# An $H_2$ Formulation for the Design of a Passive Vibration-Isolation System for Cars

Giorgio Corriga, Alessandro Giua, Giampaolo Usai

Dip. di Ingegneria Elettrica ed Elettronica, Università di Cagliari,

Piazza d'Armi, 09123 Cagliari, Italy

Phone: +39-070-675-5892 – Fax: +39-070-675-5900 – Email: {fanni,giua}@diee.unica.it

**Abstract**

Optimal design of an active suspension system for road vehicles can be solved using LQR techniques. Such a problem is equivalent, in the frequency domain, to determine the state feedback gain matrix that minimizes the $H_2$ norm of a suitable transfer matrix.

A passive suspension system can be seen as the physical realization of a suitable state feedback law whose gains are function of the system parameters. This law, and thus the characteristic elements of the passive suspension, can be determined as an approximation of the $H_2$ optimal solution. This methodology allows one to choose the best controller from a constrained subset (i.e., all possible passive suspensions of a particular form) of all possible controllers.

Published as:

G. Corriga, A. Giua, G. Usai, "An H2 Formulation for the Design of a Passive Vibration-Isolation System for Cars," *Vehicle System Dynamics*, Vol. 26, No. 5, pp. 381–393, November, 1996.

# 1 Introduction

The design of active suspensions for road vehicles has received considerable attention in the literature. The purpose of the design is that of optimizing the performances of the vehicle with regard to comfort, road holding, and rideability.

In an active suspension there are no passive elements, such as dampers and springs. The interaction between vehicle body and wheel is regulated by an actuator of variable length. The actuator is usually hydraulically controlled and applies between the body and wheel a force that represent the control action generally determined with an optimization procedure that uses LQR techniques.

Active suspensions give better performances with respect to passive suspensions. However, active suspension systems are rather complex, since they require several components such as actuators, servovalves, high-pressure tanks for the control fluid, sensors for detecting the system state, etc. The associated power, that must be provided by the vehicle engine, may reach the order of several 10 kW [1] depending on the required performances and vehicle weight. Furthermore, these suspension systems have a high cost.

As a viable alternative to a purely active suspension system, the use of an active suspension in tandem with a suitably dimensioned passive suspension has been considered [2, 3]. Such a mixed suspension system is still capable of providing the optimal control law, while sensibly reducing the power required by the active part but not its complexity.

Purely passive suspensions, however, remain the most valid engineering solutions when they can reasonably approximate the performances offered by the optimal active control. In fact they require no power consumption, they can be easily realized, and the resulting suspension system is simpler and less costly.

Using parameter optimization techniques it is possible to minimize the difference between the evolution of a passive and of an active system from a given initial condition, i.e., for a given disturbance. However, it is often the case with such a design that for different initial conditions the two evolutions are significantly far off.

In this paper, we present a design technique, based on the minimization of the $H_2$ norm of a particular transfer function. This design leads to good approximations of the active control law for different initial conditions. We show that some properties of the active suspension system are conserved when a passive suspension is used. Furthermore, we can also give an upper bound on the error criterion used to characterize the performances of the active and passive systems.

This paper is structured as follows. In section 2, following [4, 5], we recall how an optimal control problem may be solved, in the frequency domain, by computing a feedback gain matrix that minimizes the $H_2$ norm of a particular transfer matrix. In section 3 we consider the case of a road vehicle and show how a passive suspension system may be seen as a means of implementing a structured state feedback control law whose gains are simple functions of the suspension parameters [2, 3]. In particular, the optimal values of the parameters are chosen so as to better approximate the $H_2$ norm of the optimal system. In section 4 we consider an applicative example, namely the design of a quarter car suspension, and we compare the performances of the passive and active suspension systems.

## 2 Considerations on $H_2$ design

A standard approach to the control of linear time-invariant multiple input multiple output systems considers a block diagram such as the one shown in Figure 1 [4, 5]. In this figure $P(s)$ is the transfer matrix of the generalized plant, while $F(s)$ is the transfer matrix of the controller. The vector $w$ represents all external inputs, such as disturbances, sensor noise, and reference signals, while the vector $y$ is a criterion signal (usually called error signal in the literature). The vector $v$ is the set of observed variables used by the controller to compute the control input $u$. The closed loop transfer matrix between $w$ and $y$ is called *lower linear fractional transformation* (LFT) of $P$ and $F$ and is denoted $F_\ell(P, F)$.

Let us consider the linear model of a system to be controlled

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in I\!R^n$ is the state vector, $u \in I\!R^m$ is the control vector, and $z \in I\!R^p$ is the output vector.

Classical LQR problem formulation requires to find the state feedback law $u(t) = Fx(t)$, with $F \in I\!R^{m\times n}$, such that the cost functional

$$J = \int_0^\infty [z^T(t)Qz(t) + u^T(t)Ru(t)]dt, \tag{2}$$

is minimized for any initial state $x(0) = x_0$. Here $Q = Q^T \geq 0$, $R = R^T > 0$. The solution to the LQR problem is:

$$F^* = -R^{-1}B^TX, \tag{3}$$

where $X$ is the solution of the algebraic Riccati equation

$$XA + A^TX - XBR^{-1}B^TX + C^TQC = 0. \tag{4}$$

The closed loop poles are the eigenvalues of $A + BF^*$.

The equivalent frequency domain problem [5] is to find the state feedback matrix $F^*$ such that the norm[1] $\|F_\ell(P, F)\|_2$ is minimized, where the transfer matrix of the generalized plant $P(s)$ has the following expression in terms of state space data:

$$P(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \begin{array}{c} \\ \dot{x}_P \\ y_1 \\ y_2 \\ v \end{array} \begin{array}{c} x_P \quad\quad w \quad u \\ \left[\begin{array}{c|cc} A & I & B \\ \hline Q^{1/2}C & 0 & 0 \\ 0 & 0 & R^{1/2} \\ I & 0 & 0 \end{array}\right] \end{array} . \tag{5}$$

The state equation of the generalized plant is $\dot{x}_P(t) = Ax_P(t) + Bu(t) + w(t)$, i.e., it is the state equation of the system (1) with an additional disturbance input $w(t)$. For a given $u(t)$, the evolution $x(t)$ of the system under arbitrary initial conditions $x(0) = x_0$ is the same of the evolution $x_P(t)$ of the generalized plant, initially at rest, when the external input is $w(t) = x_0\ \delta(t)$.

The closed loop output vector of the generalized plant is

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} Q^{1/2}Cx_P(t) \\ R^{1/2}u(t) \end{bmatrix} \tag{6}$$

and from (2) it can be seen that whenever $x(t) = x_P(t)$

$$\|y\|_2^2 = J. \tag{7}$$

Since

$$y(s) = F_\ell(P, F)w(s), \tag{8}$$

it is possible to prove [5] that the minimization of the norm $\|F_\ell(P, F)\|_2$ leads to a minimization of (7) for any external input of the form $w(t) = x_0\ \delta(t)$.

Furthermore, let us recall another fundamental property of the $H_2$ norm: if the input $w$ is a vector white noise process with covariance matrix $E[w(t+\tau)w^T(t)] = I\ \delta(\tau)$, then the root mean square value (RMSV) output power of $y$ in equation (7) is $\|F_\ell(P, F)\|_2$. Thus this design also leads to the minimization of the RMSV of the criterion signal in response to white noise [5].

# 3 Design of a passive suspension

Let system (1) be the state space model of the quarter car active suspension system shown in Figure 2.a, where the control force $u$ is generated by an actuator.

[1] The definitions of norms are recalled in Appendix A.

Under the assumption of flat road surface, the deformation of the tyre and the position of the unsuspended mass may be taken as identical. Thus we have the following state variables.

$x_1(t)$ is the deformation of the suspension with respect to the static equilibrium configuration, taken as positive when elongating.

$x_2(t)$ is the vertical absolute velocity of the suspended mass $M_2$.

$x_3(t)$ is the movement of the non suspended mass with respect to the static equilibrium configuration, taken as positive upwards. Under the assumption of flat road surface, this is also the deformation of the tyre.

$x_4(t)$ is the vertical absolute velocity of the non suspended mass $M_1$. The tyre is represented as a purely elastic component of elastic constant. $\lambda$.

The matrices $A$, $B$, and $C$ are:

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda/M_1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1/M_2 \\ 0 \\ -1/M_1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{9}$$

The optimal control law will depend on the weight matrices $Q$ and $R$, to be chosen as a function of the desired performances. Useful suggestions for the choice of the weights can be found in the literature [6]. The control law will be in the form

$$u(t) = Fx(t) = [\ f_1 \quad f_2 \quad f_3 \quad f_4\ ][\ x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)\ ]^T. \tag{10}$$

The implementation of this control law is quite complex, since it requires the knowledge of the state vector, that can be reconstructed by an observer. Furthermore, the resulting suspension system is rather complex, requiring not only suitable software but also a set of components as discussed in the introduction. Its high cost and its limited reliability have hindered its application to commercial road vehicles.

A passive suspension, shown in Figure 2.b, can be seen as a particular state feedback law corresponding to a structured gain feedback matrix [2]

$$F_p = [\ -k \quad -f \quad 0 \quad f\ ], \tag{11}$$

where $k > 0$ is the spring elastic constant, and $f > 0$ is the characteristic coefficient of the damper.

We set our goal to that of designing a passive suspension whose control law, generated by a matrix $F_p^*$, is the best possible approximation of the control law (10), in the sense that the value of $k^*$ and $f^*$ are such to minimize the norm $\|F_\ell(P, F_p)\|_2$. Clearly we will have that

$$\|F_\ell(P, F_p^*)\|_2 \geq \|F_\ell(P, F^*)\|_2, \tag{12}$$

since the RHS of inequality (12) is a global minimum.

The performances given by the passive suspension so designed will in general be worse than those given by the active suspension in terms of the performance index (2). However, all those properties that depend on the $H_2$ norm will still be valid.

In particular, if the external input $w$ is white noise with covariance matrix $E[w(t+\tau)w^T(t)] = I\,\delta(\tau)$, then the RMSV of the power associated to the output will be minimized with respect to any other passive suspension.

Deterministically, we can say that if the generalized plant is excited with $n$ different disturbances $w_i(t) = e_i\delta(t)$, where $e_i$ is the i-th canonical basis vector, and $\delta(t)$ is the impulse, and we call $y_i(t)$ the corresponding criterion signal, then

$$\sum_{i=1}^{n} \|y_i\|_2 = \|F_\ell(P, F_p)\|_2. \tag{13}$$

Since the $\|y_i\|_2^2$ can be considered as the value $J_{p,i}$ of the performance index (2) when the passive system starts from the initial condition $x(0) = e_i$, the minimization of the $\|F_\ell(P, F_p)\|_2$ leads to the minimization of the $\sum_{i=1}^{n} J_{p,i}^{0.5}$ among all possible passive suspensions.

The passive suspension does not enjoy the fundamental property of optimal control, namely that of minimizing the performance index (2) for any arbitrary initial state $x_0$. It is possible, however, to find upper bounds for the value $J_p$ taken by (2) when the feedback matrix is $F_p^*$. Let $y(s)$ and $y_p(s)$ be the outputs of the generalized plant with the optimal active and passive suspensions when the input is $w(s) = x_0$. Then

$$\begin{aligned} y(s) &= F_\ell(P, F^*)\; x_0, \\ y_p(s) &= F_\ell(P, F_p^*)\; x_0. \end{aligned} \tag{14}$$

It is possible to prove, as we show in Appendix B, that the performance of the optimal active suspension is bounded by

$$J = \|y\|_2^2 \leq \|F_\ell(P, F^*)\|_2^2\; \|x_0\|_2^2. \tag{15}$$

while the performance of the passive suspension is bounded by

$$J_p = \|y_p\|_2^2 \leq \|F_\ell(P, F_p^*)\|_2^2\; \|x_0\|_2^2. \tag{16}$$

Note that in the previous equations the norm $\|x_0\|_2$ is the euclidean norm of a vector, while the norm $\|F_\ell(P, F_p^*)\|_2$ is a transfer function norm. These equations have a nice physical interpretation. They show that the value of the $H_2$ norm is an upper bound for the value of the performance index under arbitrary initial conditions on the unitary sphere.

When the numerical values of $\|F_\ell(P, F^*)\|_2$ and $\|F_\ell(P, F_p^*)\|_2$ are close, we may conclude that for any arbitrary initial conditions the performance indexes $J$ and $J_p$ have close upper bounds.

# 4 Applicative example

The proposed design has been applied to the quarter suspension shown in Figure 2, with system matrices given by (9) and values of the parameters [6]: $M_1 = 28.58$ kg, $M_2 = 288.90$ kg, $\lambda = 155900$ N/m.

The matrices $Q$ and $R$, taken from [6], are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}; \quad R = 0.8 \cdot 10^{-9} \ .$$

These weights, realizing a trade off between road holding and comfort, lead to good performances.

The optimal state feedback gain matrix for active control is

$$F^* = [\ -35355 \quad -4827 \quad 21879 \quad 1386\ ],$$

and gives a $\|F_\ell(P, F^*)\|_2^2 = 0.450$ that is the minimum value achievable.

To determine the optimal parameters of the passive suspension, we need to find the

$$\min_{k,f} \|F_\ell(P, F_p)\|_2. \tag{17}$$

We used the software tools available in Matlab: `fmins` is the minimization procedure, and `normh2` computes the $H_2$ norm. The optimal values of $k$ and $f$ are $k^* = 14,345$ N/m and $f^* = 1,918$ N·s/m. The corresponding state feedback gain matrix is

$$F_p^* = [\ -14345 \quad -1918 \quad 0 \quad 1918\ ],$$

and gives a $\|F_\ell(P, F_p^*)\|_2^2 = 0.522$. Note that finding the optimal passive suspension is a problem of constrained controller optimization. As such, it is almost certainly not a convex optimization problem and there may be multiple solutions that locally minimize the $\|F_\ell(P, F_p)\|_2^2$. However, in the case at hand starting from different inital values of $F_p$ corresponding to physically meaningful passive suspension, we observed that the minimization procedure always converges to the same value $F_p^*$.

Figures 3-6 shows the results of two different simulations under the assumption of flat road surface.

In the first simulation a unitary initial disturbance was applied to the suspension deformation, i.e., the initial state was $x_0 = [\ -1 \quad 0 \quad 0 \quad 0\ ]^T$. This corresponds to an abrupt compression of the suspension by an external force applied to the suspended mass, as during a sudden braking or a sharp cornering of the car.

In the second simulation a unitary initial disturbance was applied to the non suspended mass, i.e., the initial state was $x_0 = [\ 0 \quad 0 \quad -1 \quad 0\ ]^T$. This corresponds, losely, to an abrupt compression of the tyre, that can be due to a sharp discontinuity in the road profile.

The results of simulation show an acceptable loss in performance when the passive suspension is compared with the active one.

It is often the case that a passive suspension is dimensioned so that it well approximates the active suspension behavior for a given type of disturbance. This is done solving a problem of parameter identification whose solution highly depends on the chosen initial state of the system, i.e., on the type of perturbation. The behavior is usually not satisfactory when a different perturbation is considered.

In the design we have discussed, as expressed in equation (17), the passive approximation of the active system does not depend on a particular initial condition of the state $x_0$ and thus has a global validity, as shown in equations (15) and (16).

To show the influence of the choice of weights $Q$ and $R$ on the passive suspension design, we have computed $F_p^*$ for different values of the control input weight $R$ while keeping constant $Q$. These results are shown in Figures 7-9. From Figure 7 we can see that the ratio of $\|F_\ell(P, F_p^*)\|_2^2 / \|F_\ell(P, F^*)\|_2^2$ is less than 1.2 for all values of $R$. Hence the active and passive design have performance upper bounds that are close for all values of $R$. In Figure 8 we can read the value of the optimal spring elastic constant $k^*$ as a function of $R$. In Figure 9 we can read the value of the optimal damper characteristic coefficient $f^*$ as a function of $R$.

# 5 Conclusions

We presented a design technique for a passive suspension system of road vehicles. The design technique is based on the approximation of an optimal control law corresponding to a reference active suspension.

A passive suspension system can be seen, from a control point of view, as a state feedback law with a structured gain matrix whose elements are simple functions of the suspension parameters. The optimal value of the structured gain matrix — and thus of the suspension parameters — is such that a suitable $H_2$ norm is as close as possible to the norm corresponding to optimal solution.

This design leads to passive suspensions with the following properties:

- If the external disturbances can be represented as white noise with covariance matrix $I\ \delta(\tau)$, then the root mean square value of the power associated to a dummy output, chosen to characterize the disturbance effect, will be minimized with respect to any other passive suspension.
- The passive suspension does not enjoy the fundamental property of optimal control, namely that of minimizing the chosen performance index for any arbitrary initial state $x_0$. However, it has good performances for different types of disturbances, such as those corresponding to irregularities of the road or corresponding to forces acting on the suspended mass.

It was also possible to give an upper bound on the error criterion used to characterize the performances of the active and passive systems.

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# Appendix

## A - Definition of norms

Let $x \in I\!R^n$ be a vector. The 2-norm (euclidean norm) of $x$ is :

$$\|x\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \left( x^T \ x \right)^{\frac{1}{2}}$$

Let $V$ be a positive semi-definite matrix in $I\!R^{n \times n}$. Then:

$$x^T \ V \ x \leq \text{trace}\{V\} \ \|x\|_2^2$$

Let $y(t) : I\!R \rightarrow I\!R^n$ be a signal vector and $y(s)$ its Laplace transform. The $H_2$ norm of $y$ is:

$$\|y\|_2 = \left( \int_{-\infty}^{\infty} y^T(t) \ y(t) \ dt \right)^{\frac{1}{2}} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} y^H(j\omega) \ y(j\omega) \ d\omega \right)^{\frac{1}{2}}$$

where $^H$ denotes the conjugate transpose.

Let $g(t) : I\!R \rightarrow I\!R^{m,n}$ be a signal matrix and $g(s)$ its Laplace transform. The $H_2$ norm of $g$ is:

$$\|g\|_2 = \left( \int_{-\infty}^{\infty} \text{trace}\{g^T(t) \ g(t)\} \ dt \right)^{\frac{1}{2}} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\{g^H(j\omega) \ g(j\omega)\} \ d\omega \right)^{\frac{1}{2}}$$

## B - Proof of bound (15)

Let $y(t)$ be the output of the generalized plant for an input $w(t) = x_0 \delta(t)$, i.e., $y(s) = F_\ell(P, F^*) \ x_0$. As discussed in the paper, the value of the cost functional $J$ for the closed loop system starting from initial conditions $x(0) = x_0$ is:

$$\begin{aligned} J \ &= \|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} y^H(j\omega) \ y(j\omega) \ d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_0^T \ F_\ell^H(P, F^*) \cdot F_\ell(P, F^*) \ x_0 \ d\omega \\ &= x_0^T \ V \ x_0 \end{aligned}$$

where $V = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\ell^H(P, F^*) \ F_\ell(P, F^*) \ d\omega \right)$ is positive semi-definite.

Hence we can write:

$$\begin{aligned} J \leq \text{trace}\{V\} \ \|x_0\|_2^2 \ &= \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\{F_\ell^H(P, F^*) \ F_\ell(P, F^*)\} \ d\omega \right) \ \|x_0\|_2^2 \\ &= \|F_\ell(P, F^*)\|_2^2 \ \|x_0\|_2^2 \end{aligned}$$

A similar reasoning can be used to prove bound (16).

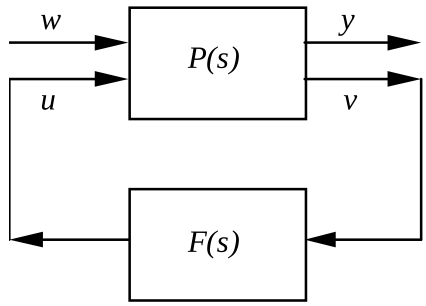


Figure 1: Linear fractional transformation scheme.

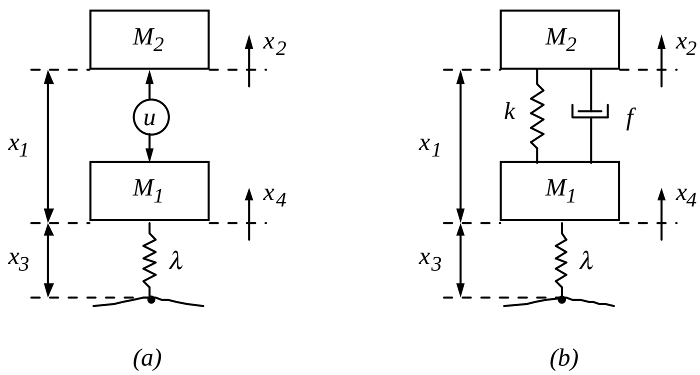


Figure 2: Model of a quarter car active $(a)$ and passive $(b)$ suspension.

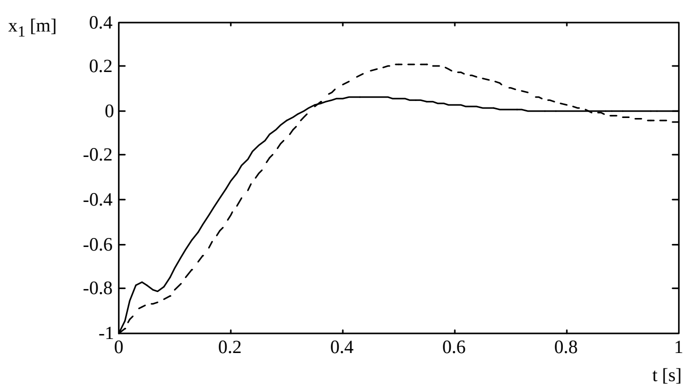


Figure 3: $x_1(t)$ for initial state $x_0 = [-1\ 0\ 0\ 0]^T$, using active (—) and passive (- - -) suspensions.

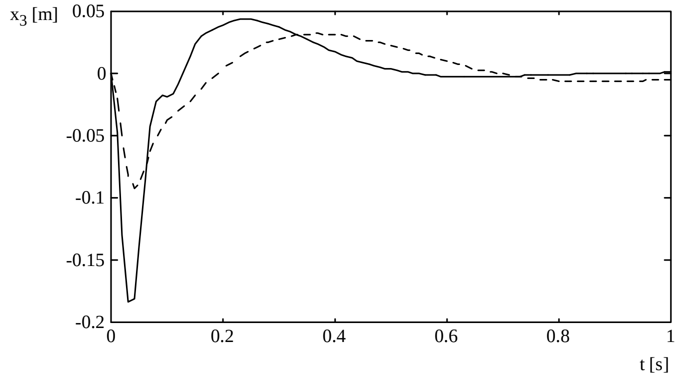


Figure 4: $x_3(t)$ for initial state $x_0 = [-1\ 0\ 0\ 0]^T$, using active (—) and passive (- - -) suspensions.

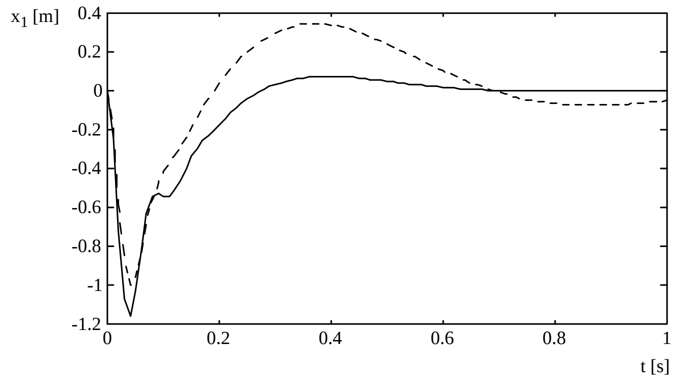


Figure 5: $x_1(t)$ for initial state $x_0 = [0\ 0\ -1\ 0]^T$, using active (—) and passive (- - -) suspensions.

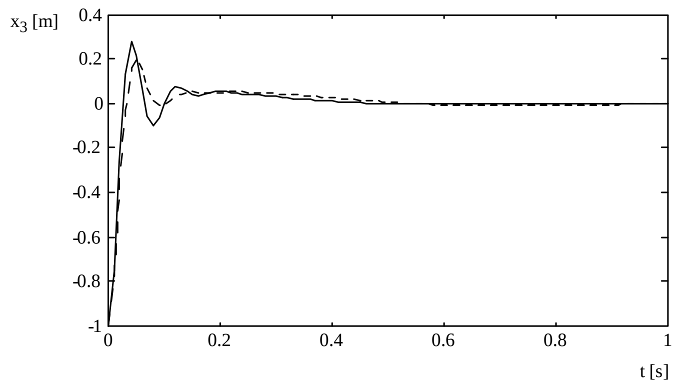


Figure 6: $x_3(t)$ for initial state $x_0 = [0\ 0\ -1\ 0]^T$, using active (—) and passive (- - -) suspensions.

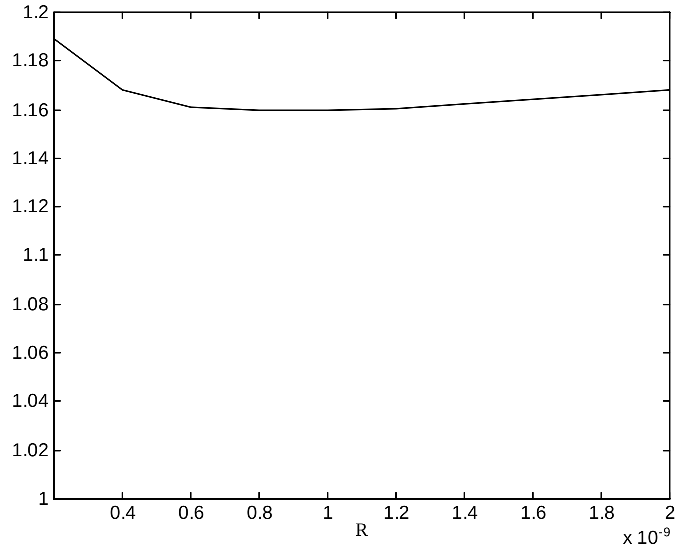


Figure 7: Ratio of $\|F_\ell(P, F_p^*)\|_2^2 / \|F_\ell(P, F^*)\|_2^2$ for different values of the control input weight R.

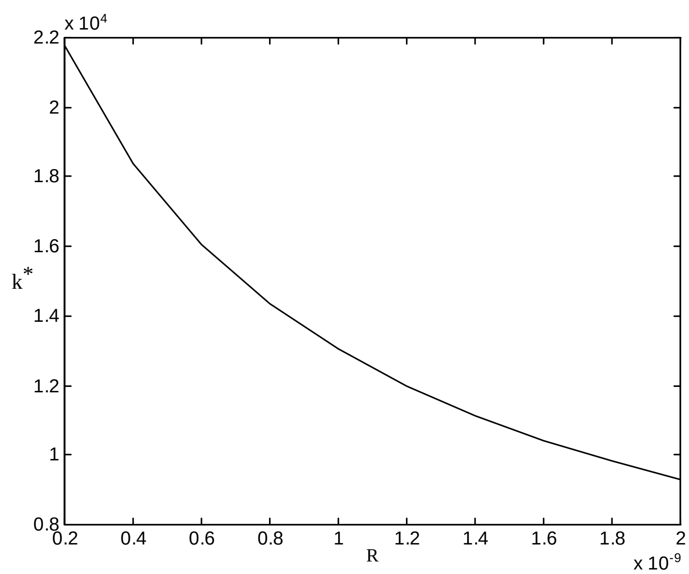


Figure 8: Value of the spring elastic constant $k^*$ in N/m for different values of the control input weight R.

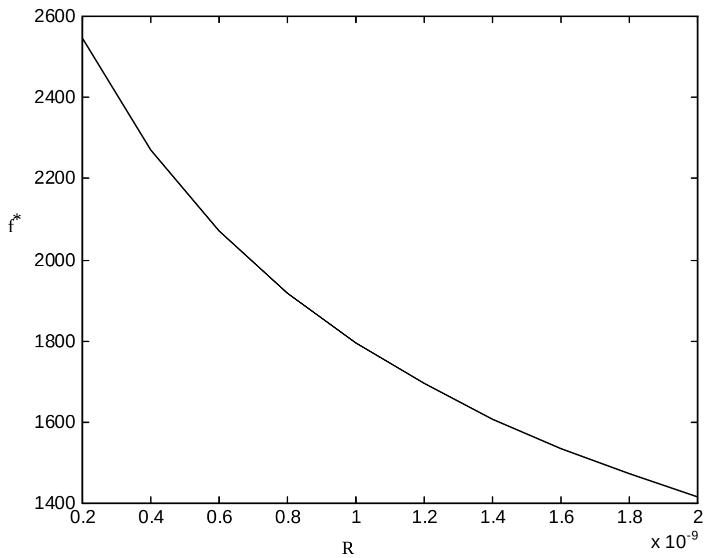


Figure 9: Value of the damper characteristic coefficient $f^*$ in Ns/m for different values of the control input weight R.