'(p_i^j) = 0 \text{ for } j \neq j_i\},$$

$I = \{i \in \{1, ..., m\} \mid a_i = 0\}$ and $j_i = v_i \mod a_i$. Since it is in the form of Eq. (8), $F' \in \mathtt{Sf}$.

As an example, consider the net in Fig. 1a with a recognizable set of final markings $F = \{3 + 2k \mid k \in \mathbb{N}\}$. The generator G' with marking M'_0, constructed following the algorithm, is shown in Fig. 2. The star-free set of final markings for G' is $F' = \{M' \mid M'(p_1) \geqslant 3,\ M'(p_1^1) = 1\}$.

If F is a finite union of sets $F_r = D(v_r, a_r)$, we have to perform the same construction, replacing in the algorithm a_i by the least common multiple $\bar{a}_i$ of the $(a_r)_i$ associated with the different F_r, and taking for F' the union of the sets,

$$F'_r = \{M' \mid (\forall i \in I_r)\ M'(p_i) = (v_r)_i \text{ and } (\forall i \notin I_r)\ M'(p_i) \geqslant (v_r)_i$$
$$\text{and if } (a_r)_i \geqslant 2 \text{ then } M'(p_i^j) = 1 \text{ for } j \in J_{ri}, \text{ and } M'(p_i^j) = 0 \text{ for } j \notin J_{ri}\},$$

where $I_r = \{i \in \{1, ..., m\} \mid (a_r)_i = 0\}$ and $J_{ri} = \{k \in \{0, ..., \bar{a}_i - 1\} \mid k \mod (a_r)_i = (v_r)_i \mod (a_r)_i\}$.

8. ($\mathscr{L}_d\mathtt{Rec} \subsetneq \mathscr{L}_d\mathtt{Rat}$). Since $\mathscr{L}_d\mathtt{Sf} = \mathscr{L}_d\mathtt{Rec}$, it is sufficient to show that the language $L = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$—that is, in $\mathscr{L}_d\mathtt{Rat}$, since it is accepted by the deterministic net in Fig. 1b with $F = \{(k, k) \mid k \in \mathbb{N}\}$—is not in $\mathscr{L}_d\mathtt{Sf}$. To show this we will use the characterization of star-free sets given in Eq. (8).

In fact, assume $L = L(G', M_0, F')$ for a deterministic PN generator G' with a star-free set $F' = \bigcup_{j=1}^r K(I_j, v_j)$. For $i > 0$, let M_i be the unique marking reached in G' by generating the string a^i. It is possible to extract from the sequence $M_1, M_2, ...$ a subsequence $M_{\alpha(1)}, M_{\alpha(2)}, ...$ with $M_{\alpha(k)} < M_{\alpha(k+1)}$ and with the property that there exists a $\bar{j} \in \{1, ..., r\}$ such that for all k the marking reached from $M_{\alpha(k)}$ by firing the string $b^{\alpha(k)}$ belongs to $K(I_{\bar{j}}, v_{\bar{j}})$. A legal move of infinite length that starts from $M_{\alpha(1)}$ is

$$M_{\alpha(1)}[\sigma_1\rangle\ M'_1[\sigma_2\rangle\ M'_2[\sigma_3\rangle\ M'_3 \ldots \tag{10}$$

with $l(\sigma_1) = b^{\alpha(1)}$ and $l(\sigma_k) = b^{\alpha(k) - \alpha(k-1)}$ for $k > 1$. It is possible to prove that each of the firing sequences σ_k for $k \geqslant 2$ strictly decreases the token count in the subset of places with index in $I_{\bar{j}}$. In fact, since $M_{\alpha(2)} > M_{\alpha(1)}$, a legal move as well is

$$M_{\alpha(2)} [\sigma_1\rangle\, M_1'' [\sigma_2\rangle\, M_2'' \ldots .$$

Now $M_1'' > M_1'$ and, since $M_1' \in K(I_{\bar{j}}, v_{\bar{j}})$ is final while M_1'' is not final, then $M_1''|_{I_{\bar{j}}} > M_1'|_{I_{\bar{j}}}$, where $|_{I_{\bar{j}}}$ denotes the projection on the subset of places with index in $I_{\bar{j}}$. Also, $M_2'' \in K(I_{\bar{j}}, v_{\bar{j}})$; hence $M_1''|_{I_{\bar{j}}} > M_1'|_{I_{\bar{j}}} = M_2''|_{I_{\bar{j}}}$. This shows that the sequence σ_2 reduces the token counts in the places with index in $I_{\bar{j}}$. A similar reasoning can be applied to all other markings $M_{\alpha(k)}$ and firing sequences σ_k for $k > 2$. Hence, the move given by (10) cannot be legal, clearly a contradiction. ∎

A further restriction of deterministic languages is given by free-labeled languages. The use of different classes of sets of final markings leads to different classes of free-labeled Petri net languages as well.

THEOREM 15. *The classes of free-labeled PN languages are ordered as shown.* (*All inclusions are strict. Classes that are not connected by a directed path are incomparable.*)

$$\begin{array}{ccccc} \mathcal{L}_f\texttt{Fin} \to \mathcal{L}_f\texttt{SCyl} \searrow & & & & \\ \nearrow & & \mathcal{L}_f\texttt{Sf} \to \mathcal{L}_f\texttt{Rec} \to \mathcal{L}_f\texttt{Rat} & & \\ \mathcal{L}_f\texttt{Triv} \to \mathcal{L}_f\texttt{Id} \quad \nearrow & & & & \end{array}$$

Proof. Note first that $\mathcal{X} \subset \mathcal{X}'$ implies $\mathcal{L}_f \mathcal{X} \subset \mathcal{L}_f \mathcal{X}'$. Hence by Proposition 7, it follows that $\mathcal{L}_f\texttt{Fin} \subset \mathcal{L}_f\texttt{SCyl} \subset \mathcal{L}_f\texttt{Sf} \subset \mathcal{L}_f\texttt{Rec} \subset \mathcal{L}_f\texttt{Rat}$ and $\mathcal{L}_f\texttt{Triv} \subset \mathcal{L}_f\texttt{Id} \subset \mathcal{L}_f\texttt{Sf}$. We are left to prove that the inclusions are strict and to prove that the other relations hold.

1. ($\mathcal{L}_f\texttt{Triv} \not\leftrightarrow \mathcal{L}_f\texttt{Fin}$) and ($\mathcal{L}_f\texttt{Id} \not\leftrightarrow \mathcal{L}_f\texttt{Fin}$). To prove this, we show that ($\mathcal{L}_f\texttt{Triv} \not\subset \mathcal{L}_f\texttt{Fin}$) and ($\mathcal{L}_f\texttt{Fin} \not\subset \mathcal{L}_f\texttt{Id}$). Consider the free-labeled net G in Fig. 3, and let $F = \{(0\ 0\ 1)\}$. One can immediately see that the language accepted by G with final set $\mathbb{N}^m$ is $L_1 = \{a^*\} \cup \{a^m b c^n \mid m \geqslant n \geqslant 0\}$, while the language accepted by G with final set F is $L_2 = \{a^m b c^m \mid m \geqslant 0\}$; hence $L_1 \in \mathcal{L}_f\texttt{Triv}$ and $L_2 \in \mathcal{L}_f\texttt{Fin}$.

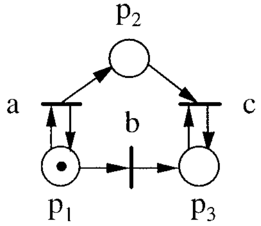

FIG. 3. A free-labeled Petri net.

We prove $L_1 \notin \mathcal{L}_f\,\mathtt{Fin}$ by contradiction. In fact, assume $L_1 = L(G', M_0, F')$ for a free-labeled PN generator G' and a finite set F'. For $i \geqslant 0$, let M_i be the unique marking reached in G' by generating the string a^i. Since the labeling of G' is free and the single transition labeled a may fire infinitely often, then $M_i \leqslant M_j$ for $i < j$. Furthermore, $L(G', M_i, F') \neq L(G', M_j, F')$ for $i \neq j$; hence $M_i < M_j$ for $i < j$. Thus there are infinitely many M_i $(i \geqslant 0)$ and all these markings must be final; hence F' must be infinite, contradicting the assumption.

We prove $L_2 \notin \mathcal{L}_f\,\mathtt{Id}$ by contradiction. In fact, assume $L_2 = L(G', M_0, F')$ for a free-labeled PN generator G' and an ideal set F'. We can define M_i $(i \geqslant 0)$ as above and we can show that $M_i < M_j$ for $i < j$. Let σ_i be the sequence of transitions such that $l(\sigma_i) = bc^i$. Now $a^i bc^i \in L_2$; hence $M_i[\sigma_i\rangle M'_i$ and $M'_i \in F'$. Choose any $j > i$; since $M_j > M_i$, we have that $M_j[\sigma_i\rangle M'_j$ and $M'_j > M'_i$; hence $M'_j \in F'$. Thus the string $a^j bc^i \notin L_2$ is also accepted, thus contradicting the assumption.

2. $(\mathcal{L}_f\,\mathtt{Id} \not\leftrightarrow \mathcal{L}_f\,\mathtt{SCyl})$. Since $\mathcal{L}_f\,\mathtt{Id} \not\leftrightarrow \mathcal{L}_f\,\mathtt{Fin} \subset \mathcal{L}_f\,\mathtt{SCyl}$ it holds that $\mathcal{L}_f\,\mathtt{SCyl} \not\subset \mathcal{L}_f\,\mathtt{Id}$. We just need to prove that $\mathcal{L}_f\,\mathtt{Id} \not\subset \mathcal{L}_f\,\mathtt{SCyl}$. This can been done using the same language L considered in the proof of Theorem 14, part 3, since $L \in \mathcal{L}_f\,\mathtt{Id}$.

3. $(\mathcal{L}_f\,\mathtt{Fin} \subsetneq \mathcal{L}_f\,\mathtt{SCyl})$ and $(\mathcal{L}_f\,\mathtt{Triv} \subsetneq \mathcal{L}_f\,\mathtt{SCyl})$. Follow from the fact that $\mathcal{L}_f\,\mathtt{Triv} \subset \mathcal{L}_f\,\mathtt{SCyl}$ and $\mathcal{L}_f\,\mathtt{Fin} \subset \mathcal{L}_f\,\mathtt{SCyl}$, while $\mathcal{L}_f\,\mathtt{Triv} \not\leftrightarrow \mathcal{L}_f\,\mathtt{Fin}$.

4. $(\mathcal{L}_f\,\mathtt{Triv} \subsetneq \mathcal{L}_f\,\mathtt{Id})$. Follows from the fact that $\mathcal{L}_f\,\mathtt{Triv} \subset \mathcal{L}_f\,\mathtt{SCyl}$ and from the incomparability of $\mathcal{L}_f\,\mathtt{Id}$ and $\mathcal{L}_f\,\mathtt{SCyl}$.

5. $(\mathcal{L}_f\,\mathtt{SCyl} \subsetneq \mathcal{L}_f\,\mathtt{Sf})$ and $(\mathcal{L}_f\,\mathtt{Id} \subsetneq \mathcal{L}_f\,\mathtt{Sf})$. Follow from the fact that $\mathcal{L}_f\,\mathtt{If} \subset \mathcal{L}_f\,\mathtt{Sf}$, $\mathcal{L}_f\,\mathtt{SCyl} \subset \mathcal{L}_f\,\mathtt{Sf}$, while $\mathcal{L}_f\,\mathtt{Id} \not\leftrightarrow \mathcal{L}_f\,\mathtt{SCyl}$.

6. $(\mathcal{L}_f\,\mathtt{Sf} \subsetneq \mathcal{L}_f\,\mathtt{Rec})$. Consider the language $L = \{w \in \{a, b\}^* \mid (\forall s \leqslant w)\ |s|_a \geqslant |s|_b \text{ and } (\exists k \in \mathbb{N})\ |w|_a = |w|_b + 2k\}$. $L \in \mathcal{L}_f\,\mathtt{Rec}$, since it is accepted by the free-labeled net in Fig. 1a with $F = \{2k \mid k \in \mathbb{N}\}$. Assume $L = L(G', M_0, F')$ for a free-labeled PN generator G' with a star-free set $F' = \bigcup_{j=1}^{r} K(I_j, v_j)$. Consider the legal move of infinite length of $G' : M_0[\sigma\rangle\, M_1[\sigma\rangle\, M_2[\sigma\rangle\, M_3 \ldots$, where $l(\sigma) = a$. $M_{i+1} = M_i + \Delta$ for some nonzero $\Delta \in \mathbb{N}^m$, and, since the markings $M_0, M_2, \ldots$ are final while the markings $M_1, M_3, \ldots$ are not final, it is clear that no two markings in the infinite sequence $M_0, M_2, \ldots$ may belong to the same $K(I_j, v_j)$. This contradicts the assumption that r be finite.

7. $(\mathcal{L}_f\,\mathtt{Rec} \subsetneq \mathcal{L}_f\,\mathtt{Rat})$. To prove this we may use the same language L considered in the proof of Theorem 14, part 8, since $L \in \mathcal{L}_f\,\mathtt{Rat}$. ∎

Finally, we show that language containment remains decidable for all these new deterministic classes. First, we prove that the complement of a deterministic PN language is a PN language.

Theorem 16. *Let $\mathcal{L}_d \mathcal{X}$ with $\mathcal{X} \in \{\mathtt{Triv}, \mathtt{Fin}, \mathtt{Id}, \mathtt{SCyl}, \mathtt{Sf}, \mathtt{Rec}, \mathtt{Rat}\}$ be a class of deterministic Petri net languages. Then co-$\mathcal{L}_d \mathcal{X} \overset{\text{def}}{=} \{\complement L \mid L \in \mathcal{L}_d \mathcal{X}\} \subset \mathcal{L}\,\mathtt{Fin}$.*

Proof. Pelz has shown that co-$\mathcal{L}_d\,\mathtt{Triv} \subset \mathcal{L}\,\mathtt{Fin}$ [20]. Let G be a deterministic PN generator, and let $L = L(G, M_0, F)$ with $F \in \mathtt{Rat}$. Let also $L' = L(G, M_0, F')$ (with $F' = \mathbb{N}^m$) be the $\mathtt{Triv}$-type language of G. Since $L \subset L'$, it follows that $\complement L = \complement L' \cup (L' \setminus L)$. Now $(L' \setminus L) = L(G, M_0, \complement F)$ and from Proposition 8 we know that $\complement F \in \mathtt{Rat}$; hence by Theorem 12 $(L' \backslash L) \in \mathscr{L}_d \mathtt{Rat} \subset \mathscr{L}\mathtt{Fin}$. Since co-$\mathscr{L}_d \mathtt{Triv} \subset \mathscr{L}\mathtt{Fin}$, $\complement L' \in \mathscr{L}\,\mathtt{Fin}$. Finally, from the closure of $\mathscr{L}\mathtt{Fin}$ under union [16] it follows that $\complement L \in \mathscr{L}\mathtt{Fin}$. ∎

Corollary 17. *The containment problem "Is $L \subset L'$?" is decidable for $L \in \mathscr{L}\mathtt{Fin}$ and $L' \in \mathscr{L}_d \mathtt{Rat}$.*

Proof. Indeed, $L \subset L'$ reduces to $L \cap \complement L' = \varnothing$. Since $\complement L' \in \mathscr{L}\mathtt{Fin}$, $\mathscr{L}\mathtt{Fin}$ being closed under intersection [16], this reduces to the emptiness problem for a language $L'' \in \mathscr{L}\mathtt{Fin}$, which reduces to the reachability problem known to be decidable [25]. ∎

Corollary 18. *Let $\mathscr{X} \in \mathscr{H}$. Then*

$$\mathscr{L}_f \mathscr{X} \subsetneq \mathscr{L}_d \mathscr{X} \subsetneq \mathscr{L}\mathscr{X} \subsetneq \mathscr{L}_\lambda \mathscr{X}.$$

Proof. It is clear that for all $\mathscr{X} \in \mathscr{H}$, $\mathscr{L}_f \mathscr{X} \subset \mathscr{L}_d \mathscr{X} \subset \mathscr{L}\mathscr{X} \subset \mathscr{L}_\lambda \mathscr{X}$. Some of these inclusions are already known to be strict. It has been shown in [14] that $\mathscr{L}\mathtt{Triv} \subsetneq \mathscr{L}_\lambda \mathtt{Triv}$ and that $\mathscr{L}\mathtt{Fin} \subsetneq \mathscr{L}_\lambda \mathtt{Fin}$. With the same reasoning it is immediate to show that $\mathscr{L}\mathtt{Id} \subsetneq \mathscr{L}_\lambda \mathtt{Id}$.

The other strict inclusions for $\mathscr{X} = \mathtt{Triv}$ have been proved in [27]. The strict inclusion $\mathscr{L}_d \mathtt{Id} \subsetneq \mathscr{L}\mathtt{Id}$ follows from a result of [10], where is was shown that the language $L = \{a^m b^n c \mid m > n \geqslant 0\} \cup \{a^+ b^*\}$ does not belong to $\mathscr{L}_d \mathtt{Id}$, while this language is accepted by the λ-free labeled net in Fig. 4 with the set of final markings $F = \uparrow(0\,0\,1\,0\,0) \cup \uparrow(0\,0\,0\,1\,0) \cup \uparrow(0\,0\,0\,0\,1)$.

The strict inclusions $\mathscr{L}_f \mathscr{X} \subsetneq \mathscr{L}_d \mathscr{X}$ for $\mathscr{X} \in \{\mathtt{Fin}, \mathtt{Id}, \mathtt{SCyl}, \mathtt{Sf}, \mathtt{Rec}, \mathtt{Rat}\}$ follow from the fact that $\mathscr{L}_d \mathscr{X}$ contains all regular languages, while the regular language $L = ab^2 + ba$ cannot be in $\mathscr{L}_f \mathscr{X}$. In fact, the string ab is not final while ba is final, but on free-labeled nets their firing yields the same marking.

The strict inclusions $\mathscr{L}_d \mathscr{X} \subsetneq \mathscr{L}\mathscr{X}$ for $\mathscr{X} \in \{\mathtt{Fin}, \mathtt{SCyl}, \mathtt{Sf}, \mathtt{Rec}, \mathtt{Rat}\}$ follow from the fact that $\mathscr{L}\mathscr{X} = \mathscr{L}\mathtt{Fin}$ (see Theorem 12) and $\mathscr{L}\mathtt{Fin}$ is not closed under complementation [16], while the complement of a language $L \in \mathscr{L}_d \mathscr{X}$ is in $\mathscr{L}\mathtt{Fin}$ (see Theorem 16).

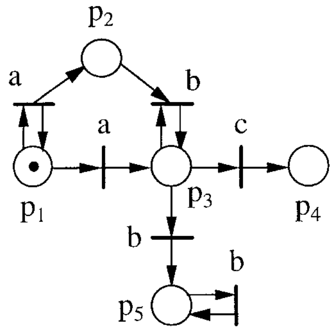

FIG. 4. A λ-free labeled net.

The strict inclusions $\mathscr{L}\mathscr{X} \subsetneq \mathscr{L}_\lambda \mathscr{X}$ for $\mathscr{X} \in \{\texttt{SCyl}, \texttt{Sf}, \texttt{Rec}, \texttt{Rat}\}$ follow from the fact that $\mathscr{L}\mathscr{X} = \mathscr{L}\texttt{Fin}$ and $\mathscr{L}_\lambda \mathscr{X} = \mathscr{L}_\lambda \texttt{Fin}$ (see Theorem 13). ∎

7. CONCLUSION

In this paper we have introduced new tractable classes of deterministic PN languages, which allow the specification of rather general conditions on accepting markings. As a future work we will study the complexity of decision procedures to check the properties of interest (controllability, etc.) of discrete event systems modeled by these classes of deterministic languages. Dually, the specification of a (finite or infinite) *set* of initial markings leads to other related interesting classes of PN languages.

REFERENCES

1. M. Ajmone Marsan, G. Balbo, G. Conte, S. Donatelli, and G. Franceschinis, "Modelling with Generalized Stochastic Petri Nets," Parallel Computing Series, Wiley, New York, 1995.
2. F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat, "Synchronization and Linearity," Wiley, 1992.
3. J. Berstel, "Transductions and Context-free Languages," Teubner, Stuttgart, 1979.
4. J. Berstel and C. Reutenauer, "Rational Series and Their Languages," EATCS Monographs in Theoretical Computer Science, Springer-Verlag, New York/Berlin, 1988.
5. J. H. Conway, "Regular Algebra and Finite Machines," Chapman & Hall, London/New York, 1971.
6. R. David and H. Alla, "Petri Nets and Grafcet," Prentice–Hall, Englewood Cliffs, NJ, 1992.
7. S. Eilenberg and M. P. Schützenberger, Rational sets in commutative monoids, *J. Algebra* **13** (1969), 173–191.
8. S. Gaubert and A. Giua, Petri net languages with infinite sets of final markings, *in* "Proc. IEEE of WODES'96, Edinburgh, UK, August 1996," p. 326–331.
9. S. Gaubert and A. Giua, Deterministic weak-and-marked Petri net languages are regular, *IEEE Trans. Automat. Control* **41**, No. 12 (1996), 1802–1803.
10. A. Giua, "On the Closure Properties of Deterministic Weak Petri Net Languages," Technical Report No. 59, Istituto di Elettrotecnica, University of Cagliari, Italy, February 1994.
11. A. Giua and F. DiCesare, Decidability and closure properties of weak Petri net languages in supervisory control, *IEEE Trans. on Automatic Control* **40**, No. 5, (1995), 906–910.
12. G. Guaiana, A. Restivo, and S. Salemi, Star-free trace languages, *Theoret. Comput. Sci.* **97** (1992), 301–311.
13. L. E. Holloway, B. H. Krogh, and A. Giua, A survey of Petri net methods for controlled discrete event systems, *in* "Discrete Event Dynamic Systems: Theory and Applications," Vol. 7, pp. 151–190, Kluwer Academic, Dordrecht/Norwell, MA, 1997.
14. M. Jantzen, On the hierarchy of Petri net languages, *R.A.I.R.O. Theoret. Inform.* **13**, No. 1 (1979), 19–30.
15. M. Jantzen and R. Valk, Formal properties of place/transition nets, *in* "Net Theory and Applications" (W. Brauer, Ed.), LNCS, No. 84, pp. 165–212, Springer-Verlag, 1980.
16. M. Jantzen, Language theory of Petri nets, *in* "Advances in Petri Nets, 1986," LNCS, No. 254-I, pp. 397–412, Springer-Verlag, 1987.

17. Y. Li and W. M. Wonham, Control of vector discrete-event systems—Part I: The base model, *IEEE Trans. on Automatic Control* **38**, No. 8 (1993), 1215–1227.

18. T. Murata, Petri nets: Properties, analysis and applications, *Proc. IEEE*, PROC-77, No. 4 (1989), 541–580.

19. M. Parigot and E. Pelz, A logical approach of Petri net languages, *Theor. Comp. Sci.* **39** (1985), 155–169.

20. E. Pelz, Closure properties of deterministic Petri nets, *in* "Proc. STACS 1987," LNCS, No. 247, pp. 371–382, Springer-Verlag, 1987.

21. J. L. Peterson, "Petri Net Theory and the Modeling of Systems," Prentice–Hall, 1981.

22. J. E. Pin, "Variétés de langages formels," Masson, Paris, 1984.

23. P. J. Ramadge and W. M. Wonham, The control of discrete event systems, *Proc. IEEE* **77**, No. 1 (1989), 81–98.

24. W. Reisig, "Petri Nets: An Introduction," Springer-Verlag, 1985.

25. C. Reutenauer, "Aspects Mathématiques des réseaux de Petri," Masson, Paris, 1988. [English edition: The Mathematics of Petri Nets. Prentice–Hall Intern., 1990]

26. R. S. Sreenivas, A note on deciding the controllability of a language K with respect to a language L, *IEEE Trans. on Automatic Control* **38**, No. 4 (1993), 658–662.

27. G. Vidal-Naquet, Deterministic Petri net languages, *in* "Application and Theory of Petri Nets" (C. Girault and W. Reisig, Eds.), Informatick-Fachberichte, No. 52, pp. 198–202, Springer-Verlag, 1982.