Discrete Event Representation of Qualitative Models Using Petri Nets

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Abstract—The paper discusses how Petri nets may be used for the qualitative modeling of physical systems. The qualitative state of a system is represented by the marking of the net. The crossing of a landmark value corresponds to the firing of a transition.

We give a formal procedure to construct a Petri net model corresponding to a given set of qualitative equations. The approach can be used to study both autonomous systems and systems with forcing inputs. The dynamic behavior of the system can be studied as sequences of reachable markings of the net and can be computed with standard Petri net execution techniques. This approach also leads to a simple framework for the study of hybrid systems, i.e., systems whose behavior is described by both continuous and discrete event dynamics. Several examples, with applications to diagnosis and control, are fully discussed.

Index Terms— Discrete event systems, hybrid systems, Petri nets, qualitative physics.

I. INTRODUCTION

THE simulation of a continuous or discrete time system can be achieved, basically, in two ways: quantitatively and qualitatively. Quantitative simulation entails an exact description of the relationships between the various quantities involved in the system functioning, in terms, e.g., of algebraic differential or difference equations. Qualitative simulation, on the other hand, exploits relationships that express qualitative connections between the quantities [2], [3], [7], [10]. Such a description does not contain as much information as a quantitative analysis. However, in those cases where the additional information provided by a quantitative description is useless for the purpose of investigation, a qualitative analysis is desirable.

Some advantages of qualitative simulation are [4]

- it permits to express incomplete knowledge and hence to handle systems that are not completely known;
- it provides general solutions for classes of systems rather than numerical solutions of each case;
- it treats homogeneously linear and nonlinear systems.

The major drawback of qualitative simulation is its fundamental ambiguity: given a qualitative model of a system and a set of qualitative inputs, more than one qualitative behavior can generally be found that follows from those initial data. This ambiguity partially depends on the choice of the quantity space used to represent the qualitative value of the variables.

By refining the partitions of the real axis that define these quantity spaces it is often possible to mitigate this ambiguity.

Another disadvantage of qualitative analysis derives from the fact that we lack effective simple mathematical tools for carrying out the simulation. Solving a set of qualitative constraints requires ingenuity and the use of heuristics.

We propose a simple way of avoiding this problem. We note that a qualitative system, with its discrete quantity state space can also be seen as a discrete event system (DES) [15]. Thus its behavior may be described by any of the models used to represent a DES. This work focuses on the use of Petri nets models [11].

Petri nets have been used in qualitative simulation by Okuda and Ushio [12], [13]. These authors noted that each *place* of a net may be associated to a state of a variable while the firing of each *transition* corresponds to crossing a landmark.

The approach presented in this paper, that is an extension of [5] and [6], modifies the approach of Okuda and Ushio in several ways. First, we assume that each *marking* (not place) of a net may be associated to a state of a variable and that a transition may represent more than one landmark crossing. Thus when we consider variables with increasing quantity spaces, we need not modify the structure of the net, but just to change the number of tokens it contains. Second, in our approach the behavior of the physical system is completely captured in the structure of the net. Thus we need not give additional interpretation rules that modify the standard Petri net execution algorithm during simulation. This also allows us to use some of the standard Petri net analysis techniques to study the properties of the model [11], [14].

Recently, there has been large interest in the control community in the study of hybrid systems, i.e., systems whose behavior is described by both continuous and discrete event dynamics [1]. In our approach it is possible to model an hybrid system as a mere discrete event system in the following way. First, a qualitative description of the continuous time behaviors is given. Second, the qualitative descriptions are captured by discrete event models. Finally, a model that integrates all these different discrete event behaviors is built using Petri nets. As an example of this, in Section V we will discuss the model of a nonlinear electric circuit.

The paper is structured as follows. In Section II, we recall the generalities on Petri nets and qualitative modeling. In Section III, we give a formal procedure to construct a Petri net model of a physical continuous time system represented by qualitative equations. We consider models with different quantity spaces and discuss the presence of forcing inputs. In

Section IV we apply this procedure to a spring-block system (discussed also in [9]) and show how reachability and invariant analysis (two well-know Petri net techniques) can be used to study the behavior of the system. In Section V we give a formal procedure to construct a Petri net model of an hybrid system, and apply this procedure to a non linear circuit. In Section VI we show how this modeling approach may be used for diagnosis and control, using a double integrator as an applied example.

II. GENERALITIES

A. Petri Nets

A *place/transition net* [11], [14] is a structure $N = (P, T, \text{Pre}, \text{Post})$, where P is a set of *places* represented by circles; T is a set of *transitions* represented by bars; Pre: $P \times T \to \mathbb{N}$ is the *pre-incidence function* that specifies the arcs directed from places to transitions; Post: $P \times T \to \mathbb{N}$ is the *post-incidence function* that specifies the arcs directed from transitions to places. The *incidence matrix* of the net is $C(p, t) = \text{Post}(p, t) - \text{Pre}(p, t)$.

A *marking* is a vector $M: P \to \mathbb{N}$ that assigns to each place of a P/T net a nonnegative integer number of tokens, represented by black dots. $\mathbb{N}^{|P|}$ will denote the set of all possible markings that may be defined on the net.

A transition $t \in T$ is *enabled* at a marking M iff $M \geq \text{Pre}(\cdot, t)$. If t is enabled at M, then t may fire yielding a new marking M' with $M' = M + \text{Post}(\cdot, t) - \text{Pre}(\cdot, t)$. We will write $M[t\rangle M'$ to denote that t may fire at M yielding M'. A marking M is *dead* if no transition is enabled at M. A *firing sequence* from M_0 is a (possibly empty) sequence of transitions $\sigma = t_1 \cdots t_k$ such that $M_0[t_1\rangle M_1[t_2\rangle M_2 \cdots [t_k\rangle M_k$. A marking M is *reachable* in $\langle N, M_0 \rangle$ iff there exists a firing sequence σ such that $M_0[\sigma\rangle M$. The set of markings reachable on a net N from a marking M is called *reachability set* of N and M and is denoted as $R(N, M)$. The reachability set can be represented by a graph, called *reachability graph*.

Example 1: Let us consider the Petri net in Fig. 1(a). The set of places is $P = \{p_1, p_2, p_3\}$, the set of transitions is $T = \{t_1, t_2, t_3\}$, the pre-incidence and post-incidence functions can be expressed as matrices

$$\text{Pre} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{Post} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The incidence matrix is

$$C = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that the double arc from t_1 to p_2 has been represented, as usual, with a single arc of weight 2.

Representing the markings of this net as vectors $M = [M(p_1) \ M(p_2) \ M(p_3)]^T$, the initial marking, shown in the figure, is $M_0 = [2 \ 0 \ 0]^T$.

To construct the reachability graph we put a node labeled M_0 in the graph. Starting from M_0, both t_1 and t_2 are enabled.
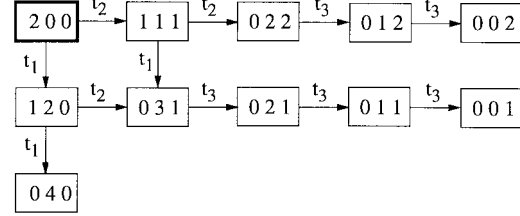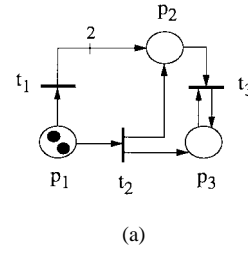
Fig. 1.   (a) A Petri net and (b) its reachability graph.

If t_1 fires we reach the marking $M' = [1 \ 2 \ 0]^T$. If t_2 fires we reach the marking $M'' = [1 \ 1 \ 1]^T$. Thus we add a node labeled M', a node labeled M'', an arc labeled t_1 from M_0 to M', and an arc labeled t_2 from M_0 to M''. We continue this construction to obtain the graph shown in Fig. 1(b).

The general algorithm to compute a reachability graph is given in [14]. Note that a reachability graph may be infinite. In this case it is still possible to construct a finite graph named *coverability graph* [14].

Let $^\bullet p = \{t \in T | \text{Post}(p, t) > 0\}$ and $p^\bullet = \{t \in T | \text{Pre}(p, t) > 0\}$. A *trap* is a set of places $\mathcal{T} \subseteq P$ such that: $\bigcup_{p \in \mathcal{T}} p^\bullet \subseteq \bigcup_{p \in \mathcal{T}} {}^\bullet p$. A *siphon* is a set of places $\mathcal{S} \subseteq P$ such that: $\bigcup_{p \in \mathcal{S}} {}^\bullet p \subseteq \bigcup_{p \in \mathcal{S}} p^\bullet$. Traps and siphons have interesting behavioral properties [11]. If a siphon is token-free under some marking, then it remains token-free under each successor marking. If a trap is marked under some marking, then it remains marked under each successor marking.

B. Qualitative Models

Qualitative modeling exploits relationships that express qualitative connections between the variables of a physical system.

A qualitative model uses *qualitative variables* with an associated *quantity space* defined as a set of disjoint intervals (possibly of zero length, in which case they reduce to points) that cover the real straight line. The qualitative value of a variable x is denoted $[x]$.

The quantity space usually employed is that comprising the intervals $(-\infty, -\epsilon), (-\epsilon, \epsilon), (\epsilon, \infty)$; thus one writes $[x] = -$, $[x] = 0$, and $[x] = +$ to denote the interval to which the value of x belongs. The laws that govern the system behavior are expressed as equations between these qualitative variables. One problem with this approach is the essential ambiguity of the qualitative sum, as defined in Table I.

The qualitative sum ambiguity can be avoided using a finer partition of the real axis that also gives a better description

TABLE I
QUALITATIVE SUM TABLES FOR QUANTITY SPACE $\{-, 0, +\}$ AND $\{-2, \cdots, 2\}$

$[x] + [y]$	$+$	0	$-$
$+$	$+$	$+$	$?$
0	$+$	0	$-$
$-$	$?$	$-$	$-$

$[x] + [y]$	2	1	0	-1	-2
2	4	3	2	1	0
1	3	2	1	0	-1
0	2	1	0	-1	-2
-1	1	0	-1	-2	-3
-2	0	-1	-2	-3	-4

Fig. 2.   Representation of (a) self-loops and (b) parallel transitions.

of the system behavior. As an example, we will often use the quantity space $\{-n, -n+1, \cdots, -1, 0, 1, \cdots, n-1, n\}$. In this case the qualitative sum $[x] + [y]$ follows the same rules of algebraic addition, as can be seen in Table I for $n = 2$.

C. Petri Nets as Qualitative Models

We will use Petri nets as qualitative models of physical systems with the following assumptions.

The qualitative value of each variable is associated with the marking of a subset of places in the Petri net. Thus we have the correspondence between qualitative states and markings. The initial state of the system will determine the initial marking M_0 of the net.

The firing of a transition will represent the change of a qualitative variable from one qualitative value to another. Note that a single transition may be enabled by several different markings, thus the same transition may represent different qualitative changes.

The set of all possible states reachable from the initial state will be given by the reachability set $R(N, M_0)$ of the net. The sequence of all possible behaviors is given by all sequences of transitions σ firable from the initial marking.

The change of value of a qualitative variable, say x, is often depending on the value of another one, say v. Thus, in the Petri net model a transition that changes the marking of the places associated to x may depend on the marking of the places associated to v. The influence of v over x may be represented by self-loops, i.e., cycles in the net graph containing only one place and one transition.

Consider a transition t self-looped with places p and p' as in Fig. 2(a). The firing of t is only possible if there is at least a token in p and at least a token in p'. The firing of t, however, does not change the number of tokens in p and p'. To avoid representing the two arcs between p and t and the two arcs between p' and t we simply assign to transition t the label $\mathbf{p} \wedge \mathbf{p}'$. This can also be generalized to a self-loop of n arcs using the label $\{\mathbf{p} = \mathbf{n}\} \wedge \{\mathbf{p}' = \mathbf{n}\}$.

We will often need to give a compact representation of a structure in which there are parallel transitions with different labels $\mathbf{F_i}$, each of which may be the $\wedge$ of single labels as
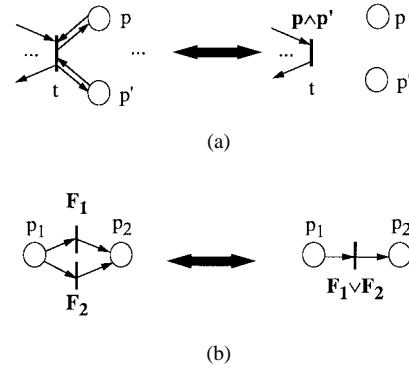
discussed above. A simpler representation of this will be a single transition with label $\mathbf{L}$: $\vee_i \mathbf{F_i}$, as shown in Fig. 2(b).

III. PETRI NET MODELING OF QUALITATIVE SYSTEMS

In this section we discuss how from a qualitative model of a continuous time system it is possible to derive a discrete event model using Petri net structures.

Let us consider a continuous time system described by the following set of state equations:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{1}$$

where $\mathbf{x}$ is the state vector with p components, $\mathbf{u}$ is the input vector with q components, $A = \{a_{i,j}\}$ is a $p \times p$ matrix, and $B = \{b_{i,k}\}$ is a $p \times q$ matrix.

The qualitative model corresponding to (1) is given by the following set of qualitative equations:

$$[\dot{\mathbf{x}}] = [A][\mathbf{x}] + [B][\mathbf{u}] \tag{2}$$

i.e., the ith qualitative equation is

$$[\dot{x}_i] = \sum_{j=1}^{p} [a_{i,j}][x_j] + \sum_{k=1}^{q} [b_{i,k}][u_k]. \tag{3}$$

We will consider two different models for state variables in the quantity spaces $\{-, 0, +\}$ and $\{-n, \cdots, 0, \cdots, +n\}$. For each of these two cases, we will give general construction algorithms to derive a Petri net model representing a given set of qualitative equations.

Once a model has been constructed, the behavior of the net can be studied with various techniques pertaining to Petri nets. In particular, reachability analysis may be used to study the evolution of the system with standard Petri net simulators [19]. Examples and discussions are presented in Section IV.

A. Model with Quantity Space $\{-, 0, +\}$

The following algorithm can be used to construct a Petri net model when the quantity space of the variables is $\{-, 0, +\}$. The coefficients $[a_{i,j}]$ and $[b_{i,k}]$ will take values in $\{-, 0, +\}$ as well.
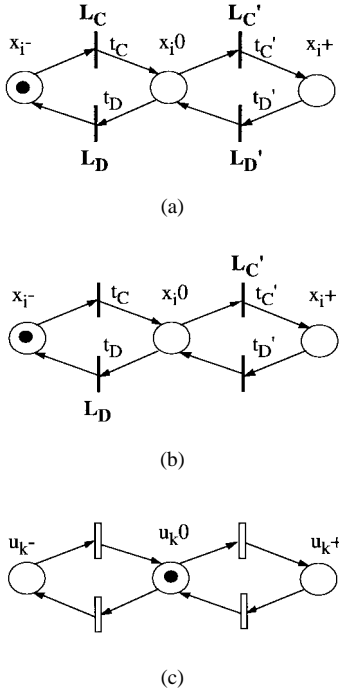
Fig. 3. Petri net model with quantity space $\{-, 0, +\}$: (a) Subnet for x_i with $[a_{i,i}] \in \{0, +\}$, (b) subnet for x_i with $[a_{i,i}] = -$, and (c) subnet for input u_k.

Algorithm 1: Consider the qualitative equations (2).

1) Associate to each state variable x_i a Petri net with three places x_i-, x_i0, and x_i+, as in Fig. 3(a). Here a token in place x_i- means that $[x_i] = -$, a token in place x_i0 means that $[x_i] = 0$, and so on. Thus, we may write $[x_i] = \text{sign}\{M(x_i+) - M(x_i-)\}$. A physically meaningful initial marking M_0 will be such that $M_0(x_i-) + M_0(x_i0) + M_0(x_i+) = 1$.

2) Associate to each input u_k a Petri net with three places u_k-, u_k0, and u_k+, as in Fig. 3(c). Here a token in place u_k- means that $[u_k] = -$, and so on. A physically meaningful initial marking M_0 will be such that $M_0(u_k-) + M_0(u_k0) + M_0(u_k+) = 1$.

3) In the net of each u_k introduce four transitions as in Fig. 3(c), whose firing will denote the crossing of a landmark value. As an example, the transition from u_k+ to u_k0 will fire when the qualitative value $[u_k]$ goes from $+$ to 0. The transitions are *controlled transitions*, i.e., they will fire according to external events and are represented as empty boxes.

4) The qualitative value of the state variable x_i will change according to the qualitative value of its derivative. Due to the ambiguity of the qualitative sum, $[\dot{x}_i]$ *may* be positive when there exists at least a positive term in the RHS of (3), i.e., when there exists at least a state variable x_j such that $[x_j] = [a_{i,j}]$, or an input u_k such that $[u_k] = [b_{i,k}]$. To represent this behavior, add several transitions in parallel from x_i- to x_i0 and from x_i0 to x_i+, one for each term in the sum at the RHS of (3). A similar reasoning can be applied when variable x_i is decreasing.

In Fig. 3(a), are represented the parallel of the increasing transitions with two single transitions t_C and t'_C labeled $\mathbf{L_C}$, $\mathbf{L'_C}$, and the parallel of the decreasing transitions with two single transitions t_D and t'_D labeled $\mathbf{L_D}$, $\mathbf{L'_D}$, following the notation defined in Section II-C.

To determine the value of the labels on the transitions we will consider two different cases.

a) $[a_{i,i}] \in \{0, +\}$. In this case let

$$\mathbf{L_C} = \mathbf{L'_C} : \left(_{j \in J} \mathbf{F_j} \right)\left(_{k \in K} \mathbf{G_k} \right)$$
$$\mathbf{L_D} = \mathbf{L'_D} : \left(_{j \in J} \mathbf{F'_j} \right)\left(_{k \in K} \mathbf{G'_k} \right)$$

where $J = \{j | j \neq i, [a_{i,j}] \neq 0\}$, $K = \{k | [b_{i,k}] \neq 0\}$, and

$$\mathbf{F_j} : \begin{cases} \mathbf{x_j+} & \text{if } [a_{i,j}] = + \\ \mathbf{x_j-} & \text{if } [a_{i,j}] = - \end{cases}$$

$$\mathbf{G_k} : \begin{cases} \mathbf{u_k+} & \text{if } [b_{i,k}] = + \\ \mathbf{u_k-} & \text{if } [b_{i,k}] = - \end{cases}$$

while

$$\mathbf{F'_j} : \begin{cases} \mathbf{x_j+} & \text{if } [a_{i,j}] = - \\ \mathbf{x_j-} & \text{if } [a_{i,j}] = + \end{cases}$$

$$\mathbf{G'_k} : \begin{cases} \mathbf{u_k+} & \text{if } [b_{i,k}] = - \\ \mathbf{u_k-} & \text{if } [b_{i,k}] = + . \end{cases}$$

Remark 1: Note that if the sets J and K are empty, there will be no transitions in the net, and this corresponds to $\mathbf{L_D} = \mathbf{L'_D} = \mathbf{L_C} = \mathbf{L'_C} = \emptyset$.

With these labels we have introduced a transition for each term in the RHS of (3), except for the term $[a_{i,i}][x_i]$. In fact, when $[a_{i,i}] = 0$, the term $[a_{i,i}][x_i]$ will be missing from the RHS of (3).

When $[a_{i,i}] = +$, we should consider several cases. If $[x_i] = 0$, again the term $[a_{i,i}][x_i]$ will not affect the RHS of (3) and thus there will be no corresponding transition in the parallel of transitions represented by t'_C and t_D. If $[x_i] = +$, the term $[a_{i,i}][x_i]$ can never contribute to give a negative value to $[\dot{x}_i]$ and thus there will be no corresponding transition in the parallel t'_D. Finally, if $[x_i] = -$, the term $[a_{i,i}][x_i]$ can never contributes to give a positive value to $[\dot{x}_i]$ hence there will be no corresponding transition in the parallel t_C.

b) $[a_{i,i}] = -$. In this case the labels $\mathbf{L'_C}$ and $\mathbf{L_D}$ are constructed as before. However, the two parallels of transitions t_C and t'_D consist of two single transitions with no label as in Fig. 3(b). In fact, when $[x_i] = +$ ($[x_i] = -$), because of the ambiguity of the qualitative sum, the term $[a_{i,i}][x_i]$ could give a negative (positive) value to $[\dot{x}_i]$ and thus the transition t'_D (t_C) could fire regardless of the marking of the other subnets.

Examples of application of this algorithm can be found in Section IV-A (autonomous system) and in Section IV-C (system with forcing inputs).

B. Model with Finer Quantity Space

We now assume that the quantity space of the variables be partitioned in finer intervals, so as to avoid the ambiguity of
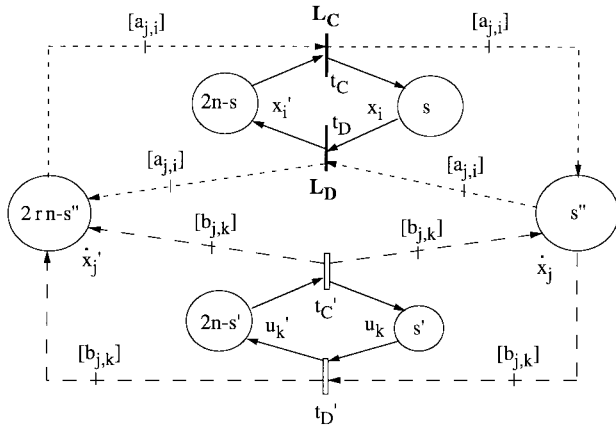
Fig. 4. Petri net model with quantity space $\{-n, \cdots, n\}$.

(a)

(b)

Fig. 5. Modified Petri net model with quantity space $\{-n, \cdots, n\}$. (a) Subnet for the case $[a_{i,i}] < 0$ and (b) subnet for the case $[a_{i,i}] > 0$.

qualitative sum, as discussed in Section II-B. In particular, each state variable x_i and each input u_k takes qualitative values in the set $\{-n, \cdots, 0, \cdots, n\}$. The coefficients $[a_{i,j}]$ and $[b_{i,k}]$ are assumed to be integers (this can be done with a suitable normalization).

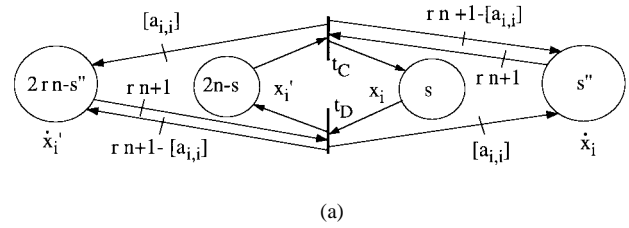*Algorithm 2:* Consider the qualitative equations (2).

1) Associate to each state variable x_i a Petri net with two places x_i and x_i', as in Fig. 4. The qualitative value of x_i is related to the marking of the net as follows: $[x_i] = M(x_i) - n$. A physically meaningful initial marking M_0 will be such that $M_0(x_i) + M_0(x_i') = 2n$. Thus, when there are, say, $n + 3$ tokens in place x_i and $n - 3$ tokens in place x_i' the qualitative value of x_i is $[x_i] = 3$.

2) Associate to each input u_k a Petri net with two places, labeled u_k and u_k', as in Fig. 4. The value of $[u_k]$ is related to the marking of this net in the same way discussed for the x_i subnet.

3) Associate to each variable $\dot{x}_j$ a Petri net with two places, labeled $\dot{x}_j$ and $\dot{x}_j'$, as in Fig. 4. Since $[\dot{x}_j]$ is defined by (3), its quantity space is $\{-rn, \cdots, 0, \cdots, rn\}$, where $r = \sum_i |[a_{j,i}]| + \sum_k |[b_{j,k}]|$. Thus, the value of $[\dot{x}_j]$ is related to the marking of the net as follows: $[\dot{x}_j] = M(\dot{x}_j) - rn$.

Since the initial value of $[\dot{x}_j]$ is a function of the qualitative values of the state variables and inputs, a physically meaningful initial marking M_0 will be such that
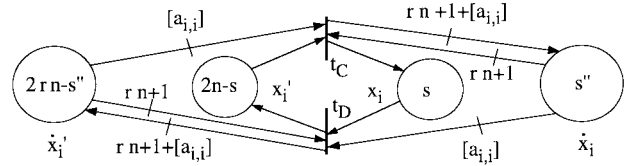
$$M_0(\dot{x}_j) = \sum_{i \in I}[a_{j,i}]M_0(x_i) - \sum_{i \in I'}[a_{j,i}]M_0(x_i')$$
$$+ \sum_{k \in K}[b_{j,k}]M_0(u_k) - \sum_{k \in K'}[b_{j,k}]M_0(u_k')$$

where $I = \{i|[a_{j,i}] > 0\}$, $I' = \{i|[a_{j,i}] < 0\}$, $K = \{k|[b_{j,k}] > 0\}$, and $K' = \{k|[b_{j,k}] < 0\}$. The initial marking of the complementary place will be $M_0(\dot{x}_j') = 2rn - M_0(\dot{x}_j)$.

4) The qualitative value of the state variable x_i will change according to the qualitative value of its derivative. Thus, two transitions will be introduced in each x_i subnet, as

in Fig. 4. The increasing (decreasing) transition t_C (t_D) may only fire when $[\dot{x}_i] > 0$ ($[\dot{x}_i] < 0$) moving a token from x_i' to x_i (from x_i to x_i'), thus it will have a label $\mathbf{L_C}: \dot{x}_i = rn + 1$ ($\mathbf{L_D}: \dot{x}_i' = rn + 1$).

Each time the value $[x_i]$ changes, according to (3) there will be a corresponding change in all the $[\dot{x}_j]$ such that $[a_{j,i}] \neq 0$. Thus, the firing of the transitions in each x_i subnet may also change the token content of the places in some $\dot{x}_j$ subnet. This can be modeled adding arcs of weight $[a_{j,i}]$ between the transitions of x_i and the places of $\dot{x}_j$. As an example, in Fig. 4 the dotted arcs correspond to a coefficient $[a_{j,i}] > 0$. The direction of the arcs should be reversed if $[a_{j,i}] < 0$. Finally, these arcs will not be present if $[a_{j,i}] = 0$.

This construction needs to be partially modified for arcs between the transitions in the x_i subnet and the places in the $\dot{x}_i$ subnet, arcs that will be present if $[a_{i,i}] \neq 0$. In fact, transition t_C associated to x_i may fire only if $[\dot{x}_i] > 0$, i.e., if there are at least $rn + 1$ tokens in place $\dot{x}_i$. If $[a_{i,i}] < 0$, the firing of t_C will remove $[a_{i,i}]$ tokens from place $\dot{x}_i$ and add $[a_{i,i}]$ tokens to place $\dot{x}_i'$. A similar reasoning can be applied to the firing of transition t_D. This behavior is captured in the construction shown in Fig. 5(a), where we have removed the labels in the transitions t_C and t_D because we have explicitly represented the self-loops. If $[a_{i,i}] > 0$, we need to use the construction shown in Fig. 5(b).

5) In each u_k subnet introduce two controlled transitions t_C' and t_D', as in Fig. 4, whose firing will denote the crossing of a landmark value.

Each time the value $[u_k]$ changes, according to (3) there will be a corresponding change in all the $[\dot{x}_j]$ such that $[b_{j,k}] \neq 0$. Thus, the firing of the transitions in each u_k subnet may also change the token content of the places in some $\dot{x}_j$ subnet. This can be modeled adding arcs of weight $[b_{j,k}]$ between the transitions of u_k and the places of $\dot{x}_j$. As an example, in Fig. 4 the dashed arcs correspond to a coefficient $[b_{j,k}] > 0$. The direction
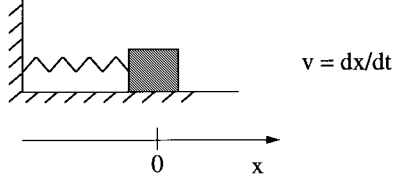
Fig. 6. Spring-block system.

of the arcs should be reversed if $[b_{j,k}] < 0$. Finally, these arcs will not be present if $[b_{j,k}] = 0$.

Remark 2: The previously described construction may be simplified if (3) contains only one term for a given $\dot{x}_i$. In fact, in this case the qualitative value of $\dot{x}_i$ is equal to the qualitative value of a state variable x_j or of an input u_k (possibly changed of sign). Thus, we need not introduce the $\dot{x}_i$ subnet.

An example of application of this algorithm can be found in Section IV-B.

IV. CASE STUDY OF CONTINUOUS SYSTEMS

In this first example we consider the dynamic system discussed in [9] and shown in Fig. 6, that consists of a block and a spring. We will present different Petri net models all describing such a system.

The dynamic equations of this system are

$$m\ddot{x} + b\dot{x} + \rho x = 0 \qquad (4)$$

where m is the block mass, b the friction coefficient, ρ the spring elastic constant, and x the block position. If v denotes the block velocity, (4) may be rewritten as:

$$\dot{x} = v, \quad \dot{v} = -\frac{b}{m}v - \frac{\rho}{m}x. \qquad (5)$$

We will assume, for sake of simplicity, that $m = 1$, and $\rho = 1$. We will also consider separately two cases: $b = 0$, i.e., conservative system, and $b = 1$, i.e., dissipative system.

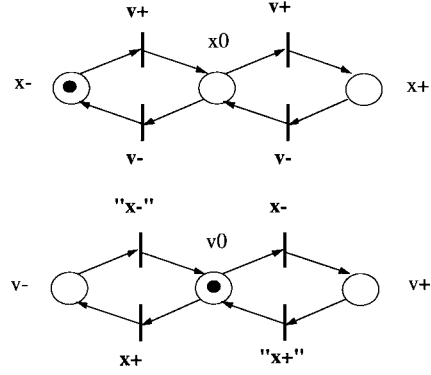If $b = 0$, the qualitative model of the system is given by the following set of qualitative equations:

$$[\dot{x}] = [v]$$
$$[\dot{v}] = -[x]. \qquad (6)$$

If $b = 1$, the qualitative model of the system is given by the following set of qualitative equations:

$$[\dot{x}] = [v]$$
$$[\dot{v}] = -[x] - [v]. \qquad (7)$$

A. Model with Quantity Space $\{-, 0, +\}$

Let us first consider the case $b = 0$. If the quantity space of x and v is the set $\{-, 0, +\}$, following Algorithm 1 (6) can be represented with the Petri net model in Fig. 7. Here a token in place x− means that $[x] = -$, a token in place x0 means that $[x] = 0$, and so on. Thus, we may write $[x] = \text{sign}\{M(x+) - M(x-)\}$ and $[v] = \text{sign}\{M(v+) - M(v-)\}$. A physically meaningful initial marking M_0 will be such that

Fig. 7. Petri net model of the spring-block system with quantity space $\{-, 0, +\}$ and its reachability graph.

$M_0(x-) + M_0(x0) + M_0(x+) = 1$ and $M_0(v-) + M_0(v0) + M_0(v+) = 1$.

The model captures the qualitative behavior described by (6). In fact, the qualitative value of x may increase (decrease) moving a token from, say, place x− to place x0, only when there is token in v+ (v−), i.e., when $[v] = +$ ($[v] = -$). Similarly, the qualitative value of v may increase (decrease) only when there is a token in place x− (x+), i.e., when $[x] = -$ ($[x] = +$).

Starting from the initial marking shown in Fig. 7, the reachability graph of the net is shown in the same figure (the dashed arcs should not be considered), where we have represented the markings as vectors: $M = [M(x-) \ M(x0) \ M(x+) \ M(v-) \ M(v0) \ M(v+)]^T$. Thus, the marking in Fig. 7 is $M_0 = [1\ 0\ 0\ 0\ 1\ 0]^T$.

The reachability graph shows that, starting from the initial state $\{[x] = -, [v] = 0\}$, the system will oscillate, never reaching the steady state $\{[x] = 0, [v] = 0\}$ corresponding to the rest marking $M_r = [0\ 1\ 0\ 0\ 1\ 0]^T$. In this case the reachability analysis is feasible, because the state space of the system consists of only nine possible markings. When the complexity of the net increases and the state space becomes very large it may be useful to resort to structural analysis, studying the properties of the net that solely depend on its structure.

In the net in Fig. 7 the set of places $\mathcal{P} = \{x-, x+, v-, v+\}$ is both a trap and a siphon. Hence, if the initial marking is M_r since $\mathcal{P}$ is a token-free siphon it will always remain token free, i.e., no transition will ever fire. Conversely, if the initial marking assigns at least a token to $\mathcal{P}$ the marking M_r will not be reachable because
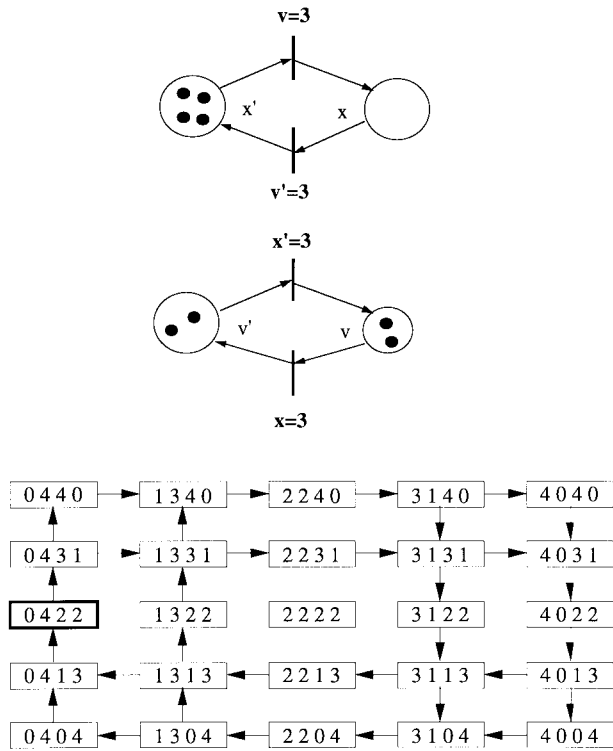
Fig. 8. Petri net model of the conservative spring-block system with quantity space $\{-2, \cdots, 2\}$ and its reachability graph.

Fig. 9. Petri net model of the dissipative spring-block system with quantity space $\{-2, \cdots, 2\}$ and its reachability graph.

under this marking the trap $\mathcal{P}$ is token-free. These simple observations lead to the same conclusions that can be drawn from the graph in Fig. 7, without having to construct the reachability graph itself.

Let us now consider the case $b > 0$. The system is now governed by the qualitative equations (7). In this case if $[v] = -$, due to the ambiguity of the qualitative sum it may be the case that $[\dot{v}] = +$ (and thus the velocity may increase) even if $[x] = 0$ or $[x] = +$. Similarly, if $[v] = +$ it may be the case that $[\dot{v}] = -$ even if $[x] = 0$ or $[x] = -$. The corresponding Petri net model, derived by Algorithm 1, is the one in Fig. 7, where the labels in quotes are removed. (This corresponds to removing some self-loops.) Starting from the initial marking of the net, the reachability graph of the net is also shown in Fig. 7 where both continuous and dashed arcs should be considered. Now, from the initial marking $M_0 = [1\ 0\ 0\ 0\ 1\ 0]^T$ it is possible to reach the rest state $M_r = [0\ 1\ 0\ 0\ 1\ 0]^T$ from which no further evolution is possible.

B. Model with Finer Quantity Space

The previous qualitative description in terms of the quantity space $\{-, 0, +\}$ may be too poor in describing the evolution of the spring-block system. We can also capture the behavior of this dynamic system with a finer quantity space $\{-n, \cdots, n\}$ for state variables $[x]$ and $[v]$. Here we assume $n = 2$.

Let us first consider the case $b = 0$ that is represented with the Petri net in Fig. 8. This model has been constructed following Algorithm 2. As discussed in Remark 2, it is not
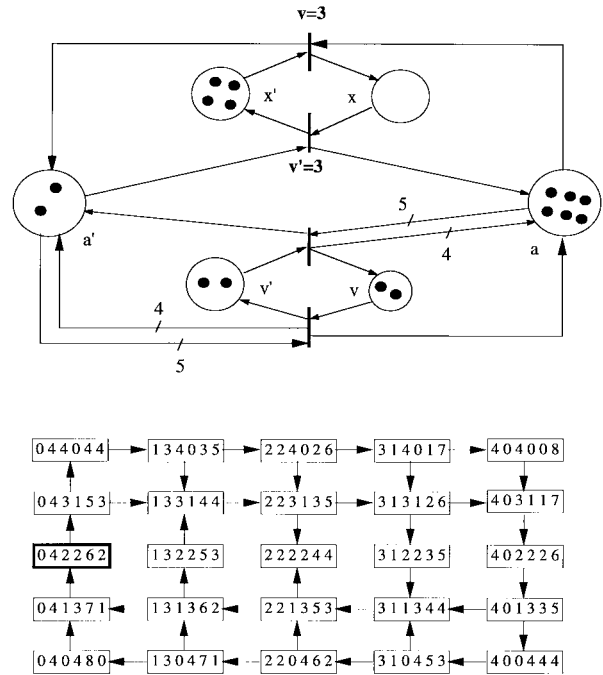
necessary to introduce the places associated to the derivatives of the state variables, because $[\dot{x}] = [v]$, and $[\dot{x}] = -[x]$.

In this model we have associated to each variable such as the position x (velocity v) two places: x and x$'$ (v and v$'$). The qualitative value of x is $[x] = M(\text{x}) - 2$; the qualitative value of v is $[v] = M(\text{v}) - 2$. Since the position x may increase only when the velocity v is greater than 0 [i.e., $M(v) \geq 3$] the transition whose firing increases the token count in place x has a selfloop of 3 arc with place v. A similar explanation can be given for the presence of all other selfloops in the figure. In the same figure is also given the reachability graph of the net from the initial marking $M_0 = [M_0(\text{x})\ M_0(\text{x}')\ M_0(\text{v})\ M_0(\text{v}')]^T = [0\ 4\ 2\ 2]^T$ that corresponds to the initial state $\{[x] = -2, [v] = 0\}$. Note again that the rest marking $M_r = [2\ 2\ 2\ 2]^T$ is never reached from an initial marking different from M_r.

Let us now consider the case $b > 0$ that is represented with the Petri net in Fig. 9. In this model we have explicitly represented the qualitative value of the acceleration $a = \dot{v}$ introducing the two places a and a$'$. Since $[a] = -[x] - [v]$, the quantity space of a is $\{-4, \cdots, 4\}$. The qualitative value of the acceleration is $[a] = M(\text{a}) - 4$. The marking of the a subnet changes every time the marking of the x and v subnet changes.

A physically meaningful initial marking M_0 will be such that: $M_0(\text{x}) + M_0(\text{x}') = 4$; $M_0(\text{v}) + M_0(\text{v}') = 4$; $M_0(\text{a}) = M_0(\text{x}') + M_0(\text{v}')$; and $M_0(\text{a}) + M_0(\text{a}') = 8$.

As an example, in Fig. 9 it is also represented the reachability graph of the net for initial marking $M_0 = [M_0(\text{x})\ M_0(\text{x}')\ M_0(\text{v})\ M_0(\text{v}')\ M_0(\text{a})\ M_0(\text{a}')]^T = [0\ 4\ 2\ 2\ 6\ 2]^T$ that corresponds to the initial state $\{[x] = -2, [v] = 0, [a] = 2\}$. The reachability
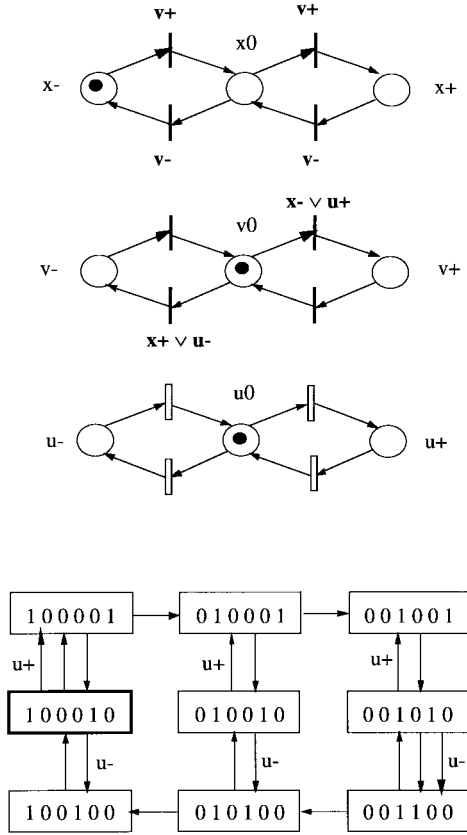
Fig. 10. Petri net model of the dissipative spring-block system with quantity space $\{-, 0, +\}$, forcing input u and its compacted reachability graph.

graph shows that the rest marking $M_r = [2\ 2\ 2\ 2\ 4\ 4]^T$, corresponding to the steady state $\{[x] = 0, [v] = 0, [a] = 0\}$, may now be reached from any other initial state.

C. Model with Forcing Inputs

In the previous examples we have considered a spring-block system without external forces. It is also possible to take into account external forcing inputs.

The dynamic equation (4) becomes

$$m\ddot{x} + b\dot{x} + \rho x = u(t) \tag{8}$$

where $u(t)$ is an external force.

The qualitative model of the system is given by the following set of qualitative equations:

$$[\dot{x}] = [v]$$
$$[\dot{v}] = -[x] - [v] + [u]. \tag{9}$$

If the quantity space of the variables is $\{-, 0, +\}$, using Algorithm 1 we obtain the Petri net model shown in Fig. 10(a).

The reachability set of this net is composed of 27 different markings, since each of the three subnets can have three different markings. We can however represent the reachability graph in a more compact form by projecting the total marking $M = [M(\text{x}-)\ M(\text{x}0)\ M(\text{x}+)\ M(\text{v}-)\ M(\text{v}0)\ M(\text{v}+)\ M(\text{u}-)\ M(\text{u}0)\ M(\text{u}+)]^T$ along the first six components. In the complete reachability graph there

will be a transition from marking $[0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1]^T$ (corresponding to the state $\{[x] = 0;\ [v] = 0;\ [u] = +\}$ to marking $[0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1]^T$ (corresponding to the state $\{[x] = 0;\ [v] = +;\ [u] = +\}$, while no transition will lead from marking $[0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0]^T$ to marking $[0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0]^T$ ($[u] = 0$ in this case) or from $[0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0]^T$ to marking $[0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0]^T$ ($[u] = -$ in this case). Thus in the compacted reachability graph, we introduce a transition labeled u+ from marking $[0\ 1\ 0\ 0\ 1\ 0]^T$ to marking $[0\ 1\ 0\ 0\ 0\ 1]^T$ to show that this transition may only fire when $[u] = +$.

The compacted reachability graph for the net in Fig. 10(a) is shown in Fig. 10(b).

V. HYBRID SYSTEMS

An hybrid system can be described in general terms as a system whose behavior is ruled by different sets of differential (or difference) equations. Each set is valid in a particular domain of the state space. Thus we distinguish a continuous time evolution within a domain (*interstate* evolution) and a discrete event evolution between domains (*intrastate* evolution).

If the interstate evolution can be described by qualitative equations, following the approach described in the previous sections, a Petri net model can be constructed. In this section we show how the different models, each valid in a given domain of the state space, can be combined to construct an overall Petri net model of the system.

Algorithm 3: Let us assume the system is ruled by different sets of qualitative equations of the form given in (2), each valid in a given domain $\mathcal{D}$.

1) Construct a Petri net model for the set of qualitative equations associated to each domain.
2) For each model, check if the state from which each transition is fireable is consistent with the domain constraint. If not, either restrict the firing of the transition adding self-loops or remove the transition if it can never fire.
3) Combine the different models in a single Petri net with set of transitions given by the union of all transitions of the different models.

A simple example will clarify this algorithm.

Consider the circuit in Fig. 11(a). We consider the quantity space $\{-, 0, +\}$. Due to the nonlinear behavior of the diode, we can distinguish two different behaviors.

In the domain $\mathcal{D}_1$: $[u] \geq [v]$, the diode is conducting and the circuit is ruled by the qualitative equation

$$[\dot{v}] = -[v] + [u]. \tag{10}$$

In the domain $\mathcal{D}_2$: $[u] < [v]$, the diode is reverse biased and the circuit is ruled by the qualitative equation

$$[\dot{v}] = -[v]. \tag{11}$$

The Petri net model corresponding to (10) is shown in the LHS of Fig. 11(b). We now take into account the restrictions imposed by the domain $\mathcal{D}_1$.
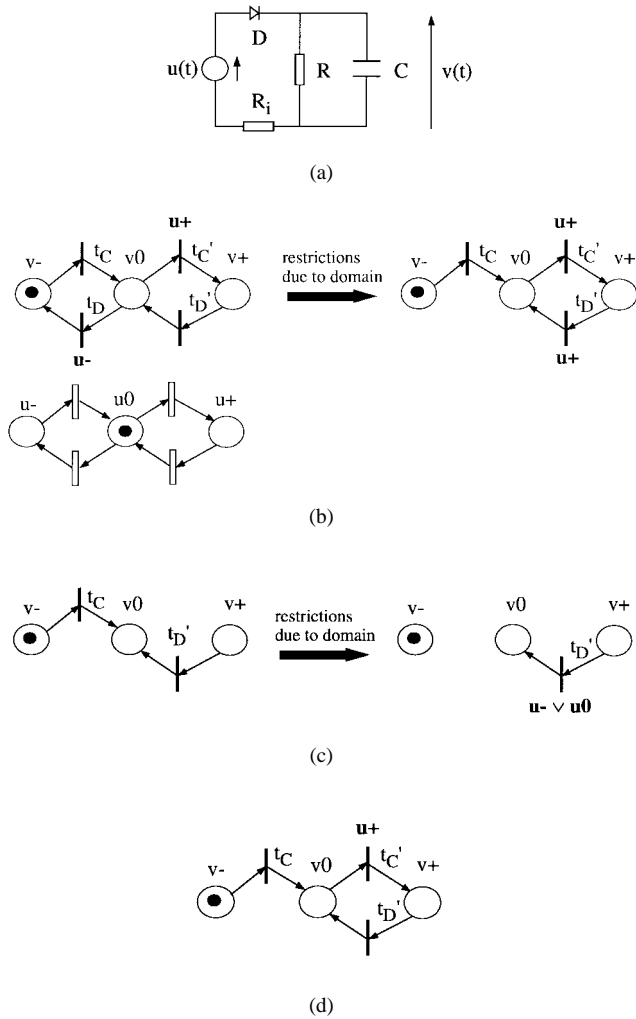
Fig. 11. (a) Nonlinear circuit, (b) Petri net model for domain $\mathcal{D}_1$, (c) Petri net model for domain $\mathcal{D}_2$, and (d) overall model.

Fig. 12. (a) Double integrator, (b) Petri net model, and (c) compacted reachability graph.

Let us consider transition t_D. This transition may fire when $[v] = 0$ and $[u] = -$. Since this state does not belong to the domain, we remove transition t_D.

Let us consider transition t'_D. This transition may fire when $[v] = +$, regardless of the value of $[u]$. However, the states $\{[v] = +, [u] = -\}$ and $\{[v] = +, [u] = 0\}$ do not belong to $\mathcal{D}_1$. Thus, to impose that t'_D may only fire from the state $\{[v] = +, [u] = +\}$, we add a self-loop with place u+, introducing the corresponding label on t'_D.

Taking into account all these restrictions we obtain the model in the RHS of Fig. 11(b).

The Petri net model corresponding to (11) is shown in the LHS of Fig. 11(c). Taking into account the restrictions imposed by the domain $\mathcal{D}_2$ we obtain the model in the RHS of Fig. 11(c).

We finally compose the two nets in a single Petri net, shown in Fig. 11(d), whose transitions are given by the union of the transitions of the two original nets. Combining the two nets, we have that there will be three transitions in parallel between place v+ and v0, (i.e., its label is **u− ∨ u0 ∨ u+**). Since regardless of the value of $[u]$ one of these transitions may fire, we remove the label, as shown in Fig. 11(d).
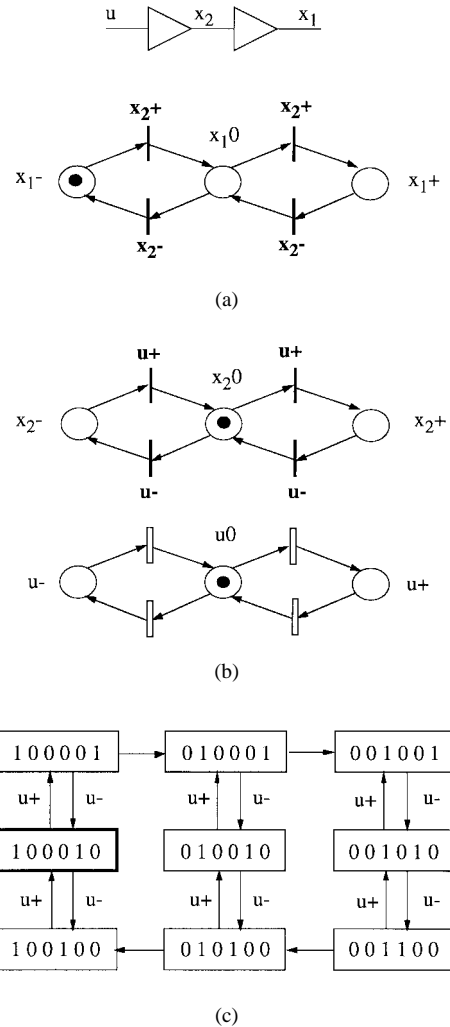
VI. APPLICATION TO CONTROL AND DIAGNOSIS

The main motivation for the research on qualitative modeling derives from the areas of diagnosis of physical systems [2], [3], [7], [10].

Discrete event system theory has been mainly focused on control [15], and Petri nets have often been used in this context [8]. There have also been some applications of discrete event theory in the diagnosis domain [16]–[18].

In our approach, qualitative systems are represented as Petri nets. Thus, all techniques developed for the diagnosis and control of discrete event systems may potentially be applied.

We discuss a simple example. Let us consider the double integrator shown in Fig. 12(a), ruled by the equations

$$\begin{cases} [\dot{x}_1] = [x_2] \\ [\dot{x}_2] = [u]. \end{cases}$$

The corresponding Petri net model (with quantity space $\{-, 0, +\}$) and its compacted reachability graph are shown in Fig. 12(b) and (c), assuming the markings as vectors $M = [M(x_1-) \; M(x_10) \; M(x_1+) \; M(x_2-) \; M(x_20) \; M(x_2+)]^T$.

Two classical control problems may be the following.

- Stability: Is it possible to find a control law such that a set of target states is reached in a finite number of steps?

  As an example, let's take as target the state $\{[x_1] = +, [x_2] = 0\}$, corresponding to the marking $[0\ 0\ 1\ 0\ 1\ 0]^T$. A possible feedback control law for solving this problem is the following: **if** $[x_1] = +$ **then** $[u] = -[x_2]$ **else** $[u] = +$, as can be checked by inspection of the reachability graph.

- Forbidden state avoidance: Is it possible to find a control law such that a set of forbidden states are never reached?

  As an example, let's take as forbidden the state $\{[x_1] = +, [x_2] = 0\}$, corresponding to the marking $[0\ 0\ 1\ 0\ 1\ 0]^T$. The maximally permissible feedback control law for solving this problem is the following: **if** $[x_1] = +$ **then** $[u] \neq -[x_2]$ **else** $[u] \in \{-, 0, +\}$.

Formal derivations of control policies for Petri net models can be found in the literature [8].

A Petri net model may also be used for model based diagnosis. Comparing the observed behavior of the system and the behavior predicted by the net, it is possible to find conflicts. It is also possible to construct fault models—by changing the qualitative equations that rule the behavior of the system—and to derive the corresponding net to check if the observed behavior can be explained by a particular fault.

As an example, we consider again the double integrator in Fig. 12(a). Let x_1 be the only measurable variable, and assume that the input has a constant value $[u] = +$. We consider that each single integrator may be malfunctioning because its input signal is stuck-at $-$ or at 0 or at $+$. Thus the reachability graph of the fault-free system and the simplified reachability graph (the nodes are not labeled) of the system affected by each of the six faults are shown in Fig. 13. The graph for the fault, say, $\dot{x}_2$ stuck-at 0, has been computed using the Petri net model corresponding to the faulty equation

$$\begin{cases} [\dot{x}_1] = [x_2] \\ [\dot{x}_2] = 0. \end{cases}$$

Let us consider an observed behavior in which the state variable x_1 changes from value $[x_1] = -$ to $[x_1] = 0$ and then to $[x_1] = -$ again. Clearly this behavior is not consistent with the reachability graph of the fault-free system. This behavior can only be explained by a stuck-at—fault on the input of the second integrator (i.e., by a fault such that $[\dot{x}_2] = -$) and corresponds to the evolution shown in the corresponding reachability graph by a thick arrow.

VII. CONCLUSIONS

The paper discussed how Petri nets may be used for the qualitative modeling of physical systems.

Given the quantitative description of a physical system behavior, the corresponding qualitative description is derived and is compiled into a Petri net structure. Different Petri net structures may be used to represent the same qualitative
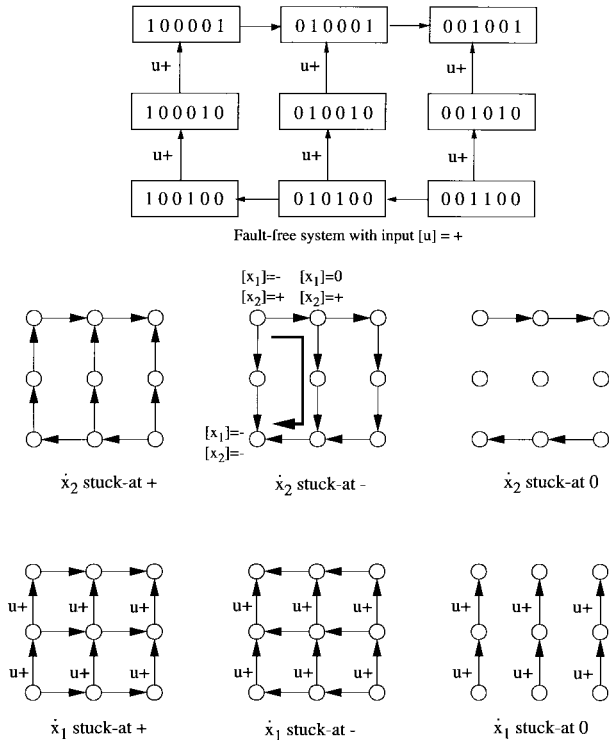

Fig. 13. Example of diagnosis of the double integrator. Reachability graph of the fault free system with $[u] = +$, and simplified reachability graphs for the faulty circuit.

behavior depending on the choice of the variable quantity space. Both systems described by homogeneous differential equations and systems with external forcing inputs have been considered.

This approach has also been extended to hybrid systems, integrating both continuous and discrete behaviors in the same net model.

Finally, examples of how this approach may be used for diagnosis and control have been given.

There are some advantages in using Petri nets to represent the qualitative behavior of a system. First, there is a simple and intuitive correspondence between the marking of the net and the state of the system. Second, the dynamic behavior of the system can be studied as sequences of reachable markings of the net. These evolutions can be computed with standard Petri net execution techniques using existing software packages. Third, the mathematical properties of Petri nets may be used to predict important characteristics of the system behavior without resorting to its simulation. In the paper it was briefly pointed out how siphons and traps prove to be important concepts to determine the existence of a steady state. This is an area open to further investigation.

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