

# Leader-Follower Formation via Complex Laplacian <sup>\*</sup>

Zhiyun Lin, Wei Ding, Gangfeng Yan, Changbin Yu, Alessandro Giua<sup>†</sup>

## Abstract

The paper introduces complex-valued Laplacians for graphs whose edges are attributed with complex weights and studies the leader-follower formation problem based on complex Laplacians. The main goal is to control the shape of a planar formation of point agents in the plane using simple and linear interaction rules related to complex Laplacians. We present a characterization of complex Laplacians that preserve a specific planar formation as an equilibrium solution for both single integrator kinematics and double integrator dynamics. Planar formations under study are subject to translation, rotation, and scaling in the plane, but can be determined by two co-leaders in leader-follower networks. Furthermore, when a complex Laplacian does not result in an asymptotically stable behavior of the multi-agent system, we show that a stabilizing matrix, which updates the complex weights, exists to asymptotically stabilize the system while preserving the equilibrium formation. Also, algorithms are provided to find stabilizing matrices. Finally, simulations are presented to illustrate our results.

Published as:

Z. Lin, W. Ding, G. Yan, C. Yu, A. Giua, Leader-Follower Formation via Complex Laplacian, *Automatica*, Vol. 49, No. 6, pp. 1900-1906, June 2013. DOI: [10.1016/j.automatica.2013.02.055.1111](https://doi.org/10.1016/j.automatica.2013.02.055.1111).

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<sup>\*</sup>The work is in part supported by National Natural Science Foundation of China under Grant 61273113, Shandong Academy of Science Development Fund for Science and Technology, the Pilot Project for Science and Technology in Shandong Academy of Science, and Australian Research Council's DP130103610. C. Yu is also supported by an ARC Queen Elizabeth II Fellowship under DP110100583 and also by the Overseas Expert Program of Shandong Province.

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# 1 Introduction

Formation control is a fundamental research problem in networked multi-agent systems due to their civil (Leonard et al., 2007) and military applications (Murray, 2007). Roughly, three main approaches to formation control have been discussed in recent literature. The first approach describes a formation in terms of inter-agent distance measures (Eren et al., 2002; Yu et al., 2007; Hendrickx et al., 2007; Bai et al., 2008) and uses gradient descent control laws resulted from distance-based artificial potentials (Olfati-Saber and Murray, 2002a; Cao et al., 2008; Yu et al., 2009; Krick et al., 2009; Guo et al., 2010; Cao et al., 2011). The second approach describes a formation in terms of inter-agent bearing measures (Eren, 2007) and uses angle only control laws (Basiri et al., 2010; Guo et al., 2011). The third approach describes a formation in terms of inter-agent relative positions and uses consensus-based control laws with input bias (Lin et al., 2004; Lafferriere et al., 2005; Ren, 2007), which are related to real-valued Laplacians.

The majority of existing algorithms considers the formation representation in terms of inter-agent distance measures. Such an approach can be naturally applied to undirected graphs using the concept of graph rigidity (Olfati-Saber and Murray, 2002a; Eren et al., 2002; Krick et al., 2009), in which two neighboring agents work together to reach the specific distance between them. For directed graphs, a further concept termed persistence is introduced to characterize a planar formation (Hendrickx et al., 2007; Yu et al., 2007). Nevertheless, it is challenging to synthesize a control law and analyze the stability property for a group of agents with a directed interaction graph (Yu et al., 2009). Due to this reason, most works on directed formation are limited to directed acyclic graphs (Cao et al., 2008; Guo et al., 2010; Cao et al., 2011). Angle-based control for formations in terms of inter-agent bearing measuring is relatively new and has not been fully explored. In Basiri et al. (2010), only three agents are considered to achieve a triangle with angle-only constraints, but global asymptotic convergence results are established. Compared with formations in terms of inter-agent distance constraints and inter-agent angle constraints, formations in terms of relative positions require less links and it is more straightforward extending from undirected graphs to directed graphs. The consensus-based control laws with input biases (Lin et al., 2004; Lafferriere et al., 2005; Ren, 2007) are affine and thus could lead to global stability, but the approach has the drawback that all the agents should have a common sense of direction since the input biases are defined in a common reference frame.

In this paper, we introduce a new approach based on complex Laplacians to study the formation control problem in the plane. The inspiration is from Pavone and Frazzoli (2007); Ding et al. (2009); Ren (2009); Ding et al. (2010), where pursuit strategies with offset angles are investigated to exhibit more interesting collective motions. For a network of  $n$  interacting agents modeled as a directed graph, we represent a planar formation as an  $n$ -dimensional complex vector called the *formation basis* and introduce a complex Laplacian of the directed graph to characterize the planar formation. That is, the formation basis is another linearly independent eigenvector of the complex Laplacian associated with the zero eigenvalues in addition to the eigenvector of ones. In this way, a planar formation has four degrees of freedom subject to translation, rotation, and scaling. However, if two agents in the group can specify their locations, then the four degrees of freedom are taken and the planar formation is completely determined. For this purpose, we consider a leader-follower network with two co-leaders. On the other hand, the complex Laplacian corresponding to the sensing graph of networked agents naturally leads to a simple distributed control law, which is locally implementable without requiring a common reference frame. That is, for single integrator kinematics, the velocity control of each follower agent is the complex combination of the relative positions of its neighbors using the complex weights on the incoming edges. A complex weight multiplying the relative position of a neighbor actually means that the agent moves along the line of sight rotated by an offset angle with certain speed gain (the magnitude of the complex weight). This complex

Laplacian based control law can also be generalized for the case of double integrator dynamics, which is investigated in the paper as well.

However, unlike real Laplacians, a complex Laplacian might have eigenvalues in the left complex plane, a situation that would lead to the instability of the overall system. To the best of our knowledge there exists no known result on the direct design of a complex Laplacian such that all its eigenvalues have nonnegative real parts, and we propose in this paper an alternative but systematic design approach based on multiplicative inverse eigenvalue problems (MIEP) (Friedland, 1975; Chu, 1998). That is, we update the Laplacian and re-assign the eigenvalues by pre-multiplying a diagonal matrix, called a *stabilizing matrix*. Sufficient conditions for the existence of a stabilizing matrix are developed and algorithms are also provided to find it. Compared with the existing approaches for formation control, our complex Laplacian based approach exhibits the following advantages: the distributed control law is linear ensuring global stability; the system behavior is simple for analysis in both undirected and directed formations; the strategy does not require global information but intermediate relative position measurements.

*Notations:*  $\mathbb{C}$  and  $\mathbb{R}$  denote the set of complex and real numbers, respectively.  $\iota = \sqrt{-1}$  denotes the imaginary unit.  $\mathbf{1}_n$  represents the  $n$ -dimensional vector of ones and  $I_n$  denotes the identity matrix of order  $n$ .

## 2 Preliminaries

### 2.1 Graph theory

A *digraph*  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  consists of a non-empty node set  $\mathcal{V} = \{1, 2, \dots, n\}$  and an edge set  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ . We let  $\mathcal{N}_i$  denote the in-neighbor set of node  $i$ , i.e.,  $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$ , and let  $n_i$  denote the cardinality of  $\mathcal{N}_i$ . In the paper, we assume that a digraph does not have self-loops, which means  $i \notin \mathcal{N}_i$  for any node  $i$ .

For a digraph  $\mathcal{G}$ , we associate to each edge  $(j, i)$  a complex number  $w_{ij} \neq 0$ , called *complex weight*. Then we can define a complex-valued Laplacian  $L$ , for which the off-diagonal entry  $L(i, j) = -w_{ij}$  if  $j \in \mathcal{N}_i$  and 0 otherwise, and the diagonal entry  $L(i, i) = \sum_{j \in \mathcal{N}_i} w_{ij}$ .

### 2.2 Planar formation

A tuple of  $n$  complex numbers  $\xi = [\xi_1, \xi_2, \dots, \xi_n]^T$  is called a *formation basis* for  $n$  agents in the plane, which defines a geometric pattern in a specific coordinate system. Usually two agents are expected not to overlap each other, so we assume that  $\xi_i \neq \xi_j$  for  $i \neq j$  in the paper. A formation with four degrees of freedom (translation, rotation, and scaling) is defined by  $F_\xi = c_1 \mathbf{1}_n + c_2 \xi$ , where  $c_1, c_2 \in \mathbb{C}$ . When  $|c_2| = 1$ , the formation is obtained from the basis via only translations and rotations, a case which we are more familiar with.

Denote  $z = [z_1, \dots, z_n]^T \in \mathbb{C}^n$  the aggregate position vector of  $n$  agents. We say that the  $n$  agents *form a planar formation*  $F_\xi$  with respect to basis  $\xi$  if there exist complex constants  $c_1$  and  $c_2$  such that  $z = c_1 \mathbf{1}_n + c_2 \xi$ . The  $n$  agents are said to *asymptotically reach a planar formation*  $F_\xi$  if there exist complex constants  $c_1$  and  $c_2$  such that  $\lim_{t \rightarrow \infty} z(t) = c_1 \mathbf{1}_n + c_2 \xi$ .

### 3 Main results

We consider a group of  $n$  agents in the plane labeled  $1, \dots, n$ , consisting of leaders and followers. Suppose that there are two leaders in the group (without loss of generality, say 1 and 2) and all the others are followers. The positions of the  $n$  agents are denoted by complex numbers  $z_1, \dots, z_n \in \mathbb{C}$ . We use a digraph  $\mathcal{G}$  of  $n$  nodes to represent the sensing graph, in which  $\{1, 2\}$  are leader agents,  $\{3, \dots, n\}$  are follower agents, and an edge  $(j, i)$  indicates that agent  $i$  can measure the relative position of agent  $j$ , namely,  $(z_j - z_i)$ . Because in a leader-follower network, the leader agents do not interact with the follower agents and do not need to access the information from the follower agents, the sensing graph  $\mathcal{G}$  has the following property.

**(P1):** Leader nodes (1 and 2) do not have incoming edges.

Thus, the Laplacian of  $\mathcal{G}$  takes the following form.

$$L = \left[ \begin{array}{c|c} 0_{2 \times 2} & 0_{2 \times (n-2)} \\ \hline L_{lf} & L_{ff} \end{array} \right]. \quad (1)$$

#### 3.1 Single-integrator kinematics

Suppose that each agent is governed by a single-integrator kinematics

$$\dot{z}_i = v_i, \quad (2)$$

where  $z_i \in \mathbb{C}$  represents the position of agent  $i$  in the plane and  $v_i \in \mathbb{C}$  represents the velocity control input. Consider the sensing graph  $\mathcal{G}$  and suppose that the agents are driven by the following control laws.

$$\begin{aligned} v_i &= 0, & i &= 1, 2; \\ v_i &= \sum_{j \in \mathcal{N}_i} w_{ij}(z_j - z_i), & i &= 3, \dots, n, \end{aligned} \quad (3)$$

where the complex weights  $w_{ij} = k_{ij}e^{i\alpha_{ij}}$ , with  $k_{ij} > 0$  and  $\alpha_{ij} \in [-\pi, \pi)$ , are design parameters. Note that the choice of complex weights is not unique, and given a formation basis  $\xi \in \mathbb{C}^n$  satisfying  $\xi_i \neq \xi_j$ , for each agent  $i$  we may choose any set of complex weights  $w_{ij}$  that satisfy the linear equality  $\sum_{j \in \mathcal{N}_i} w_{ij}(\xi_j - \xi_i) = 0$ .

Let  $z = [z_1, z_2, \dots, z_n]^T \in \mathbb{C}^n$ . Then the overall dynamics of the  $n$  agents can be written as

$$\dot{z} = -Lz, \quad (4)$$

where  $L$  is the complex-valued Laplacian of  $\mathcal{G}$  defined in (1).

Denote  $\bar{z}_1 = z_1(0)$  and  $\bar{z}_2 = z_2(0)$ . Next we show a necessary and sufficient condition such that any equilibrium state of (4) forms a planar formation  $F_{\bar{\xi}}$ .

**Theorem 3.1** *Assume that  $\xi \in \mathbb{C}^n$  satisfies  $\xi_i \neq \xi_j$  for  $i \neq j$ . Then every equilibrium state of (4) forms*

a planar formation  $F_\xi = c_1 \mathbf{1}_n + c_2 \xi$  with

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} \quad (5)$$

if and only if  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ .

**Proof:** (Sufficiency) From the condition  $L\xi = 0$ , we know that  $L$  has a zero eigenvalue associated to eigenvector  $\xi$ . On the other hand, the Laplacian  $L$  always has a zero eigenvalue associated to eigenvector  $\mathbf{1}_n$ . The two eigenvectors  $\mathbf{1}_n$  and  $\xi$  are linearly independent because  $\xi_i \neq \xi_j$ . Moreover, it follows from the condition  $\det(L_{ff}) \neq 0$  that  $\text{rank}(L) = n - 2$  and  $L$  has only two zero eigenvalues. So the null space of  $L$  is  $\{c_1 \mathbf{1}_n + c_2 \xi : c_1, c_2 \in \mathbb{C}\}$  and thus every equilibrium state forms a planar formation  $F_\xi = c_1 \mathbf{1}_n + c_2 \xi$ . Notice that  $z_1(t) = \bar{z}_1$  and  $z_2(t) = \bar{z}_2$ . Therefore (5) follows.

(Necessity) Suppose on the contrary that  $L\xi \neq 0$ . Then  $L(c_1 \mathbf{1}_n + c_2 \xi) \neq 0$  for any nontrivial  $c_1$  and  $c_2$ , which means a state corresponding to a planar formation  $F_\xi$  cannot be an equilibrium state of (4). On the other hand, suppose on the contrary that  $\det(L_{ff}) = 0$ . Thus we could find a vector  $\eta_f \in \mathbb{C}^{(n-2)}$  such that  $L_{ff}\eta_f = 0$ . As a result  $\eta = [0 \ 0 \ \eta_f^T]^T$  is in the null space of  $L$ . It can be checked that  $\mathbf{1}_n$ ,  $\xi$  and  $\eta$  are linearly independent since  $\xi_i \neq \xi_j$  in  $\xi$ . Thus, the equilibrium state  $\eta$  does not correspond to any planar formation  $F_\xi$  generated from the formation basis  $\xi$ . ■

**Remark 3.1** From Theorem 3.1 it can be seen that the equilibrium formation of the  $n$  agents is uniquely determined by the two leaders' locations. If the two leader agents do not remain stationary but asymptotically converge to two different locations, then the limit positions of two co-leaders specify the planar formation  $F_\xi$ . Hence, by controlling the motions of two co-leaders, the group formation can be rotated, translated, and scaled.

Next we come to study whether the  $n$  agents can asymptotically reach a planar formation, i.e., we study the stability of system  $\dot{z} = -Lz$ . Before presenting our results on stability, we provide a result on the invariance property for the operation of pre-multiplying an invertible diagonal matrix  $D$ . It is an important property ensuring that the equilibrium formations are preserved.

**Theorem 3.2** Every equilibrium state of system (4) forms a planar formation  $F_\xi$  if and only if every equilibrium state of the following system

$$\dot{z} = -DLz \quad (6)$$

forms a planar formation  $F_\xi$  for all invertible diagonal matrix  $D = \text{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{C}^{n \times n}$ .

**Proof:** Since  $D$  is diagonal and invertible, it follows that the null space of  $DL$  is the same as the one of  $L$ . So the two systems have the same equilibrium set. ■

When  $L$  is pre-multiplied by  $D$ , the complex weights on the edges having their heads at agent  $i$  are multiplied by a nonzero complex number  $d_i$ . Therefore, the interaction rule is still locally implementable using relative position information. Generally, for a complex-valued Laplacian  $L$  satisfying the conditions of Theorem 3.1,  $L$  may have eigenvalues with both negative and positive real parts and thus system (4) may not be asymptotically stable with respect to the equilibrium subspace  $\ker(L)$ . However, we show in the next result that if certain conditions are satisfied, there always exists an invertible diagonal matrix  $D$  such that  $DL$  has all other eigenvalues with positive real parts in addition to the two eigenvalues at the origin and thus  $\dot{z} = -DLz$  is asymptotically stable with respect to the equilibrium subspace  $\ker(L)$ . Such a matrix  $D$  is called a *stabilizing matrix*.

**Theorem 3.3** Consider a formation basis  $\xi \in \mathbb{C}^n$  satisfying  $\xi_i \neq \xi_j$  for  $i \neq j$  and suppose a complex

Laplacian  $L$  of  $\mathcal{G}$  satisfies  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ . If there exists a permutation matrix  $P$  such that all the leading principal minors of  $PL_{ff}P^T$  are nonzero, then a stabilizing matrix  $D$  for system (4) exists.

The proof requires the following result related to the multiplicative inverse eigenvalue problem.

**Theorem 3.4** (Ballantine (1970)) *Let  $A$  be an  $n \times n$  complex matrix all of whose leading principal minors are nonzero. Then there is an  $n \times n$  complex diagonal matrix  $D$  such that all the eigenvalues of  $DA$  are positive and simple.*

**Proof of Theorem 3.3:** Since there is a permutation matrix  $P$  such that all the leading principal minors of  $PL_{ff}P^T$  are nonzero, then by Theorem 3.4 we obtain that there exists a diagonal matrix  $D'$  such that  $D'PL_{ff}P^T$  has all eigenvalues with positive real parts. Note that since  $P^{-1} = P^T$ , then  $D'PL_{ff}P^T$  is obtained from  $P^TD'PL_{ff}$  via a similarity transformation, i.e.,  $D'PL_{ff}P^T = P(P^TD'PL_{ff})P^T$ . So it follows that  $P^TD'PL_{ff}$  has the same eigenvalues as  $D'PL_{ff}P^T$ . Also, note that  $P^TD'P$  is a diagonal matrix as well, and we denote as  $D'' = P^TD'P$ . Let

$$D = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & D'' \end{bmatrix}. \quad (7)$$

Then the matrix  $DL$  has two eigenvalues at the origin and others with positive real parts. As a result,  $D$  is a stabilizing matrix. ■

We now discuss how the choice of a complex Laplacian affects the existence of a permutation matrix  $P$  as required in Theorem 3.3. Note that the set of matrices  $L$  satisfying  $L\xi = 0$  is a linear subspace. Thus any such  $L$  can be written as  $L = x_1A_1 + \dots + x_kA_k$  where  $A_1, \dots, A_k$  form the basis of the subspace and  $x_1, \dots, x_k$  are appropriate coefficients. Let  $L'$  denote a leading principal sub-matrix of  $L$ . Then  $L'$  can also be expressed in a similar form, i.e.,  $L' = x_1A'_1 + \dots + x_kA'_k$  with  $A'_i$  ( $i = 1, \dots, k$ ) taking the corresponding leading principal sub-matrix from  $A_i$ . Let us recall a property given below.

**Property 3.1** (Loasz, 1989) *For given matrices  $A'_1, \dots, A'_k$ , if the determinant of matrix  $A(x_1, \dots, x_k) = x_1A'_1 + \dots + x_kA'_k$  is not zero for a choice of variables  $x_1, \dots, x_k$ , then it is not zero for almost all choices of variables.*

We then can conclude that if a choice of complex Laplacian satisfying  $L\xi = 0$  makes any leading principal minor of  $L$  nonzero (namely,  $\det(L') \neq 0$ ), then almost all complex Laplacians satisfying  $L\xi = 0$  also have the property that any leading principal minor is nonzero. Thus when computing a matrix  $L$  that satisfies  $L\xi = 0$ , it is not difficult to determine one whose leading principal minors are all nonzero, if any exists.

In the case in which some leading principal minors are null, however, we may change them by permutation. Consider for example a graph given in Fig. 1 with two leaders labeled as 1 and 2. Suppose for a given formation basis  $\xi$ ,  $L_{ff}$  is of the form given in eq. (8). It is noticed that the 3rd order leading principal minor of  $L_{ff}$  is 0 for any choice of  $x_1, \dots, x_8$ . However, if we relabel the nodes 3, 4, 5 as 6, 7, 8, and relabel the nodes 6, 7, 8 as 3, 4, 5, then it can be checked that for almost all  $x_1, \dots, x_8$ , all the leading principal minors of the new  $L_{ff}$  corresponding to the new labels are nonzero.

Next we give an algorithm to find  $D$ , in which the notation  $L_{ff}^{(1 \sim i)}$  is used to denote the sub-matrix formed by the first  $i$  rows and columns of  $L_{ff}$ .

### Algorithm 3.1

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**Input:**  $L_{ff}$  with all nonzero leading principal minors.

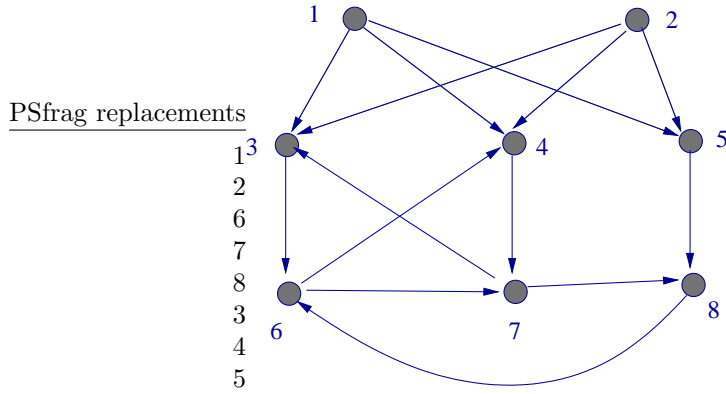


Figure 1: An example of a leader-follower network.

$$L_{ff} = \left[ \begin{array}{ccc|ccc} (1 + \iota)x_1 & 0 & -\iota x_1 & -x_1 & 0 & 0 \\ (2 + \iota)x_2 & -4x_2 & 0 & 0 & (2 - \iota)x_2 & 0 \\ 0 & -(2.4 + 0.8\iota)x_3 & \iota x_3 & 0 & 0 & (2.4 - 0.2\iota)x_3 \\ \hline 0 & x_4 & 0 & x_5 & 0 & 0 \\ x_6 & 0 & 0 & 0 & x_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_8 \end{array} \right] \quad (8)$$

**Output:** *Stabilizing matrix*  $D$ .

**Procedure:**

**for**  $i = 1, \dots, n - 2$  **do**

*Find*  $d_{i+2}$  *to assign the eigenvalues of*  $\text{diag}(d_3, \dots, d_{i+2})L_{ff}^{(1 \sim i)}$  *in the open right half complex plane.*

**end for**

*Construct*  $D = \text{diag}\{1, 1, d_3, \dots, d_n\}$ .

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The algorithm finds the diagonal entries of  $D$  one by one in an iterative way. First,  $d_3$  can be chosen such that  $d_3 L_{ff}^{(1 \sim 1)}$  has an eigenvalue with positive real part, say  $\lambda'_1$ . Notice that for a given  $d_3$ , the matrix  $\text{diag}(d_3, d_4)L_{ff}^{1 \sim 2}$  becomes solely dependent on the single variable  $d_4$ . We denote by  $\lambda_1(d_4)$  and  $\lambda_2(d_4)$  the eigenvalues of the matrix  $\text{diag}(d_3, d_4)L_{ff}^{1 \sim 2}$ . It is known that  $\lambda_1(d_4)$  and  $\lambda_2(d_4)$  are continuous functions with respect to the variable  $d_4$ . Moreover, when we choose  $d_4 = 0$ , it can be checked that the eigenvalues of  $\text{diag}(d_3, 0)L_{ff}^{1 \sim 2}$  are respectively 0 and  $\lambda'_1$  from the previous step, namely,  $\lambda_1(0) = \lambda'_1$  and  $\lambda_2(0) = 0$ . Therefore, by varying  $d_4$  around 0, we could make  $\lambda_1(d_4)$  in the neighborhood of  $\lambda'_1$  and  $\lambda_2(d_4)$  in the neighborhood of 0. Moreover, notice the fact  $\det(L_{ff}^{1 \sim 2})d_3 d_4 = \lambda_1(d_4)\lambda_2(d_4)$ , so it is known that  $d_4$  can be selected around 0 to make  $\lambda_2(d_4)$  not only in the neighborhood of 0 but also in the right-hand plane. The process repeats until all  $d_i$ 's are found. One can thus conclude that when a matrix  $L$  satisfies the assumption in Theorem 3.3, Algorithm 3.1 can always provide a solution  $D$  without any exemption.

### 3.2 Double-integrator dynamics

Suppose that each agent is governed by a double-integrator dynamics

$$\begin{cases} \dot{z}_i = v_i \\ \dot{v}_i = a_i \end{cases} \quad (9)$$

where the position  $z_i \in \mathbb{C}$  and the velocity  $v_i \in \mathbb{C}$  are the states, and the acceleration  $a_i \in \mathbb{C}$  is the control input. Consider the sensing graph  $\mathcal{G}$  and suppose that each agent takes the control law

$$\begin{aligned} a_i &= -\gamma v_i, & i &= 1, 2; \\ a_i &= \sum_{j \in \mathcal{N}_i} w_{ij} (z_j - z_i) - \gamma v_i, & i &= 3, \dots, n, \end{aligned} \quad (10)$$

where  $w_{ij} = k_{ij} e^{i\alpha_{ij}}$  is a complex weight with  $k_{ij} > 0$  and  $\alpha_{ij} \in [-\pi, \pi)$ , and  $\gamma > 0$  is a real number representing the damping gain.

Write  $z = [z_1, \dots, z_n]^T$  and  $v = [v_1, \dots, v_n]^T$ . Then the overall system of the  $n$  agents under the interaction rule (10) can be written as

$$\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -L & -\gamma I_n \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} \quad (11)$$

where  $L$  is the Laplacian of  $\mathcal{G}$  defined in (1).

The interaction rule (10) similar to (3) can also be locally implemented, which requires only the relative positions of neighbors and its own velocity. Denote  $\bar{z}_1 = \lim_{t \rightarrow \infty} z_1(t)$  and  $\bar{z}_2 = \lim_{t \rightarrow \infty} z_2(t)$ . Next we show that the conditions in Theorem 3.1 is also a necessary and sufficient condition such that the equilibrium states  $(\bar{z}, \bar{v})$  of system (11) form a planar formation  $F_\xi$ , i.e.,  $\bar{z} = c_1 \mathbf{1}_n + c_2 \xi$  and  $\bar{v} = 0$ , where  $c_1$  and  $c_2$  can be obtained from (5). Moreover, we show that the equilibrium formations are invariant to the operation of pre-multiplying an invertible diagonal complex matrix  $D$ .

**Theorem 3.5** *Assume that  $\xi \in \mathbb{C}^n$  satisfies  $\xi_i \neq \xi_j$  for  $i \neq j$ . Then the following are equivalent.*

1.  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ .
2. Every equilibrium state of system (11) forms a planar formation  $F_\xi$ .
3. Every equilibrium state of the following system

$$\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ -DL & -\gamma I_n \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} \quad (12)$$

*forms a planar formation  $F_\xi$  for all invertible diagonal matrices  $D = \text{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{C}^{n \times n}$ .*

**Proof:** By simply checking system (12), one can obtain that the equilibrium states satisfy  $L\bar{z} = 0$  and  $\bar{v} = 0$ . Thus, the conclusion follows from the same argument in Theorem 3.1 and Theorem 3.2.  $\blacksquare$

According to Theorem 3.5, the equilibrium formation for the double integrator model is characterized — as was the case with the single integrator model — by  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ . Also, similar to the single-integrator model, the eigenvalues of system (11) may be distributed everywhere in the complex plane such that the trajectories of system (11) may not converge to the equilibrium formation. Hence,



an invertible diagonal matrix  $D$  is utilized to assign the eigenvalues of

$$\begin{bmatrix} 0_{n \times n} & I_n \\ -DL & -\gamma I_n \end{bmatrix}$$

in the open left half complex plane in addition to the two eigenvalues at the origin, i.e., to make the  $n$  agents asymptotically reach a planar formation  $F_\xi$  with the interaction law (12). If such a matrix  $D$  exists, it is called a *stabilizing matrix*. Next, we show the existence condition of a stabilizing matrix.

**Theorem 3.6** *Consider a formation basis  $\xi \in \mathbb{C}^n$  satisfying  $\xi_i \neq \xi_j$  and suppose a complex Laplacian  $L$  of the sensing graph  $\mathcal{G}$  satisfies  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ . If there is a permutation matrix  $P$  such that all the leading principal minors of  $PL_{ff}P^T$  are nonzero, then a stabilizing matrix  $D$  for system (11) exists.*

**Proof:** Denote

$$A = \begin{bmatrix} 0_{n \times n} & I_n \\ -DL & -\gamma I_n \end{bmatrix}.$$

Let  $\sigma_i$  be an eigenvalue of the matrix  $DL$  corresponding to eigenvector  $x$ , i.e.,  $DLx = \sigma_i x$ . Moreover, let  $\lambda_i$  be a root of

$$\lambda_i^2 + \gamma\lambda_i + \sigma_i = 0 \quad (13)$$

and define

$$y = \lambda_i x. \quad (14)$$

Considering (13) and (14), we obtain that

$$-DLx - \gamma y = -\sigma_i x - \gamma\lambda_i x = \lambda_i^2 x = \lambda_i y. \quad (15)$$

Eqs. (14) and (15) together imply that  $\lambda_i$  is an eigenvalue of  $A$  corresponding to the eigenvector  $[x^T \ y^T]^T$ .

Since  $L$  satisfies  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ , it is known that  $DL$  has two zero eigenvalues for all invertible diagonal matrices  $D$ . Without loss generality, denote  $\sigma_1 = \sigma_2 = 0$ . For  $\sigma_1 = \sigma_2 = 0$ , the roots of the characteristic equation (13) are  $\lambda_{i,1} = 0$ ,  $\lambda_{i,2} = -\gamma < 0$ ,  $i = 1, 2$ . Thus, to show the existence of a stabilizing matrix  $D$ , it remains to show that  $\sigma_i$  ( $i = 3, \dots, n$ ) can be assigned such that the roots of the complex-coefficient characteristic equation (13) have negative real parts. According to Chen and Tsai (1993), the roots are in the open left half complex plane if and only if

$$\frac{\operatorname{Re}(\sigma_i)}{(\operatorname{Im}(\sigma_i))^2} > \frac{1}{\gamma^2}.$$

By our assumption that there is a permutation matrix  $P$  such that all the leading principal minors of  $PL_{ff}P^T$  are nonzero, it follows from the same argument as in the proof of Theorem 3.3 that there exists a diagonal matrix  $D'$  such that the eigenvalues of  $D'PL_{ff}P^T$  all have positive real parts. Denote the eigenvalues of  $D'PL_{ff}P^T$  by  $\sigma'_3, \dots, \sigma'_n$ . Then we choose  $D$  as

$$D = \begin{bmatrix} I_2 & 0 \\ 0 & \epsilon P^T D' P \end{bmatrix} \quad (16)$$

where  $\epsilon > 0$  is a scalar. Thus, the eigenvalues of  $DL$  are  $\sigma_1 = \sigma_2 = 0$ ,  $\sigma_i = \epsilon\sigma'_i$ ,  $i = 3, \dots, n$ . Then it

can be checked that for sufficiently small  $\epsilon > 0$

$$\frac{\operatorname{Re}(\sigma_i)}{(\operatorname{Im}(\sigma_i))^2} = \frac{\operatorname{Re}(\sigma'_i)}{\epsilon(\operatorname{Im}(\sigma'_i))^2} > \frac{1}{\gamma^2}, \quad i = 3, \dots, n. \quad (17)$$

Therefore, a stabilizing matrix  $D$  is derived, making a group of  $n$  agents asymptotically reach the planar formation  $F_\xi$ .  $\blacksquare$

From the proof of Theorem 3.6, we know that a stabilizing matrix can also be obtained for the double-integrator case with a minor modification of Algorithm 3.1. We present it below.

### Algorithm 3.2

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**Input:**  $L_{ff}$  with all nonzero leading principal minors.

**Output:** Stabilizing matrix  $D$ .

**Procedure:**

**for**  $i = 1, \dots, n - 2$  **do**

Find  $d_{i+2}$  to assign the eigenvalues of  $\operatorname{diag}(d_3, \dots, d_{i+2})L_{ff}^{(1 \sim i)}$  in the open right half complex plane.

**end for**

Select an  $\epsilon$  satisfying (17).

Construct  $D = \operatorname{diag}\{1, 1, \epsilon d_3, \dots, \epsilon d_n\}$ .

---

**Remark 3.2** The static formation results can be simply extended to reach and maintain a formation shape while moving. When the synchronized velocity  $v_0(t)$  (or acceleration  $a_0(t)$ ) is available to all the followers, then from the change of origin, the following control laws are obtained.

$$\begin{cases} v_i = v_0(t), & i = 1, 2; \\ v_i = \sum_{j \in \mathcal{N}_i} w_{ij}(z_j - z_i) + v_0(t), & i = 3, \dots, n. \end{cases} \quad (18)$$

$$\begin{cases} a_i = -\gamma v_i + a_0(t), & i = 1, 2; \\ a_i = \sum_{j \in \mathcal{N}_i} w_{ij}(z_j - z_i) - \gamma v_i + a_0(t), & i = 3, \dots, n. \end{cases} \quad (19)$$

When the leaders' velocity or acceleration is not accessible by all followers, estimation schemes (see for example Guo et al. (2010)) can be adopted to estimate the piece of information and then the estimated one can be substituted in (18) or (19).

## 4 Simulations

### 4.1 Single-integrator kinematics

First, we consider an example of single-integrator model. The system consists of five agents where agent 1 and 2 are two co-leaders, and agent 3, 4, and 5 are followers. The sensing graph  $\mathcal{G}$  is given in Fig. 2 where the blue lines with arrows represent the edges.

Consider a planar formation with a formation basis (Fig. 2) defined as  $\xi = \begin{bmatrix} 0 & -2\iota & 2 - 2\iota & -2 - 2\iota & -4\iota \end{bmatrix}^T$ . To achieve a planar formation with the defined formation basis  $\xi$ , the follower agents take the interaction

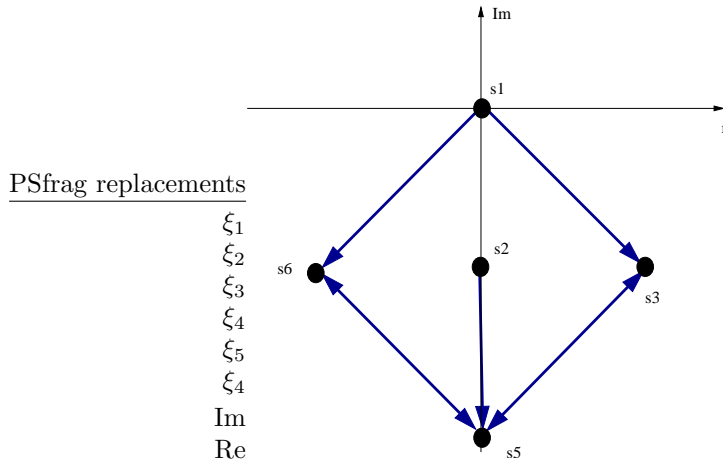


Figure 2: The sensing graph  $\mathcal{G}$  and a formation basis  $\xi$ .

law (3) with the complex-valued Laplacian

$$L = \left[ \begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 + \iota & 0 & -\iota \\ -\iota & 0 & 0 & 1 + \iota & -1 \\ 0 & -2\iota & -2(1 - \iota) & -(1 + \iota) & 3 + \iota \end{array} \right]. \quad (20)$$

It can be checked that  $L\xi = 0$  and  $\det(L_{ff}) \neq 0$ . Hence, by Theorem 3.1, the equilibrium state of the system exactly corresponds to the planar formation  $F_\xi = c_1 \mathbf{1}_n + c_2 \xi$  for some  $c_1, c_2 \in \mathbb{C}$ . It is calculated that  $L$  contains several eigenvalues with negative real parts, which means the system is not stable with respect to the equilibrium formations. However, we can check that all the leading principal minors of  $L_{ff}$  are nonzero. Therefore, according to Theorem 3.3, a stabilizing matrix  $D$  exists to stabilize the system. For example, taking  $D = \text{diag} \left\{ 1, 1, e^{-\iota \frac{3\pi}{4}}, 3e^{-\iota \frac{\pi}{4}}, 3 \right\}$ , we then have the eigenvalues of  $DL$  all with positive real parts in addition to the two zero eigenvalues. Therefore, the five agents asymptotically reach a planar formation  $F_\xi$  with the complex-valued Laplacian  $DL$ . Suppose that two co-leaders have the synchronized velocity  $v_0(t) = 2t \cos(0.1t) + \iota 0.5t \sin(0.1t)$ . A simulation of moving formation is shown in Fig. 3.

## 4.2 Double-integrator dynamics

Next we consider the double-integrator dynamics of the same example. The sensing graph  $\mathcal{G}$  and the formation basis  $\xi$  are the same as in Fig. 2. To achieve a planar formation  $F_\xi$ , the agents take the interaction law (10) where  $\gamma = 5$  and  $L$  is the same as the one in (20). For this  $L$ , the system is unstable. However, a stabilizing matrix  $D$  exists by Theorem 3.6 since the leading principal minors are nonzero as we checked. Indeed, for this example, the same  $D$  used to stabilize the case of single-integrator kinematics also stabilizes the case of double-integrator dynamics. If the two co-leaders move with the synchronized acceleration  $a_0(t) = 2t \cos(0.1t) + \iota 1.5t \sin(0.1t)$ , the five agents asymptotically reach a moving planar formation  $F_\xi$  with the same velocity and acceleration. A simulation result is presented in Fig. 4 validating the conclusion.

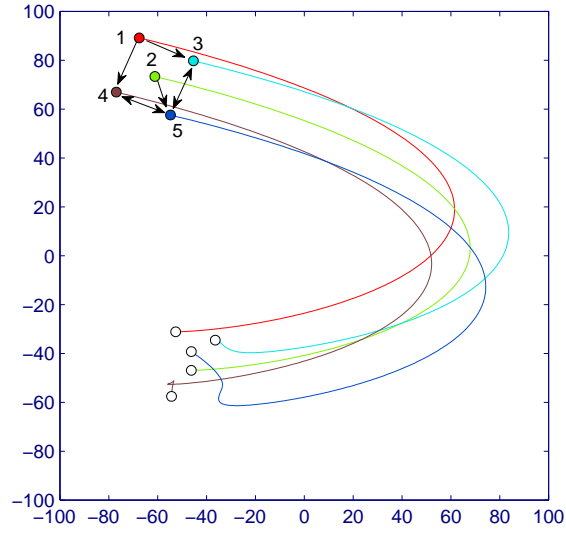


Figure 3: Reaching a moving formation (single integrators).

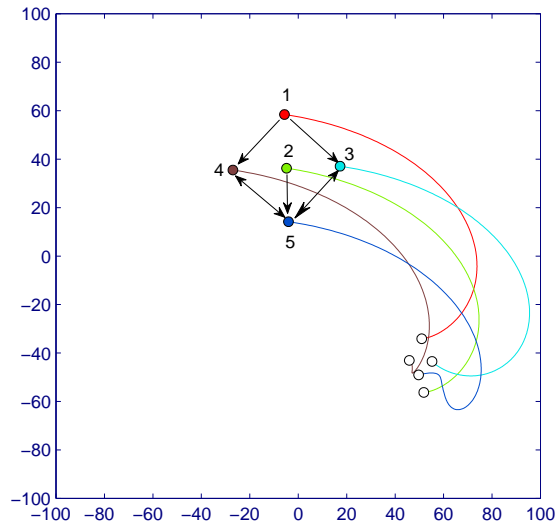


Figure 4: Reaching a moving formation (double integrators).

## 5 Conclusion and open problems

This paper introduces complex Laplacians for directed graphs whose edges are attributed with two labels (gain and offset angle, combined in one complex weight). A simple and locally implementable linear control law related to complex Laplacian is investigated to control the shape of a planar formation. Necessary and sufficient conditions are obtained such that complex Laplacians characterize a specific planar formation. Unlike real Laplacians that have all eigenvalues in the closed right half complex plane, complex Laplacians may distribute their eigenvalues in the whole complex plane, which may lead to instability of the overall system. To overcome the difficulty, we show that the eigenvalues of a complex Laplacian can be re-assigned by pre-multiplying an invertible diagonal matrix. However, some important

issues in practical applications such as agent failure, collision avoidance, and limited sensing capability have not been addressed. How to avoid collisions and maintain the links by simply adjusting the complex weights in the control law will be an interesting problem. Moreover, the dynamic topology case resulting from limited sensing capability or agent failures will be another concern.

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