Stabilization of Switched Systems via Optimal Control

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Abstract

We consider switched systems composed of linear time invariant unstable dynamics and we deal with the problem of computing an appropriate switching law such that the controlled system is globally asymptotically stable. On the basis of our previous results in this framework, we first present a method to design a feedback control law that minimizes a linear quadratic (LQ) performance index when an infinite number of switches is allowed and at least one dynamics is stable. Then, we show how this approach can be useful when dealing with the stabilization problem of switched systems characterized by unstable dynamics, by applying the proposed procedure to a "dummy" system, augmented with a stable dynamics. If the system with unstable dynamics is globally exponentially stabilizable, then our method provides the feedback control law that minimizes the chosen quadratic performance index, and that guarantees the closed loop system to be globally asymptotically stable.

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1 Introduction

In this paper we show how it is possible to design a stabilizing law $i(t)$ for linear time-invariant (LTI) switched systems

$$\dot{x}(t) = A_{i(t)}x(t), \qquad i(t) \in \mathcal{S} = \{1, \ldots, s\}, \tag{1}$$

where for all $i \in \mathcal{S}$, A_i are unstable [7].

Note that here all subsystems are autonomous, i.e., the only control action is the switching function $i(t)$.

The proposed procedure is based on the solution of an optimal control problem for the above switched system.

In [25] we have presented a technique to solve an optimal control problem with quadratic performance index for systems of the form (1) assuming that a finite number of switches N is allowed. The solution takes the form of a state feedback law, i.e., the optimal value of $i(t^+)$ can be chosen as a function of the current continuous state $x(t)$, of the current dynamics $i = i(t)$ and of the number of remaining switches k. The feedback law is described by a set of partitions of the state space $\mathcal{C}_{i,k}$ that, for a given current dynamics i and for a given number of remaining switches k, assigns to each continuous state $x(t)$ the optimal value of $i(t^+)$.

In [25] we dealt with the case of finite N, hence we assumed that at least one dynamics is stable to ensure that the considered optimal control problem has a finite cost. In such a case the system is trivially stabilizable: just use the stable dynamics. In this paper, on the contrary, we consider the case in which all dynamics are unstable and thus an infinite number of switches is required to stabilize the system.

Our stabilization procedure is based on two ideas.

Firstly, we show that an optimal control law for infinite switches can be easily computed solving an optimal control problem for a finite number N of switches, provided N is large enough. We also show that in this case the optimal control law is still given by a partition of the state space $\mathcal{C}_\infty$ that, however, does not depend on the current location $i(t)$ and on the number of remaining switches (that is obviously infinite).

Secondly, to relax the assumption that at least one dynamics is stable, we extend system (1) by adding an arbitrary dummy stable dynamics $s + 1$. We also show that if the cost associated to dynamics $s + 1$ is sufficiently large, then the system is stabilizable if and only if no region of partition $\mathcal{C}_\infty$ is associated to dynamics $s + 1$.

It is important to stress that the presented stabilization procedure is not necessarily related to the optimal control technique presented in [25]. In fact, it can be applied in tandem to any procedure that solves an optimal control problem providing a feedback law with a finite number of switches, such as the one presented by Shaikh and Caines in [26].

Moreover we observe that using an optimal control approach to stabilize a switched system has already been investigated by other authors. See e.g. the recent survey by Margaliot [23].

Other interesting approaches to the optimal control of switched systems have been proposed by Xu and Antsaklis [30], by Egerstedt, Wardi et al. [3, 10], and by Axelsson et al. [1]. However such procedures cannot be directly applied within this framework in their present form because they do not determine a state feedback control law.

The main advantage of the stabilizing approach we present is that it provides a systematic procedure to compute a stabilizing law when it does exist. In fact, although there is a rich literature on stability analysis of hybrid systems, there are very few results on the design of stabilizing laws. The literature in this area is surveyed in the next subsection.

Finally, we observe that a disadvantage of the optimal control approach [25] we use, that is common to several other approaches (see e.g. the paper by Hedlund and Rantzer [12]), is that it requires a discretization of the state space. For large dimensional systems this may be computationally burdensome. Moreover, the discretization may also affect the optimality of the solution. However, here we are concerned with stability thus this problem is less important. This issue has been extensively addressed in [25] and it is not discussed in this paper. Note, however, that as stated above, the use of the procedure in [25] is not a strict requirement for the presented stabilization procedure.

1.1 Literature review

Many papers on stability and stabilizability of switched linear systems have been published in the last two decades. In this section we provide a short overview of the most important contributions in this topic with particular attention to those results that are closely related to this paper. In particular, we focus on autonomous switched linear systems, i.e., systems with no control input [9, 17]. For a more exhaustive survey on these topics we address to the recent work by Lin and Antsaklis [22].

The first problem that has been investigated in this framework is that of stability under arbitrary switching. Solutions to this problem have been proposed based on common quadratic Lyapunov functions (CQLF) [15–17, 20, 24, 27] or on switched quadratic Lyapunov functions [8]. In [22] some necessary and sufficient conditions are given. In particular, it is shown that the asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems, thus allowing to use a series of conditions that exist in this framework [2].

Several results have also been proposed in the literature under the assumption that the switching signals satisfy certain constraints, namely under restricted switching. In many applications this is definitely realistic and it is unnecessarily conservative to impose that stability should hold under arbitrary switching. Such restrictions may either arise in the order in which current modes can be active, or in the minimum time that each mode should remain active once it has been activated. Many contributions in this framework are based on the multiple Lyapunov functions

(MLF). See, e.g., [9, 18] and a series of references therein.

Another fundamental issue, related to stability analysis, is the stabilization problem, i.e., the synthesis of a stabilizing law. A really rich literature on this have been produced in the last years. See, e.g. [5, 8, 9] just to mention a few. However, the main limitation of such approaches is that they only give sufficient conditions for the existence of a stabilizing law.

Necessary and sufficient conditions are given in [29] in the case of a switched system commuting between two subsystems, when the performance index under consideration is the quadratic stability of the switched systems. The main feature of this property is that it requires for uncertain systems a quadratic Lyapunov function which guarantees asymptotic stability for all uncertainties under consideration, and is thus a kind of robust stability with very good property, yet usually needs more restrictive conditions [31]. Iterative algorithms for constructing such common Lyapunov functions can be found in [19].

Antsaklis *et al.* in [14], using a geometric approach, were able to obtain necessary and sufficient conditions for asymptotic stabilizability of switched systems with an arbitrarily large number of *second-order* LTI unstable systems. When the switched system is asymptotically stabilizable, they also provide an approach to compute a stabilizing law.

Finally, Lin and Antsaklis [21] derived a necessary and sufficient condition for the existence of a switching control law (in static feedback form) for asymptotic stabilization of continuous-time switched linear systems.

1.2 Paper structure

The paper is structured as follows. In Section 2 the problem statement is formally introduced and some preliminary results used in the rest of the paper are provided. Section 3 discusses the optimal control problem in the case of a finite number of switches. In particular this section is focused on the procedure we proposed in [25]. The optimal control problem in the case of an infinite number of switches is discussed in detail in Section 4. Section 5 presents the main contribution of this paper that consists in a procedure to stabilize switched systems with unstable dynamics. A numerical example is presented in Section 6. Conclusions are finally drawn in Section 7 where our future work in this framework is also discussed.

2 Problem formulation

Consider the *nonautonomous system*

$$\dot{x}(t) = f(x, t) \tag{2}$$

where $f : D \times [0, \infty) \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $D \times [0, \infty)$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

System (1) is a particular case of (2), denoted as *switched systems*, when $f(x,t) \triangleq A_{i(t)}x(t)$ for $t > 0$.

In this paper we assume a continuous evolution of the state, i.e., when a switch occurs at time τ, $x(\tau^-) = x(\tau^+)$.

Now, let us consider system (1) with initial continuous state $x(0) = x_0$ and initial discrete state $i(0) = i_0$.

Let $i(t) : [0, +\infty) \to \mathcal{S}$ be a piecewise constant function that represents our control variable. For the ease of notation, the following symbolism will be adopted in the reminder:

$$i(t) = \{(i_k, \delta_k)\}_{k \geq 0}$$

meaning that the system evolves with dynamics A_{i_0} during the time interval $[0, \delta_0)$; then, it switches to dynamics A_{i_1} at time δ_0 and evolves with such a dynamics until time $\delta_0 + \delta_1$, and so on.

Moreover, we denote as $\{x_1, x_2, \ldots, x_{k+1}, \ldots\}$ the set of continuous states reached after the time intervals δ_0, $\delta_0 + \delta_1$, …, $\sum_{i=0}^{k} \delta_i$, …, respectively.

Definition 1 The switched system (1) is said *stabilizable* if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, given any initial condition $x(0)$ with $\|x(0)\| \leq \delta(\varepsilon)$, a switching control law exists such that for all $t > 0$ the state $x(t)$ of the controlled system satisfies the condition: $\|x(t)\| \leq \varepsilon$.

Analogous definitions hold for asymptotic (or exponential) stabilizability. ■

For the class of systems we are dealing with the following important results hold.

Lemma 1 Let us consider the switched linear system (1) with initial state (x_0, i_0) and controlled with $i(t) = \{(i_0, \delta_0), (i_1, \delta_1), \ldots, (i_k, \delta_k), \ldots\}$.

If the system is initialized at $(\lambda \cdot x_0, i_0)$ and the same control law $i(t)$ is applied, then the set of states reached after the time intervals $\delta_0,\ \delta_0 + \delta_1,\ \ldots, \sum_{i=0}^{k} \delta_i, \ldots$, are respectively $\lambda \cdot x_1,\ \lambda \cdot x_2,\ \ldots, \lambda \cdot x_{k+1}, \ldots$, for any $\lambda \in \mathbb{R}$.

Proof: The validity of the statement trivially follows from the linearity of the systems dynamics. □

As a consequence of the above lemma, *local* stability properties imply *global* stability properties.

Corollary 1 A switched linear system that is asymptotically (resp., exponentially) stabilizable is also *globally* asymptotically (resp., exponentially) stabilizable.

Proof: Assume that a stabilizing control law exists such that the closed loop system is asymptotically (resp., exponentially) stabilizable when the initial state lies within a given neighborhood of the origin. By Lemma 1, an asymptotically (resp., exponentially) stabilizing control law also exists for any initial state. □

For sake of conciseness, in the rest of the paper we will deal with asymptotic or exponential stability, omitting the redundant qualification *global*.

The main goal of this paper is that of computing an appropriate switching law $i(t)$, when it does exist, such that the controlled system (1) is asymptotically stable.

Note that if at least one dynamics A_i is stable, then the system (1) is obviously exponentially stabilizable. Hence, we are interested in dealing with the case in which all dynamics are unstable.

3 The optimal control problem with a finite number of switches

The proposed stabilizing procedure is based on the solution of an optimal control problem of the following form:

$$\begin{aligned} V_N^*(x_0, i_0) \triangleq \min_{I,\mathcal{T}} F(I,\mathcal{T}) \triangleq & \int_0^\infty x'(t) Q_{i(t)} x(t) dt \\ \text{s.t.} \quad & \dot{x}(t) = A_{i(t)} x(t), \;\; x(0) = x_0, \quad i(0) = i_0 \\ & i(t) = i_k \text{ for } \tau_k \leq t < \tau_{k+1}, k = 0, \ldots, N \\ & \tau_0 = 0, \;\; \tau_{N+1} = +\infty \\ & i_k \in \mathcal{S}, \qquad\qquad\qquad k = 0, \ldots, N \end{aligned} \tag{3}$$

where:

- N, denoting the maximum number of allowed switches, is finite and fixed a priori;
- Q_i, $i \in \mathcal{S}$, are positive definite weighting matrices;
- x_0 is the initial continuous state, and i_0 is the initial mode, where x_0 and i_0 are both given;
- $\mathcal{T} \triangleq \{\tau_1, \ldots, \tau_N\}$ and $I \triangleq \{i_1, \ldots, i_N\}$ denote, respectively, the set of switching times and the sequence of indices associated with discrete modes.

In order to make the problem solvable with finite cost V_N^*, we assume the following:

Assumption 1 *There exists at least one index $i \in \mathcal{S}$ such that A_i is stable.* ■

In [25] we showed that the optimal control law for the optimization problem (3) takes the form of a state-feedback, i.e., it is only necessary to look at the current system state in order to determine if a switch from linear dynamics $A_{i_{k-1}}$ to A_{i_k}, should occur.

More precisely, for a given mode $i \in \mathcal{S}$ when k switches are still available, it is possible to construct a partition $\mathcal{C}_k^i$ of the state space $\mathbb{R}^n$ into s regions $\mathcal{R}_j$'s, $j = 1, \cdots, s = |\mathcal{S}|$. We call *table* the partition $\mathcal{C}_k^i$. Whenever $i_{N-k} = i$ we use table $\mathcal{C}_k^i$ to determine if a switch should occur: as soon as the continuous state x reaches a point in the region $\mathcal{R}_j$ for a certain $j \in \mathcal{S} \setminus \{i\}$ we will switch to mode $i_{N-k+1} = j$; no switch will occur if the continuous system's state x belongs to $\mathcal{R}_i$.

To prove this result, in [25] we showed constructively how the tables $\mathcal{C}_k^i$ can be computed off-line using a dynamic programming argument. We first showed how the tables $\mathcal{C}_1^i$ $(i \in \mathcal{S})$ for the last switch can be determined. Then, we shown by induction how the tables $\mathcal{C}_k^i$ can be computed once the tables $\mathcal{C}_{k-1}^i$ are known.

Remark 1 In order to provide a graphical representation of $\mathcal{C}_k^i$ we associate a different color to each dynamics A_j, $j \in \mathcal{S}$. The region $\mathcal{R}_j$ of $\mathcal{C}_k^i$ is represented according to the defined color mapping. ■

Note that regions $\mathcal{R}_j$'s are homogeneous, namely if $x \in \mathcal{R}_j$ then $\lambda x \in \mathcal{R}_j$ for all $\lambda \in \mathbb{R}$. This implies that they can be computed by simply looking at the unit semisphere. A term that has also been used in the literature to define the special form of these regions is *conic*.

Now, let $y \in \mathbb{R}^n$ be a generic vector on the unit semisphere and let $\mathcal{D}$ be an appropriate set of points in the unit semisphere, that define the considered state space discretization.

The procedure to compute the switching regions is briefly summarized in the following algorithm. For a complete derivation of it, as well as for a detailed description of an efficient and systematic procedure to define the set $\mathcal{D}$, we refer the reader to [25].

Algorithm 1 (Tables construction)
Input: $A_i \in \mathbb{R}^{n \times n}$, $Q_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{S}$, N, $t_{\max}$, $\mathcal{D}$.
Output: $\mathcal{C}_k^i(y)$, $k = 0, 1, \ldots, N$, $i \in \mathcal{S}$, $y \in \mathcal{D}$.
Notation: $\bar{A}_i(t) = e^{A_i t}$, $\bar{Q}_i(t) = \int_0^t \bar{A}_i(\tau)' Q_i \bar{A}_i(\tau) d\tau$, $Z_i = \lim_{t \to \infty} \bar{Q}_i(t)$[1].

1. *Initialization:* $k = 0$ *remaining switches*

   for $i = 1 : s$

   for all $y \in \mathcal{D}$

   $$\text{Cost assignment: } T_0(i, y) = \begin{cases} y' Z_i y & \text{if } A_i \text{ is stable} \\ +\infty & \text{otherwise} \end{cases}$$

   end (y)

   end (i)

2. *for* $k = 1 : N$

   for $i = 1 : s$

   for all $y \in \mathcal{D}$

   Computation of the remaining cost:

   for $t = 0 : \Delta t : t_{\max}$

   $z = \bar{A}_i(t) y$, $\lambda = ||z||$

[1]Note that if dynamics A_i is stable, Z_i is the solution of the Lyapunov equation $A_i' Z_i + Z_i A_i = -Q_i$.

for $j \in \mathcal{S}$

$$T(i, y, j, t) = y'\bar{Q}_i(t)y + \lambda^2 T_{k-1}(j, z/\lambda)$$

end (j)

end (t)

Cost assignment: $T_k(i, y) = \min_{j,t} T(i, y, j, t)$

Color assignment: $(j^*, t^*) = \arg\min_{j,t} T(i, y, j, t), \quad \mathcal{C}_k^i = \begin{cases} j^* & \textit{if } t^* = 0 \\ i & \textit{otherwise} \end{cases}$

end (y)

end (i)

end (k)

In simple words, at Step 1 of Algorithm 1 we compute for all $y \in \mathcal{D}$ the costs $T_0(i, y)$ associated to the evolution that starts from y and remains in dynamics A_i indefinitely, with $i = 1, \ldots, s$. Obviously, if A_i is not stable, then such a cost is infinite. Otherwise it is equal to $y'Z_iy$ where Z_i is the solution of the Lyapunov equation $A_i'Z_i + Z_iA_i = -Q_i$.

At Step 2 we start computing regions $\mathcal{C}_1^i$, $i = 1, \ldots, s$, that correspond to only one available switch. Then, the other regions are computed for increasing number of available switches, namely $\mathcal{C}_2^i$, $\mathcal{C}_3^i$ and so on, until $\mathcal{C}_N^i$, for all $i = 1, \ldots, s$.

This recursive procedure requires computing for all $i = 1, \ldots, s$, and for all $y \in \mathcal{D}$ the cost $T(i, y, j, t)$ of the evolution that starts from y with dynamics A_i, switches after a time interval of length t to dynamics A_j, and then evolves according to the optimal evolution, depending on the number of switches still available. Such a cost is the sum of two terms. The first term $y'\bar{Q}_i(t)y$ is equal to the cost of evolving with dynamics A_i for a time interval of length t. The second term $\lambda^2 T_{k-1}(j, z/\lambda)$ is equal to the cost of the optimal evolution starting from $z = e^{A_i t}y$, i.e., from the point reached when the commutation to A_j occurs and $k - 1$ switches are still available. Note that the second term is known (by a dynamic programming argument) because the algorithm keeps memory of the values of the optimal costs starting from any dynamics A_j and any point in $\mathcal{D}$. Such an information is sufficient since regions are conic. Indeed, if $T_{k-1}(j, x)$ is the optimal cost of the evolution starting from a point x on the unitary semisphere with dynamics A_j when $k - 1$ switches are still available, then $\lambda^2 T_{k-1}(j, x)$ is equal to the optimal cost of the evolution starting from $z = \lambda x$ with dynamics A_j when $k - 1$ switches are still available.

In the last part of Step 2 colors are assigned. If the current dynamics is A_i and the current state is y, then the color associated with y is equal to i if it is better to still evolve with dynamics A_i. Otherwise the color corresponding to the dynamics $j^* \neq i$ to which it is better to switch is assigned to y.

The value of $t_{\max}$ is chosen very large to approximate the infinite time horizon[2]. The value

[2]Note that the use of a *finite* time horizon $t_{\max}$ does not affect the validity of the proposed approach. In fact,

of Δt is chosen quite small to reduce the effects of time-discretization on the optimality of the solution.

The computational cost of the proposed approach is $(r^{n-1}Ns^2) \cdot c$ where c is the number of operations required to find the optimal value of $t^* \in [0, t_{\max}]$ that minimizes the remaining cost; n is the dimension of the state space; r is the number of samples in each direction (i.e., r^{n-1} is the cardinality of $\mathcal{D}$). Therefore, the complexity is a quadratic function of the number of possible dynamics and depends on the value of $t_{\max}$ that influences c.

For a more detailed discussion of the computational complexity of the approach we address the reader to [25].

4 The optimal control problem with an infinite number of switches

In this section we discuss how, under appropriate assumptions, the above procedure can be extended to the case of $N = \infty$. In particular, we consider an optimal control problem of the form (3) where

(i) for at least one $i \in \mathcal{S}$, A_i is stable;

(ii) for all $i \in \mathcal{S}$, $Q_i > 0$.

Let us preliminary state a monotonicity result.

Property 1 Let $N, N' \in \mathbb{N}$. If $N < N'$, then for any continuous initial state x_0, and for all $i, j \in \mathcal{S}$,

$$+\infty > V_N^*(x_0, i) \geq V_{N'}^*(x_0, j).$$

Proof: We first observe that by assumption (i) $V_N^*(x_0, i)$ is finite for any $N \geq 1$. In fact, regardless of the value of the initial mode i, we can always switch to the stable dynamics whose cost to infinity is finite. Now, we prove the second inequality by contradiction. Assume that $\exists\ j \in \mathcal{S}$ such that $V_{N'}^*(x_0, j) > V_N^*(x_0, i)$. Then it is obvious that the same evolution that generates $V_N^*(x_0, i)$ is also admissible for (3) starting from (x_0, j) when a larger value N' of switches is allowed (it is sufficient to switch immediately from mode j to mode i). This leads to a contradiction. □

Proposition 1 Given a continuous initial state x_0, for any $\varepsilon' > 0$, there exists $\bar{N} = \bar{N}(\varepsilon', x_0) \in \mathbb{N}$ such that for all $N > \bar{N}$, $V_{\bar{N}}^*(x_0, i) - {V_N}^*(x_0, j) < \varepsilon'$ for all $i, j \in \mathcal{S}$.

Proof: By definition $V_N^*(x_0, i) \geq 0$ for all $i \in \mathcal{S}$, hence V_N^* is a lower bounded non-increasing sequence (by Property 1). By the Axiom of Completeness it converges in $\mathbb{R}$, hence it is a Cauchy sequence. □

even in the presence of unstable dynamics a simple rule may be used to determine a "non restrictive" value of $t_{\max}$ [25].

In the previous proposition $\bar{N}$ was a function of x_0. To eliminate this dependency we consider the relative error.

Proposition 2 For any $\varepsilon > 0$ there exists $\bar{N} = \bar{N}(\varepsilon) \in \mathbb{N}$ such that for all $N > \bar{N}$, for all $i, j \in \mathcal{S}$ and for any continuous initial state x_0, $x_0 \neq 0$

$$\frac{V^*_{\bar{N}}(x_0, i) - V_N{}^*(x_0, j)}{V^*_N(x_0, i)} < \varepsilon.$$

Proof: We first observe that by assumption (ii) $V^*_N(x_0, i)$ is lower bounded by a strictly positive number. Moreover, the optimal costs are quadratic functions of x_0, i.e., if $x_0 = \lambda y_0$, then $V^*_N(\lambda y_0, i) = \lambda^2 V^*_N(y_0, i)$. Finally, by Proposition 1 $\forall$ y_0 and $\forall$ $\varepsilon' > 0$, $\exists$ $\bar{N}(y_0)$ such that $\forall$ $N > \bar{N}(y_0)$, $V^*_{\bar{N}}(y_0, i) - V_N{}^*(y_0, j) < \varepsilon'$. Hence if we define

$$\bar{N} = \max_{y_0 \,:\, ||y_0||=1} \bar{N}(y_0) \quad \Rightarrow$$

$$\frac{V^*_{\bar{N}}(x_0, i) - V_N{}^*(x_0, j)}{V^*_N(x_0, i)} = \frac{\lambda^2 [V^*_{\bar{N}}(y_0, i) - V_N{}^*(y_0, j)]}{\lambda^2 V^*_N(y_0, i)} \leq \frac{\varepsilon'}{\min\limits_{y_0 \,:||y_0||=1} V^*_N(y_0, i)} = \varepsilon.$$

□

Thus, one may use a given fixed relative tolerance ε to approximate two cost values, i.e.,

$$\frac{V^*_N(x, i) - V_{N'}{}^*(x, j)}{V^*_N(x, i)} < \varepsilon \implies V^*_N(x, i) \cong V^*_{N'}(x, j).$$

We can now prove the main result of this section.

Proposition 3 Given a fixed relative tolerance ε, if $\bar{N} = \bar{N}(\varepsilon)$ is chosen as in Proposition 2 then for all $N > \bar{N} + 1$ it holds that $\mathcal{C}^i_N = \mathcal{C}^i_{\bar{N}+1}$.

Proof: By definition (see also [25])

$$V^*_N(x_0, i) = \min_{j \in \mathcal{S}} \min_{\varrho \geq 0} \left\{ x'_0 \bar{Q}_i(\varrho) x_0 + V^*_{N-1}(x(\varrho), j) \right\}$$

where $x(\varrho) = e^{A_i \varrho} x_0$ and $\bar{Q}_i(\varrho) = \displaystyle\int_0^{\varrho} e^{A'_i t} Q_i e^{A_i t} dt$. Now, being by assumption $N - 1 > \bar{N}$, by Proposition 2 we may approximate

$$V^*_{N-1}(x(\varrho), j) \cong V^*_{\bar{N}}(x(\varrho), j) \quad \Rightarrow$$

$$\begin{aligned} V^*_N(x_0, i) &= \min_{j \in \mathcal{S}} \min_{\varrho \geq 0} \left\{ x'_0 \bar{Q}_i(\varrho) x_0 + V^*_{N-1}(x(\varrho), j) \right\} \\ &\cong \min_{j \in \mathcal{S}} \min_{\varrho \geq 0} \left\{ x'_0 \bar{Q}_i(\varrho) x_0 + V^*_{\bar{N}}(x(\varrho), j) \right\} = V^*_{\bar{N}+1}(x_0, i). \end{aligned}$$

Therefore, *by virtue of the above equations*, the optimal arguments (ϱ^*, j^*) used to compute $\mathcal{C}^i_N$ and $\mathcal{C}^i_{\bar{N}+1}$ are the same. □

The above result allows one to compute with a finite procedure the optimal tables for a switching law when N goes to infinity. In such a case, in fact, it holds that for all $i \in \mathcal{S}$, $\mathcal{C}^i_\infty = \lim_{N\to\infty} \mathcal{C}^i_N = \mathcal{C}^i_{\bar{N}+1}$.

Proposition 4 Given a fixed relative tolerance ε, if $\bar{N} = \bar{N}(\varepsilon)$ is chosen as in Proposition 2 then for all $i, j \in \mathcal{S}$ it holds that $\mathcal{C}^i_{\bar{N}+1} = \mathcal{C}^j_{\bar{N}+1}$.

Proof: It trivially follows from the fact that, by Proposition 2, $V^*_{\bar{N}+1}(x_0, i) = V^*_{\bar{N}+1}(x_0, j)$ for all $i, j \in \mathcal{S}$, and from the uniqueness of the optimal tables as discussed in [25]. □

This result also allows one to conclude that for all $i \in \mathcal{S}$,

$$\mathcal{C}_\infty = \lim_{N\to\infty} \mathcal{C}^i_N,$$

i.e., *all tables converge to the same one.*

To construct the table $\mathcal{C}_\infty$ the value of $\bar{N}$ is needed. We do not provide so far any analytical way to determine $\bar{N}$, therefore our approach consists in constructing tables until a convergence criterion is met.

Table $\mathcal{C}_\infty$ can be used to compute the optimal feedback control law that solves an optimal control problem of the form (3) with $N = \infty$. More precisely, when an infinite number of switches is available, we only need to keep track of the table $\mathcal{C}_\infty$. If the current continuous state is x and the current dynamics is A_i, on the basis of the knowledge of the color of $\mathcal{C}_\infty$ in x, we decide if it is better to still evolve with the current dynamics A_i or switch to a different dynamics, that is uniquely determined by the color of the table in x.

5 Stabilizability of unstable switched systems

In this section we deal with the problem of stabilizing a switched system (1) whose linear dynamics A_i are not stable. In particular, we show that a solution to this problem — when it does exist — can be obtained by solving an optimal control problem of the form (3) with $N = \infty$. The idea is that of applying the switching table procedure to a “dummy” problem that satifies the assumption that at least one dynamics A_i is asymptotically stable. We first present the following preliminary result.

Proposition 5 Let us consider an optimal control problem (OP) of the form (3) with $N = \infty$, and whose possible dynamics are A_i, $i \in \mathcal{S}$, and the corresponding weighting matrices are Q_i, $i \in \mathcal{S}$. If the table $\mathcal{C}_\infty$ only contains colors associated to a subset of indices $\mathcal{S}' \subset \mathcal{S}$, then $\forall x_0 \in \mathbb{R}^n$, the optimal control law that results by solving (OP) is also optimal for the optimal control problem (OP’) of the same form (3) with $N = \infty$, and whose possible dynamics are A_i, $i \in \mathcal{S}'$ and the corresponding weighting matrices are Q_i, $i \in \mathcal{S}'$.

Proof: The validity of the statement follows from the definition of the table $\mathcal{C}_\infty$ and the possibility of using it to derive an optimal feedback control law for (OP). If a color corresponding to a certain dynamics A_j does not appear in $\mathcal{C}_\infty$, this means that it is never convenient to switch to dynamics A_j, or to evolve with A_j if it is the initial dynamics, regardless of the current continuous state and the current mode. $\square$

The above result suggests one way to compute a stabilizing switching law for switched systems whose dynamics are unstable.

Definition 2 Let us consider an optimal control problem of the form (3) with $N = \infty$. Assume that all dynamics A_i, $i \in \mathcal{S}$, are not stable and the corresponding weighting matrices Q_i, $i \in \mathcal{S}$, are positive definite.

Let $\bar{A} \in \mathbb{R}^{n \times n}$ be an Hurwitz matrix and $\bar{Q} \in \mathbb{R}^{n \times n}$ be a positive definite matrix. Let $A_{s+1} = \bar{A}$ and $Q_{s+1} = q \cdot \bar{Q}$ where $q \in \mathbb{R}^+$.

We define *augmented optimal control problem* an optimal control problem of the form (3) with $N = \infty$, and whose possible dynamics are A_i, $i \in \bar{\mathcal{S}}$, with $\bar{\mathcal{S}} = \mathcal{S} \cup \{s+1\}$, and the weighting matrices are Q_i, $i \in \bar{\mathcal{S}}$. $\blacksquare$

We now prove an obvious monotonicity result for the augmented optimal control problem, namely that increasing the weight of the dummy dynamics $\bar{A}$ also increases the optimal cost of any evolution that uses $\bar{A}$.

Proposition 6 Let us consider an optimal control problem (OP) of the form (3) with $N = \infty$. Assume that all possible dynamics A_i, $i \in \mathcal{S}$, are unstable and the corresponding weighting matrices Q_i, $i \in \mathcal{S}$, are positive definite. Let $V_\infty^*(x_0)$ be the optimal value of the cost of (OP) when the initial continuous state is x_0.

Let us consider an augmented optimal control problem ($\overline{\text{OP}}$) with $A_{s+1} = \bar{A}$ and $Q_{s+1} = q \cdot \bar{Q}$ where $q \in \mathbb{R}^+$ and $\bar{Q} > 0$. Let $\overline{V}_\infty^*(x_0, q)$ be the value of the cost of ($\overline{\text{OP}}$) when the initial continuous state is x_0 and the optimal evolution is based on tables computed using Algorithm 1.

The cost $\overline{V}_\infty^*(x_0, q)$ is a strictly increasing function of q for all values of q such that the stable dynamics A_{s+1} appears in the optimal evolution of the augmented optimal control problem.

Proof: We prove this by contradiction. Let us consider two different augmented optimal control problems ($\overline{\text{OP}}$') and ($\overline{\text{OP}}$") that differ for their value of q (while $\bar{Q}$ is the same). In particular, let q' and q'' be the values of the coefficient q associated to ($\overline{\text{OP}}$') and ($\overline{\text{OP}}$") respectively, and let $q' > q''$. Assume that $\overline{V}_\infty^*(x_0, q') = \overline{V}_\infty^*(x_0, q'')$. If we consider the evolution that is optimal for ($\overline{\text{OP}}$') and evaluate the cost using the weights of ($\overline{\text{OP}}$"), we find out that the resulting value of the cost is less than $\overline{V}_\infty^*(x_0, q'')$, that leads to a contradiction. $\square$

Now, we prove the main result.

Theorem 1 Given a switched system (1), let us consider an optimal control problem of the form (3) with $N = \infty$ and weighting matrices $Q_i > 0$, $i \in \mathcal{S}$. Then, let us define an augmented optimal control problem with stable dynamics $A_{s+1} = \bar{A}$ and corresponding weighting matrix $Q_{s+1} = \bar{q} \cdot \bar{Q}$, where $\bar{Q} > 0$ and $\bar{q} \in \mathbb{R}^+$. Let $\bar{\mathcal{S}} = \mathcal{S} \cup \{s+1\}$.

(i) The switched system (1) is *exponentially stabilizable* $\Longrightarrow \exists\ \bar{q} \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$, computed using Algorithm 1 and solving the augmented optimal control problem, does not contain the color associated to $\bar{A}$.

(ii) The switched system (1) is *asymptotically stabilizable* $\Longleftarrow \exists\ \bar{q} \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$, computed using Algorithm 1 and solving the augmented optimal control problem, does not contain the color associated to $\bar{A}$.

Proof: Let $\overline{V}^*_\infty(x_0, q)$ be the cost of the augmented optimal control problem when the multiplicative weight of matrix $\bar{Q}$ is equal to q, the initial continuous state is x_0, and the evolution is based on tables computed using Algorithm 1.

(i) By Proposition 6 we know that $\overline{V}^*_\infty(x_0, q)$ is an increasing function of q for all values of q such that A_{s+1} appears in the optimal evolution. Moreover, since regions are computed using Algorithm 1, then each dynamics whose color appears in $\mathcal{C}_\infty$ should be used for a time interval whose length is greater than or equal to Δt (see Algorithm 1, Step 2, Computation of the remaining cost). Therefore, given a value of Δt used to implement Algorithm 1, if $q \to \infty$, then the cost resulting from using dynamics A_{s+1} goes to infinity as well.

Now, if the original system is exponentially stabilizable, it means that a control law that is exponentially stable exists to which it corresponds a finite cost. This implies that for all initial states x_0, if increasing values of q are associated to the dummy dynamics in the augmented optimal control problem, a value of q, let's say $q'(x_0)$, is reached such that $\mathcal{C}_\infty$ does not contain the dummy dynamics.

The result holds if we let

$$\bar{q} = \max_{x_0 \in \mathbb{R}^n} q'(x_0) = \max_{||y_0||=1} q'(y_0),$$

where the second equality follows from the fact that $\overline{V}^*_\infty(x_0, q)$ is a quadratic function of x_0, i.e., if $x_0 = \lambda y_0$ then $\overline{V}^*_\infty(\lambda y_0, q) = \lambda^2 \overline{V}^*_\infty(y_0, q)$.

(ii) Now, to prove that the switched system (1) is asymptotically stabilizable, we introduce a Lyapunov-like function $V(x) = V^*_\infty(x)$, i.e., the value of this function coincides with the optimal cost of an evolution starting from x, when an infinite number of switches is allowed.

We first prove that there exist two functions $V_{\min}(x)$ and $V_{\max}(x)$ such that $V_{\min}(x) \leq V(x) \leq V_{\max}(x) < +\infty$.

- $(V(x) \leq V_{\max}(x) < +\infty)$. By assumption $\exists\ \bar{q}$ such that the switching table $\mathcal{C}_\infty$, computed applying the switching table procedure to the augmented optimal control problem with $Q_{s+1} = \bar{q} \cdot \bar{Q}$, does not contain the color associated to the stable $A_{s+1} = \bar{A}$.

  By Proposition 5 the control law that results using table $\mathcal{C}_\infty$ is also optimal for the optimal control problem with unstable dynamics A_i's and weighting matrices Q_i's, with $i \in \mathcal{S}$. Therefore, if we define $V_{\max}(x) = x^T Z_{s+1} x$ where Z_{s+1} is the solution of the Lyapunov equation $A^T_{s+1} Z_{s+1} + Z_{s+1} A_{s+1} = -Q_{s+1}$, we have $\overline{V}^*_\infty(x, \bar{q}) \leq V_{\max}(x) < +\infty$ for all

$x \in \mathbb{R}^n$. Moreover, being $V^*_\infty(x) \le \overline{V}^*_\infty(x,\bar{q})$ for all $x \in \mathbb{R}^n$, it follows that $V^*_\infty(x) \le V_{\max}(x) < +\infty$.

- $(V(x) \ge V_{\min}(x))$. Let us define

$$V_{\min}(x) \triangleq \frac{\lambda_Q}{2\sigma}||x||^2, \qquad \lambda_Q = \min_{i\in\mathcal{S}} \min \lambda\{Q_i\} > 0$$

  is the minimum among the smallest eigenvalues of the weighting matrices Q_i's, and

$$\sigma = \max_{i\in\mathcal{S}} \sigma_i, \qquad \sigma_i = ||A_i|| \text{ is the largest singular value of } A_i,$$

  is the maximum among the largest singular values of A_i's. By definition

$$V(x) = \int_0^\infty x^T_{op}(t) Q_{i_{op}(t)} x_{op}(t) dt$$

  where $x_{op}(t)$ and $i_{op}(t)$ denote an optimal evolution starting at x.

  Let us rewrite $x_{op}(t) = y_{op}(t)||x_{op}(t)||$, thus[3]

$$\begin{aligned} V(x) &= \int_0^\infty y^T_{op}(t) Q_{i_{op}(t)} y_{op}(t)||x_{op}(t)||^2 dt \ge \int_0^\infty y^T_{op}(t)\lambda_Q y_{op}(t)||x_{op}(t)||^2 dt \\ &= \lambda_Q \int_0^\infty ||x_{op}(t)||^2 dt \ge \lambda_Q \int_0^\infty e^{-2\sigma t}||x||^2 dt \ge \frac{\lambda_Q}{2\sigma}||x||^2 \triangleq V_{\min}(x). \end{aligned}$$

Now, we prove that the switched system (1) optimally controlled is stable because given an arbitrary $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $||x(t_0)|| \le \varepsilon$ for all $t \ge t_0$ if $||x(t_0)|| \le \delta$.

- (Stability of the optimally controlled system). Let C be the maximal value for which the curve $V_{\min}(x) = C$ is all contained within the closed ball of radius ε. Then, choose δ as the minimal value of $||x||$ for x belonging to the curve $V_{\max}(x) = C$. We prove that any optimal evolution $x_{op}(t)$ that starts in $x_{op}(t_0) = x_0$ within the closed ball of radius δ (this initial state is such that $V_{\max}(x_0) \le C$) remains in the closed ball of radius ε. Assume, by contradiction that for $t \ge t_0$ we have $x_{op}(t) = x'$, with $||x'|| > \varepsilon$. Then, $V(x') \ge V_{\min}(x') > C \ge V_{\max}(x_0) \ge V(x_0)$, thus contradicting the monotonicity property[4] that states that the cost must decrease along any optimal trajectory.

Finally, the *asymptotic* stabilizability of the switched system trivially follows from the stability of the optimally controlled system and the fact that $V^*(x) < +\infty$, hence by virtue of the item above $||x|| \to 0$. □

[3] In [11] we showed that given a dynamical system $\dot{x}(t) = Ax(t)$, with initial condition $x(0) = x_0$, for all $t \ge 0$ holds $||x(t)|| = ||e^{At}x_0|| \ge e^{-\sigma t}||x_0||$, where $\sigma = ||A||$ is the largest singular value of matrix A.

[4] The monotonicity property trivially follows from the fact that the cost is the integral of a positive definite function.

We are aware of the small gap in the above result: a switched system may be asymptotically (but not exponentially) stabilizable but if no finite-cost optimal control law exists, we cannot compute a stabilizing law. The results of Hespanha [13] and Sun [28] seem to imply that for the class of systems we consider exponential stability and asymptotic stability coincide. If such is the case, Theorem 1 can be more succinctly restated as a necessary and sufficient condition for stabilizability. However this is still an open issue.

The above theorem provides a systematic way to deal with the problem of determining an asymptotic stabilizing switching law for switched system (1) with linear unstable dynamics, that can be summarized as follows.

— We associate to the switched system to stabilize an optimal control problem of the form (3) with $N = \infty$.

— We define an augmented optimal control problem with a Hurwitz matrix $A_{s+1} = \bar{A}$ and weighting matrix $Q_{s+1} = q \cdot \bar{Q}$, where $\bar{Q}$ is any positive definite matrix and q is a very large positive real number.

— We construct the switching table $\mathcal{C}_\infty$ solving the augmented optimal control problem.

— If this table does not contain the color associated to the stable dynamics A_{s+1}, by Theorem 1, item (ii), we may conclude that the switched system (1) is asymptotically stabilizable. In such a case, we compute the stabilizing feedback control law that minimizes the chosen quadratic performance index using table $\mathcal{C}_\infty$.

We do not provide an a priori rule to establish if the switched system is stabilizable and in such a case, an analytical way to compute an appropriate value of q. Nevertheless in all numerical examples taken from the literature, we found out that if the system is stabilizable it was sufficient to use a large value of q ($10^{10} \div 10^{20}$) and to take

$$\bar{Q} = I_n \cdot \max_{i \in \mathcal{S}} \max_{r,c=1,\dots,n} Q_i(r,c)$$

(where I_n denotes the n-th order identity matrix) to compute stabilizing laws .

6 A numerical example

Let us consider a variant of a very well-known switched system [4] (1), with $s = 3$ and

$$A_1 = \begin{bmatrix} 1 & -10 \\ 100 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 39.97 & -77.5 \\ 32.5 & -37.97 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -37.97 & -77.50 \\ 32.50 & 39.97 \end{bmatrix}.$$

Note that dynamics A_2 and A_3 are obtained from dynamics A_1 by an axis rotation of 120 and 240 degrees respectively. All dynamics A_i's are unstable.

To determine a stabilizing switching law we first associate to the switched system (1) an optimal control problem of the form (3) with $N = \infty$. In particular, we take $Q_i = I_2$, $i = 1, 2, 3$, where I_2 denotes the second order identity matrix.

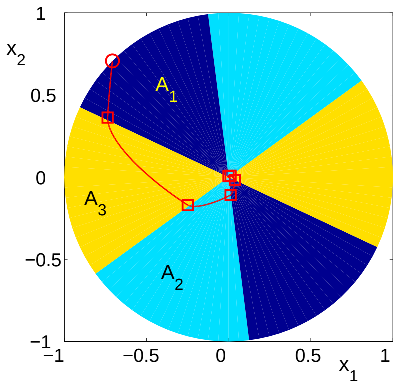

Figure 1: Example studied in Sction 6: table $\mathcal{C}_\infty$ and an optimal trajectory.

We define an augmented optimal control problem with the stable dynamics $A_4 = -A_1$ and weighting matrix $Q_4 = \bar{q} \cdot \bar{Q}$, where $\bar{q} = 10^5$ and $\bar{Q} = I_2$.

We construct the table $\mathcal{C}_\infty$. More precisely, we apply the procedure to construct the tables $\mathcal{C}_N^i$ for finite values of N and we find out that, for a sufficiently large value of N, namely $N = 15$, the tables converge to the same one. Table $\mathcal{C}_\infty$ is reported in Figure 1.

We can immediately observe that the color associated to the stable dynamics A_4 never appears. This means that, regardless of the initial state, the optimal trajectory of the augmented optimal control problem is obtained by infinitely switching among unstable dynamics A_i, $i = 1, 2, 3$.

This allows one to conclude that the switched system (1) with $i \in \{1, 2, 3\}$ is globally asymptotically stabilizable. Moreover, the table $\mathcal{C}_\infty$ can be used to compute the stabilizing feedback control law that minimizes the chosen quadratic performance index.

An example of an optimal trajectory is reported in Figure 1 when the initial state is $x_0 = [-1 \quad 1]^T/\sqrt{2}$, $i_0 = 1$. The set of optimal switching times $\mathcal{T}^*$, the set of optimal switching sequence $\mathcal{I}^*$ and the optimal cost $V_\infty^*(x_0)$ are:

$$\mathcal{T}^* = 10^{-2} \cdot \{0.48, 3.84, 3.78, 3.42, 3.72, 3.48, 3.18, \ldots\}, \qquad \mathcal{I}^* = \{1, 3, 2, 1, 3, 2, 1, \ldots\},$$

and $V_\infty^*(x_0) = 0.0208$. Note that the system, because of the homogeneous regions, presents a periodic behaviour.

Note that here we presented a numerical second order example because it enables us to visualize graphically the switching regions. Netherless, this does not mean that the control procedure can only be applied to second order systems. Examples of fourth order systems have been extensively presented in [6].

7 Conclusions and future work

In this paper we extended our previous results on the optimal control of switched systems with a finite number of admissible switches and at least one stable dynamics. More precisely, we first showed that a feedback control law that minimizes a given quadratic cost can also be computed when the number of allowed switches goes to infinity and only one dynamics is stable. Then, we showed that this approach can also be efficiently applied when all LTI dynamics are not stable, by simply solving an appropriate optimal control law, called the *augmented optimal control problem* that contains a "dummy" stable dynamics. In particular, we showed that if the switched system with unstable dynamics is globally exponentially stabilizable, then an optimal feedback control law can be computed, that guarantees the closed-loop system to be globally asymptotically stable.

Two interesting problems are still open in our approach. We have been able to prove that the state space partitions $\mathcal{C}_{i,k}$ computed under the assumption that a finite number of switches N is allowed, all converge to the same state space partition $\mathcal{C}_\infty$ provided that $N \geq \bar{N}$. However we do not provide a rule to determine the value of $\bar{N}$, neither an upper bound on it. Nevertheless from a practical point of view this is not a limitation: our procedure can be recursively applied off-line for increasing values of N until convergence is met.

The second open problem is that of providing a systematic criterion to determine the weighting matrix that should be associated to the dummy stable dynamics $s + 1$ in the optimal control problem. Only a heuristic rule is given, that in all numerical cases we examined provided good results.

References

[1] H. Axelsson, Y. Wardi, M. Egerstedt, and E.I. Verriest. Gradient descent approach to optimal mode scheduling in hybrid dynamical systems. *Journal of optimization theory and applications*, 136(2):167–1864, 2008.

[2] A. Bhaya and F. Mota. Equivalence of stability concepts for discrete time-varying systems. *International Journal of Robust and Nonlinear Control, year=1994, volume=4, pages=725-740,*.

[3] M. Boccadoro, M. Egerstedt, and Y. Wardi. Optmial control of switching surfaces in hybrid dynamic systems. In *Proc. 7th Int. Workshop on Discrete Event Systems*, pages 127–133, 2004.

[4] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automatic Control*, 43(4):475–482, 1998.

[5] D. Cheng, L. Guo, Y. Lin, and Y. Wang. Stabilization of switched linear systems. *IEEE Trans. Automatic Control*, 50(5):661–666, 2005.

[6] D. Corona. *Optimal Control of Affine Hybrid Automata.* PhD thesis, Cagliari, Italy, 2005.

[7] D. Corona, A. Giua, and C. Seatzu. Stabilization of switched systems via optimal control. In *Proc. 16th IFAC World Congress*, Prague, Czech Republic, 2005.

[8] J. Daafouz, R. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched-lyapunov function approach. *IEEE Trans. Automatic Control*, 47(11):1883–1887, 2002.

[9] R. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2000.

[10] M. Egerstedt, Y. Wardi, and F. Delmotte. Optimal control of switching times in swithced dynamical systems. In *Proc. 44th IEEE Conf. on Decision and Control and European Control Conference ECC 2005*, pages 2138–2143, Maui, Hawaii, USA, Dec 2003.

[11] A. Giua, C. Seatzu, and C. Van Der Mee. Optimal control of switched autonomous linear systems. In *Proc. 40th IEEE Conf. on Decision and Control*, pages 2472–2477, Orlando, Florida, USA, Dec 2001.

[12] S. Hedlund and A. Rantzer. Convex dynamic programming for hybrid systems. *IEEE Trans. Automatic Control*, 47(9):1536–1540, September 2002.

[13] J.P. Hespanha. Uniform stability of switched linear systems: extensions of LaSalle's invariance principle. *Automatica*, 49(4):470–482, 2004.

[14] B. Hu, X. Xu, P.J. Antsaklis, and A.N. Michel. Robust stabilizing control law for a class of second-order switched systems. *Syst. Control Letters*, 38:197–207, 1999.

[15] T. Laffey and H. Smigoc. Tensor conditions for the existence of a common solution to the Lyapunov equation. *Linear Algebra Applications*, 420:672–685, 2007.

[16] D. Liberzon. *Switching in Systems and Control.* Boston, MA: Birkhauser, 2003.

[17] D. Liberzon, J.P. Hespanha, and A.S. Morse. Stability of switched linear systems: a Lie-algebraic condition. *Systems and Control Letters*, 37, 1999.

[18] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

[19] D. Liberzon and R. Tempo. Gradient algorithms for finding common Lyapunov functions. In *Proc. 44th IEEE Conf. on Decision and Control and European Control Conference ECC 2005*, pages 4782–4787, Hawaii, USA, December 2003.

[20] D. Liberzon and R. Tempo. Common lyapunov functions and gradient algorithms. *IEEE Trans. Automatic Control*, 49(6):990–994, 2004.

[21] H. Lin and P.J. Antsaklis. Switching stabilizability for continuous-time uncertain switched linear systems. *IEEE Trans. Automatic Control*, 52(4):633–646, 2007.

[22] H. Lin and P.J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Trans. Automatic Control*, 54(2):308–322, 2009.

[23] M. Margaliot. Stability analysis of switched systems using variational principles: An introduction. *Automatica*, 42(12):2059–2077, 2006.

[24] M. Margaliot and D. Liberzon. Lie-algebraic stability conditions for nonlinear switched systems and differential inclusions. *Systems and Control Letters*, 55(1):8–16, 2006.

[25] C. Seatzu, D. Corona, A. Giua, and A. Bemporad. Optimal control of continuous-time switched affine systems. *IEEE Trans. Automatic Control*, 51(5):726–741, 2006.

[26] M.S. Shaikh and P.E. Caines. Optimality zone algorithms for hybrid systems computation and control: from exponential to linear complexity. pages 1403–1408, Seville, Spain, December 2005.

[27] R. Shorten, K. Narendra, and O. Mason. A result on common quadratic lyapunov functions. *IEEE Trans. Automatic Control*, 48(1):110–113, 2003.

[28] Z. Sun. Combined stabilizing strategies for switched linear systems. *IEEE Trans. Automatic Control*, 51(4):666–674, 2006.

[29] M. Wicks, P. Peleties, and R. DeCarlo. Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems. *European Journal of Control*, 4:140–147, 1998.

[30] X. Xu and P.J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Trans. Automatic Control*, 49(1):2–16, 2004.

[31] G. Zhai, H. Lin, and P.J. Antsaklis. Quadratic stabilizability of switched linear systems with polytopic uncertainties. *Int. J. Control*, 76(7):747–753, 2003.