IPA for Continuous Stochastic Marked Graphs *

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Abstract

This paper presents a unified framework for the Infinitesimal Perturbation Analysis (IPA) gradient-estimation technique in the setting of marked graphs. It proposes a systematic approach for computing the derivatives of sample performance functions with respect to structural and control parameters. The resulting algorithms are recursive in both time and network flows, and their successive steps are computed in response to the occurrence and propagation of certain events in the network. Such events correspond to discontinuities in the network flow-rates, and their special characteristics are due to the properties of continuous transitions and fluid places. Following a general outline of the framework we focus on a simple yet canonical example, and investigate throughput and workload-related performance criteria as functions of structural and control variables. Simulation experiments support the analysis and testify to the potential viability of the proposed approach.

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1 Introduction and Problem Statement

In the past ten years there has been a mounting interest in performance optimization of fluid-queueing networks having the structure of stochastic hybrid systems (see [5] and references therein). Such systems have bi-level dynamics with time-driven dynamics at the lower level and event-driven dynamics at the upper level, and the two kinds of dynamics interact with each other through the controlled timing of the occurrence of discrete events [4]. In various optimization problems of interest the system's dynamic behavior depends not only on time but also on a structural or control parameter denoted by θ. Furthermore, when θ represents a collection of independent parameters, it can be viewed as a variable in a Euclidean space R^N. Let $L(\theta)$ denote a random function defined on a suitable probability space (Ω, F, P) whose realizations correspond to the evolution of the network's dynamics over a finite-horizon interval, and let $J(\theta) := E(L(\theta))$ where $E(\cdot)$ denotes expectation in (Ω, F, P). Certain optimization problems arising in system design or control concern minimizing such a function $J(\theta)$ subject to a given set of constraints.

The expected-value function $J(\theta)$ often lacks a closed-form expression, and hence its evaluation has to be accomplished via sample-path techniques that are based on $L(\theta)$. Note that $L(\theta)$ is a realization of $J(\theta)$ and it can be viewed as a sample-performance function. In the event that $\theta \in R^N$ is a continuous variable and gradient-techniques are sought for minimizing $J(\theta)$, $\nabla J(\theta)$ could be approximated by the gradients of the sample performance functions, $\nabla L(\theta)$, or averages thereof. Such gradient terms have to be computed along particular sample paths, possibly the same sample paths that are used to realize $L(\theta)$. A general framework for computing these sample gradients is Infinitesimal Perturbation Analysis (IPA).

IPA originally was developed in the context of queueing networks [13, 3], and recently much of its investigation has focused on fluid queues [4, 20]. Fluid queueing networks, called stochastic flow models, provide a natural setting for IPA since their sample gradients often admit very simple algorithms amounting to little more than counting processes [4]. Furthermore, these algorithms exhibit a discernable measure of low sensitivity to modeling variations and hence can be applied, with good precision, to performance functions defined on sample paths not only of fluid queues but also of discrete queues approximated by them [5]. Another appealing property of fluid queues is that their IPA gradients are statistically unbiased in a far-larger class of networks and functions than those associated with discrete queues. For these reasons it has been suggested (see [6] and references therein) that the IPA gradients developed in the fluid-queue setting could be used in design optimization as well as in real-time control. Their theoretical underpinnings have been established in [4, 20, 5], and algorithms for general network topologies were developed in [21, 6, 23, 14]. Examples in various application areas including telecommunications, manufacturing, traffic networks, and supply chains, were presented in [16, 24, 17, 22, 9, 7, 25].

The purpose of this paper is to investigate the application of IPA to another category of stochastic hybrid systems, namely a class of decision-free continuous Petri nets called marked graphs. The key element in such system-models that is absent from queueing networks is the transition, which is used to capture the notions of concurrency and synchronization in dynamical systems. In this setting we develop an abstract algorithm, suitable to a wide range of network topologies, and apply it to a simple yet canonical example. The algorithm is based on an extension of the framework, developed in [21, 23] for IPA in a general class of stochastic hybrid systems and applied to queueing networks, to the present setting of Petri nets. The example concerns throughput in a single-join system with respect to a flow-control variable, and it serves to illustrate the analysis of the fluid transition from the standpoint of IPA.

The rest of the paper is organized as follows. Section 2 defines the IPA setting for continuous marked graphs, and Section 3 develops a general-purpose prescriptive framework for computing the sample derivatives. Section 4 considers an example of workload minimization, and Section 5 concludes the paper. The results presented herein summarize, build upon, and generalize those presented in three conferences, namely [10, 11, 12].

2 Continuous Petri Nets

Hybrid Petri nets have "fluidized" tokens flowing on their arcs in addition to the usual discrete tokens. They were extensively analyzed (see [19, 1], and [18, 8] for more recent surveys), and investigated from the standpoint of control and optimization in [2]. Their discrete tokens often represent discontinuities in the flow rates of the fluidized tokens, and the flow rates are described via piecewise-analytic functions like (piecewise) constant, linear, or quadratic. The present paper assumes that the fluid-flow rates are piecewise continuous without restricting their parametric forms, and this will suffice for the purpose of analyzing the IPA gradients of performance functions of interest. Furthermore, we do not have to resort to discrete tokens to represent discontinuities of the flow rates, but rather use the hybrid-system modeling framework where such discontinuities are caused by the occurrence of discrete events [4]. Accordingly, the fluid-flow rates can be described by functions $v(t) : R^+ \rightarrow R^+$ of a temporal variable $t \geq 0$. To emphasize the absence of explicit discrete tokens from our modeling framework, we call the Petri nets in question *continuous* rather than *hybrid*.

Recently a class of decision-free continuous Petri nets, namely marked graphs, has been studied from the standpoint of IPA [22, 10, 11, 12]. A marked graph is a net where each place has a single input arc and a single output arc. For this class of nets, reference [22] derived algorithms for the sample derivatives. This paper has a similar objective, but it is different from [22] in the following two ways:

1. Reference [22] assumes that the maximum transitions' firing speeds (defined below) are piecewise constant functions of time, while this paper allows for general functions. Consequently, the algorithm in [22] is described in terms of evolution equations which are not available for the more-general flow-rate functions considered here.
2. The variational parameter in [22] is either the transitions' maximum firing speeds or the initial marking. This paper allows for more general parameters, including a variable of the probability law underlying the transitions' maximum speeds, the duration of these speeds at given values, as well as various network control parameters such as threshold flow control.

As a result the analysis and algorithms developed here, while not as elegant as those in [22], have a much wider scope in terms of network models and potential applications. Furthermore, they cannot be derived by the specialized techniques that are used in [22].

In the marked graphs that we consider, transitions are denoted by upper-case letters like T and U while places are denoted by lower-case letters such as p and q. A typical transition T is characterized by its maximum firing (flow) speed, or rate, also called its *capacity*. This is a function of time, denoted by $V_T(t)$. Generally $V_T(t)$ is a random function defined on a suitable probability space, but we will focus on its sample realizations, also denoted by $V_T(t)$. The actual firing rate, denoted by $v_T(t)$, must satisfy the inequalities $0 \leq v_T(t) \leq V_T(t)$. For a typical place p, the fluid level contained in it (often called the workload, or marking) is denoted by $m_p(t)$. For every transition T we denote by $in(T)$ and $out(T)$ the sets of its input places and output places, respectively. As in earlier studies of IPA in the Petri-net setting ([22, 12]), we assume that each place p has a single input transition and a single output transition, and we denote these transitions by $in(p)$ and $out(p)$, respectively. This assumption, as mentioned in [22], qualifies the net as a decision-free marked graph. No further restrictions are made on the topology of the networks, and they may be closed, open, or neither closed nor open. A transition T is called a source transition if $in(T) = \emptyset$, and a sink transition whenever $out(T) = \emptyset$.

Suppose that the marked graph evolves over a given time-interval $[0, t_f]$. For each transition T, the process $\{V_T(t)\}_{t=0}^{t_f}$, representing the maximum firing rate of T, often is an exogenous process, but also can be a controlled process as we shall see later. We assume that the transition fires at the highest-possible rate, defined as follows. At every time $t \in [0, t_f]$, if none of the places $p \in in(T)$ is empty, then $v_T(t) = V_T(t)$; while if some of the places $p \in in(T)$ are empty, then $v_T(t)$ is equal to the lowest firing rate among all transitions $U = in(p)$ where p is empty. Formally, define $\varepsilon_T(t) := \{p \in in(T) : m_p(t) = 0\}$, namely the set of input places to T which are empty at time t. Then, $v_T(t)$ is determined via the following equation,

$$v_T(t) \;=\; \begin{cases} V_T(t), & \text{if } \varepsilon_T(t) = \emptyset \\ \min\{v_{in(p)}(t) : p \in \varepsilon_T(t)\}, & \text{if } \varepsilon_T(t) \neq \emptyset. \end{cases} \tag{1}$$

For each place p, the workload process $\{m_p(t)\}_{t=0}^{t_f}$ evolves according to the following flow equation,

$$\dot{m}_p(t) \;=\; v_{in(p)}(t) - v_{out(p)}(t), \tag{2}$$

with some given initial condition $m_p(0)$.

We note that the *min* term in Equation (1) captures the concept of synchronization that is inherent in Petri nets and absent from queueing models. As we shall see, it will be closely related to the computation of the sample derivatives, developed and discussed below.

The stochastic processes comprising our system can be classified as *exogenous* vs. *derived.* An exogenous process does not depend on any other system's process, and the totality of exogenous processes defines the probability law of the system in the sense that, every other process can be expressed as a function of them. Any process that is not exogenous is said to be derived. We say that the network is *uncontrolled* if the processes $\{V_T(t)\}$ are exogenous while the processes $\{v_T(t)\}$ and $\{m_p(t)\}$ are derived from them via Equations (1) and (2). On the other hand, we say that the network is *controlled* if some of the processes $\{V_T(t)\}$ are controlled by other processes, and hence classified as derived. For example, given two exogenous processes $\{\psi_1(t)\}$ and $\{\psi_2(t)\}$, associated with a particular transition T, suppose that either $V_T(t) = \psi_1(t)$ or $V_T(t) = \psi_2(t)$ according to whether a certain condition on the network processes holds or not. For instance, in the case of threshold-flow control, the condition can be that $m_p(t) \leq r$ for given place p and $r > 0$. Notice that in this case, the processes $\{\psi_1(t)\}$ and $\{\psi_2(t)\}$ are exogenous whereas the process $\{V_T(t)\}$ is derived. More abstractly, let $\mathbf{v}(t)$ and $\mathbf{m}(t)$ denote the vectors of transitions' firing rates and places' workloads, respectively; let $\{\psi(t)\}$ be an (possibly multi-variable) exogenous process; for every transition T, let $\zeta_T : (\mathbf{v}, \mathbf{m}, \psi) \to R^+$ be a function; and suppose that $V_T(t)$ is given by

$$V_T(t) \;=\; \zeta_T\big(\mathbf{v}(t), \mathbf{m}(t), \psi(t)\big). \tag{3}$$

In the forthcoming discussion we assume that all the exogenous processes in the network are either $\{V_T(t)\}$, or the processes $\{\psi(t)\}$ upon which $V_T(t)$ depends via (3). This implies that all of the processes $\{v_T(t)\}$ and $\{m_p(t)\}$ are derived. Furthermore, we say that Equations (1)-(3) are *consistent throughout the network* if, for every initial network workloads $m_p(0)$ (for every place p), and every realization of the exogenous processes, w.p.1, these equations have a unique joint solution at every time $t \in [0, t_f]$. In this case all of the derived processes in the network are defined via these equations, and said to be derived from them. In particular, in the case of uncontrolled networks, all of the processes $\{V_T(t)\}$ are exogenous, and the processes $\{v_T(t)\}$ and $\{m_p(t)\}$ are derived from them via Equations (1) and (2). For this case, Reference [22] pointed out the following sufficient conditions for consistency of Equations (1) and (2) throughout the network: every elementary circuit contains at least one place p such that $m_p(0) > 0$. For controlled networks there are no such general results, and consistency throughout the network has to be ascertained for each particular network or classes of networks.

We point out that an analysis of the general framework for controlled networks, defined via Equation (3), is beyond the scope of a single paper. Instead, we consider here the case of uncontrolled networks in detail, and then address a particular case of threshold-based flow control.

Now suppose that the probability law underscoring the network processes is a function not only of time, but also of a variable $\theta \in R^N$, and hence the various network processes are denoted by $\{V_T(\theta,t)\}_{t=0}^{t_f}$, $\{v_T(\theta,t)\}_{t=0}^{t_f}$, and $\{m_p(\theta,t)\}_{t=0}^{t_f}$. Their sample-path realizations as functions of time (t) will be denoted by $V_T(\theta,t)$, $v_T(\theta,t)$, and $m_p(\theta,t)$, respectively. With this notation, Equations (1)-(3) are to be understood in the following way: fix $\theta \in R^n$, and let the traffic processes evolve in the time-interval $[0,t_f]$ according to these equations. Sample performance functions of frequent interest in applications, like throughput and delay, are related to the following two functions defined, respectively, for transitions T and places p (see [11]):

$$L_T(\theta) \ := \ \int_0^{t_f} v_T(\theta,t)dt, \tag{4}$$

and

$$L_p(\theta) \ := \ \int_0^{t_f} m_p(\theta,t)dt. \tag{5}$$

Their sample gradients, $\nabla L_T(\theta)$ and $\nabla L_p(\theta)$, are the IPA gradients that we investigate in this paper. The next section develops an abstract algorithmic framework for them while the subsequent section presents an example.

3 General Framework for IPA

This section considers the IPA gradients of the sample performance functions $L_T(\theta)$ and $L_p(\theta)$ defined by Equations (4) and (5). To somewhat simplify the exposition we assume that θ is a one-dimensional variable so that the IPA gradient is called the *IPA derivative* and denoted by $\frac{dL}{d\theta}(\theta)$. Furthermore, we will implicitly assume that θ is constrained to a closed, bounded interval $\Theta \subset R$. In the early part of the discussion we assume that all of the mentioned derivatives exist; later we present assumptions guaranteeing this and, as in [21, 23, 6], verify them for particular examples. This order of presentation helps us focus on the main ideas while addressing some technical details in the context of specific examples where their exposition is considerably simpler. We also assume that for a given $\theta \in \Theta$, transition T, and place p, the function $v_T(\theta,\cdot)$ is piecewise continuous and piecewise continuously differentiable, and the function $m_p(\theta,\cdot)$ is continuous and piecewise continuously differentiable. These assumptions, too, will be verified from basic conditions. The discontinuity (jump) time-points of $v_T(\theta,\cdot)$ are functions of θ, and hence are denoted by $t_{k,T}(\theta)$, $k=1,\ldots,K$, in increasing order, for some (random) K, while their derivatives with respect to θ are denoted by $\frac{dt_{k,T}}{d\theta}(\theta)$.

3.1 IPA derivatives

Consider first the IPA derivative $\frac{dL_T}{d\theta}(\theta)$. Taking derivatives in (4) we obtain its following general form,

$$\frac{dL_T}{d\theta}(\theta) \;=\; \int_0^{t_f} \frac{\partial v_T}{\partial \theta}(\theta,t)dt \;+$$
$$\sum_{k=1}^{K}\Big(v_T(\theta,t_{k,T}(\theta)^-) - v_T(\theta,t_{k,T}(\theta)^+)\Big)\frac{dt_{k,T}}{d\theta}(\theta). \tag{6}$$

The terms $\frac{\partial v_T}{\partial \theta}(\theta,t)$ typically can be computed directly and easily from the sample path (this will be demonstrated on the example discussed in the sequel), and hence the main challenge is to compute the sum-terms in the Right-Hand Side (RHS) of Equation (6). These terms also arise in the IPA derivative $\frac{dL_p}{d\theta}(\theta)$ for place p. Indeed, taking derivatives in (5) we obtain,

$$\frac{dL_p}{d\theta}(\theta) \;=\; \int_0^{t_f} \frac{\partial m_p}{\partial \theta}(\theta,t)dt, \tag{7}$$

and the integrant in this equation has the following form. If t lies in the interior of an empty period at p then $\frac{\partial m_p}{\partial \theta}(\theta,t)=0$. On the other hand, if $m_p(\theta,t)>0$, let $\xi(\theta):=\max\{\tau\leq t: m_p(\theta,\tau)=0\}$, then by (2)

$$m_p(\theta,t) \;=\; \int_{\xi(\theta)}^{t}\Big(v_{in(p)}(\theta,\tau) - v_{out(p)}(\theta,\tau)\Big)d\tau. \tag{8}$$

When taking derivatives with respect to θ in (8), we have to consider the points $\tau\in(\xi(\theta),t)$ where the integrant is discontinuous. Let us denote such points via two finite, monotone-increasing sequences as follows: $t_{j,in(p)}(\theta)$, $j=j_1,\ldots,j(t)$, are the jump-points of the function $v_{in(p)}(\theta,\cdot)$ in the interval $(\xi(\theta),t)$, and $t_{\ell,out(p)}(\theta)$, $\ell=\ell_1,\ldots,\ell(t)$ are the jump-points of the function $v_{out(p)}(\theta,\cdot)$ in the interval $(\xi(\theta),t)$. Suppose that $v_{in(p)}(\theta,\cdot)$ and $v_{out(p)}(\theta,\cdot)$ are continuous at the point t. Then (8) implies that

$$\frac{\partial m_p}{\partial \theta}(\theta,t) \;=\; \int_{\xi(\theta)}^{t}\Big(\frac{\partial v_{in(p)}}{\partial \theta}(\theta,\tau) - \frac{\partial v_{out(p)}}{\partial \theta}(\theta,\tau)\Big)d\tau$$
$$+\sum_{j=j_1}^{j(t)}\Big(v_{in(p)}(\theta,t_{j,in(p)}(\theta)^-) - v_{in(p)}(\theta,t_{j,in(p)}(\theta)^+)\Big)\times\frac{dt_{j,in(p)}}{d\theta}(\theta)$$
$$-\sum_{\ell=\ell_1}^{\ell(t)}\Big(v_{out(p)}(\theta,t_{\ell,out(p)}(\theta)^-) - v_{out(p)}(\theta,t_{\ell,out(p)}(\theta)^+)\Big)\times\frac{dt_{\ell,out(p)}}{d\theta}(\theta)$$
$$-\big(v_{in(p)}(\theta,\xi(\theta)^+) - v_{out(p)}(\theta,\xi(\theta)^+)\big)\frac{d\xi}{d\theta}(\theta). \tag{9}$$

We see that the terms $\Big(v_T(\theta,t_{k,T}(\theta)^-) - v_T(\theta,t_{k,T}(\theta)^+)\Big)\times\frac{dt_{k,T}}{d\theta}(\theta)$ also arise as key elements in the IPA derivatives $\frac{dL_p}{d\theta}(\theta)$. We next describe a recursive way to compute them along a sample path, and this will constitute the heart of the IPA algorithm.

Let us define the notation $\Delta v_T(\theta,t):=v_T(\theta,t^-)-v_T(\theta,t^+)$ and $\Delta V_T(\theta,t):=V_T(\theta,t^-)-V_T(\theta,t^+)$, and we observe that $\Delta v_T(\theta,t)\neq 0$ ($\Delta V_T(\theta,t)\neq 0$, resp.) only if t is a jump point of the function $v_T(\theta,\cdot)$ ($V_T(\theta,\cdot)$, resp.). Equations (6) and (9) require the computation of $\Delta v_T(\theta,t_T(\theta))\frac{dt_T}{d\theta}(\theta)$, where $t_T(\theta)$ serves as a generic notation for a jump point of $v_T(\theta,\cdot)$. We point out that the computation of the product-term $\Delta v_T(\theta,t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ typically is considerably simpler than the computation of each one of its multiplicands.

3.2 Event classification

The IPA algorithm and its computation of the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ are based on the notion of events in the following way. Events occur at an element of the network, be it transition or place, and they are associated with discontinuities in the flow-rate functions $v_T(\theta, \cdot)$ at some transition T. Throughout a sample path the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ evolve in a recursive manner according to the occurrence of events at the various network elements. References [21, 23, 6] defined and classified events according to three categories: exogenous, endogenous, and induced. Broadly speaking, exogenous events are discontinuities (jumps) in exogenous processes, endogenous events occur when one of the network's flow processes reaches or departs from a certain given value, and induced events are jumps in one of the network processes that are triggered by another event in the network. The literature has not yet completely settled on these definitions, and we slightly modify those in [21, 23, 6], especially regarding endogenous events, to better suit our description of the network processes.[1] The following definition is clarified by the ensuing remarks.

Definition 3.1 *1. An exogenous event is a jump (discontinuity) in an exogenous process $\{\phi(\theta, \cdot)\}$. An exogenous event is said to occur at transition T if it is a jump in $V_T(\theta, \cdot)$.*[2] *We further classify such events as type-1 exogenous if $\{\phi_T\}$ does not depend on θ, and type-2 exogenous if $\{\phi_T\}$ is a function of θ.*

2. Let $\gamma(\theta)$ be a continuously-differentiable, non-negative-valued function.

(a). A type-1 endogenous event at place p with respect to the function $\gamma(\theta)$ is said to occur at time t if $m_p(\theta, t) = \gamma(\theta)$ while $m_p(\theta, t^-) \neq \gamma(\theta)$.

(b). A type-2 endogenous event at place p with respect to the function $\gamma(\theta)$ is said to occur at time t if $m_p(\theta, t^+) \neq \gamma(\theta)$ while, for some $\Delta t > 0$, $m_p(\theta, \tau) = \gamma(\theta)$ $\forall \tau \in (t - \Delta t, t]$.

(c). An endogenous event is said to occur at a transition T if it is an endogenous event at any place $p \in in(T)$.[3]

3. Suppose that an event at a transition U, occurring at time $t_U(\theta)$, triggers a discontinuity of $v_T(\theta, \cdot)$ at a transition T at time $t_T(\theta) \geq t_U(\theta)$. We say that the latter event at T is induced by the former event at U, and we say that the pair of events at transitions U and T are triggering-induced.

We make the following remarks.

Remark 3.2 *1. We have assumed that exogenous processes are either $\{V_T(\theta, t)\}$ or the processes $\{\psi(\theta, t)\}$ upon which $V_T(\theta, t)$ depends via Equation (3). Therefore, according to Definition 3.1.1, exogenous events must be jumps in $V_T(\theta, t)$ for some transition T.*

2. An exogenous process $\{V_T\}$ may or may not be a function of θ. If $V_T(\theta, t) = V_T(t)$ is not a function of θ then discontinuities in this process are type-1 exogenous events, and in this case $\frac{dt_T}{d\theta}(\theta) = 0$. On the other hand, if $V_T(\theta, t)$ is a function of θ then jumps in this process are type-2 exogenous, and in this case the term $\frac{dt_T}{d\theta}(\theta)$ often is easily computable from the sample path at (simulated) time $t_T(\theta)$. Furthermore, we assume that the maximum flow rate at every transition is computable at every time t, and hence the term $\Delta V_T(\theta, t_T(\theta))$ can be computed as well at time $t_T(\theta)$. If $\{V_T(\theta, t)\}$ is controlled by another process then it is not an exogenous process, and its discontinuities are induced events. This will be discussed later and exemplified in Section 4.

[1] The term *network process* refers to either $\{v_T(\theta, t)\}$, $\{V_T(\theta, t)\}$ (for any transition T), or $\{m_p(\theta, t)\}$ (for any place p). We will use the generic notation $\{\phi(\theta, t)\}$ for such processes.

[2] See Remark 3.2.1, below, for clarifications.

[3] This ensures that all events are associated with transitions, thereby simplifying the ensuing presentation. Correspondingly, we will denote the occurrence time of an endogenous event at a place p by $t_T(\theta)$, where $T := out(p)$.

3. *Endogenous events associated with the function $\gamma(\theta) = 0$ signify the boundary points of empty periods in a network's places. Such events often cause a jump in $v_T(\theta, \cdot)$ for $T = out(p)$, and hence we say that they occur at T in order to simplify the notation by associating all events with transitions. Functions $\gamma(\theta)$ other than zero are used in threshold-based flow control.*

4. *Note that in type-1 endogenous events the workload $m_p(\theta, t)$ becomes equal to $\gamma(\theta)$, while in type-2 endogenous events $m_p(\theta, t)$ ceases to be equal to $\gamma(\theta)$ after having had that value for a positive amount of time. Thus, if $m_p(\theta)$ crosses that value $\gamma(\theta)$ and is equal to it at a single point, the related event is classified as type-1 endogenous.*

5. *Consider a place $p \in out(U) \cap in(T)$ for a pair of transitions U and T, namely p is between U and T. Suppose that $m_p(\theta, t) = 0$ for all t in an interval $[t_1, t_2]$, and hence $v_U(\theta, t) = v_T(\theta, t)$. Then a jump in $v_U(\theta, \cdot)$ triggers a jump in $v_T(\theta, \cdot)$ at the same time, and this is an induced event. It is possible to have a chain of such events, $e_1, \ldots, e_n$, such that e_1 is a triggering event, and for all $i = 2, \ldots, n$, e_i is induced by e_{i-1}. Such a chain is called an induced chain. Can such a chain be infinite or loop around itself? For the case of uncontrolled networks, the arguments in [22] imply that the answer is "no" as long as Equations (1) and (2) are consistent throughout the network.*

The following assumption, or minor variants thereof, are routinely made in the literature on IPA (see [5, 21, 23, 6]), as explained below.

Assumption 3.3 *Fix $\theta \in \Theta$. W.p.1, the following statements are in force.*

1. *No type-1 endogenous event can be induced. Furthermore, if such an event occurs at a place p then the function $v_{in(p)}(\theta, \cdot)$ is continuous at its occurrence time.*

2. *A type-1 endogenous event cannot occur at the same time as other events that are not in the induced chain initiated by it.*

3. *An exogenous event cannot occur at the same time as other events that are not in the induced chain initiated by it.*

4. *No empty period consists of a single time-point.*

5. *For every transition T, place $p \in in(T)$, and transition $U := in(p)$; for every open interval I contained in an empty period of p, it is impossible to have $v_U(\theta, t) = V_T(\theta, t)$ $\forall t \in I$ unless $\frac{\partial v_U}{\partial \theta}(\theta, t) = \frac{\partial V_T}{\partial \theta}(\theta, t)$ $\forall t \in I$ as well. Furthermore, for another place $\tilde{p} \in in(T)$ and $\tilde{U} := in(\tilde{p})$, and for every open interval I contained in empty periods of both p and $\tilde{p}$, it is impossible to have $v_U(\theta, t) = v_{\tilde{U}}(\theta, t)$ $\forall t \in I$ unless $\frac{\partial v_U}{\partial \theta}(\theta, t) = \frac{\partial v_{\tilde{U}}}{\partial \theta}(\theta, t)$ $\forall t \in I$ as well.*

Remark 3.4 *As argued in [21, 23, 6]), this assumption guarantees that the various network-flow processes, event-times, and sample performance functions are differentiable, at a given $\theta \in \Theta$, w.p.1. Parts 1-4 of the assumption are reasonable if there is sufficient statistical mixing in the network processes. Part 5 guarantees that the flow derivatives $\frac{\partial v_T}{\partial \theta}(\theta, t)$ exist. To see what happens if part 5 is not satisfied, consider the following example concerning a transition T, place $p \in in(T)$, and $U := in(p)$. Suppose that $v_U(\theta, t) = 1$, and $V_T(\theta, t) = \theta$, throughout an empty period of p, denoted by I. Let $\theta = 1$. Then, under a slight increase of θ, I would remain an empty period, while under slight decrease of θ, I would no longer be an empty period. Consequently, $\frac{\partial v_T}{\partial \theta^+}(\theta, t) = 0$ while $\frac{\partial v_T}{\partial \theta^-}(\theta, t) = 1$ for all $t \in I$. Although $v_T(\theta, t)$ is not differentiable in θ (at a given $t \in I$), it has one-sided derivatives.*

Under general conditions, as argued in [21], if part 5 of the assumption is not satisfied then the aforementioned sample-based functions have one-sided derivatives, and in this case the analysis in the sequel *is valid in that context. Furthermore, though it may be hard to ascertain in the abstract setting of the present discussion, part 5 may be quite simple to check for specific examples, as we shall see in Section 4.*

3.3 Computation of the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ and specification of the IPA derivatives

The computation and propagation of the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ is recursive in time and network flows, and can be carried out according to the occurrence of the various events. Driving the computation are the exogenous events and the quantities $\Delta V_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ related to them. It is assumed that these quantities are computable at the simulated times $t_T(\theta)$, and moreover that $\Delta V_T(\theta, t)$ is computable as well for every $t \in [0, t_f]$.

The main element of the recursion consists of a relation of the term $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ to either the analogous term $\Delta V_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ derived from an exogenous event, to a similar term $\Delta v_U(\theta, t_U(\theta))\frac{dt_U}{d\theta}(\theta)$ at another transition U, or to the sum of such terms computed in the past.

These relationships are linear, and in most cases with unity coefficients. For example, in the second case (above), $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta) = \alpha\Delta v_U(\theta, t_U(\theta))\frac{dt_U}{d\theta}(\theta)$ for a proportionality factor α. In most cases $\alpha = 1$, meaning that the perturbation in U is transferred to T. In other cases $\alpha = 0$, meaning that the perturbation in U is canceled at T. In few cases, associated with type-2 endogenous events, $\alpha \notin \{0, 1\}$. In such cases, which are the most complicated from a computational standpoint, α can be expressed as the ratio of transitions flow rates (as we shall see), which must be computed just before and after the events occur. However, we always have the inequality $|\alpha| \leq 1$, implying that, if these situations arise infrequently, α can be adequately approximated by 0, 1, 0.5, or a unit variate. We point out that such terms also arise in the setting of queueing networks [21], where due to their rarity could be ignored by the IPA algorithms.

We next present these linear recursive relationships according to the various events. Consider first exogenous events.

Proposition 3.5 *Consider an exogenous event occurring at a transition T at time $t_T(\theta)$. Then*

$$\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta) = \alpha\Delta V_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta), \tag{10}$$

where α has the following values: (i) If $\varepsilon_T(t_T(\theta)) = \emptyset$, namely $m_p(\theta, t_T(\theta)) > 0$ $\forall p \in in(T)$, then $\alpha = 1$. (ii) If $t_T(\theta)$ lies in the interior of an empty period of some place $p \in in(T)$, then $\alpha = 0$. (iii) If the event in question, namely the jump in $V_T(\theta, \cdot)$, results in the end of an empty period at some $p \in in(T)$, then

$$\alpha = \frac{v_T(\theta, t_T(\theta)^-) - V_T(\theta, t_T(\theta)^+)}{V_T(\theta, t_T(\theta)^-) - V_T(\theta, t_T(\theta)^+)} \tag{11}$$

and in this case $0 \leq \alpha \leq 1$.

Proof. Part (i) follows directly from (1), since $v_T(\theta, t) = V_T(\theta, t)$ for all t in some neighborhood of $t_T(\theta)$. Part (ii) follows from (1) as well since there exists $p \in T$ such that, with $U := in(p)$, $v_T(\theta, t) = v_U(\theta, t)$ for all t in a neighborhood of $t_T(\theta)$. (iii). Note that $\varepsilon_T(\theta, t_T(\theta)^+) = \emptyset$ and hence, by (1), $v_T(\theta, t_T(\theta)^+) = V_T(\theta, t_T(\theta)^+)$. Consequently $\Delta v_T(\theta, t_T(\theta)) = v_T(\theta, t_T(\theta)^-) - V_T(\theta, t_T(\theta)^+)$, and by dividing and multiplying this term by $\Delta V_T(\theta, t_T(\theta))$, Equation (11) follows. Furthermore, by Assumption 3.3(3), for every $p \in in(T)$, there is no event at the transition $U = in(p)$, and hence the only way $t_T(\theta)$ is the end-time of an empty period at p is if $v_T(\theta, t_T(\theta)^-) > v_T(\theta, t_T(\theta)^+)$. But $\varepsilon_T(\theta, t_T(\theta)^+) = \emptyset$ and hence $v_T(\theta, t_T(\theta)^+) = V_T(\theta, t_T(\theta)^+)$, while $v_T(\theta, t_T(\theta)^-) \leq V_T(\theta, t_T(\theta)^-)$ by definition of the maximum transition-flow rate. All of this implies that $0 \leq \alpha \leq 1$. $\square$

Consider next the case where an event at transition U triggers an induced event at a transition T, immediately downstream from it, through an empty period in a place between them.

Proposition 3.6 *Suppose that an event occurs at a transition U at a time $t_U(\theta)$ while a place $p \in out(U)$ is empty, and let $T := out(p)$. This event triggers an event at T at the same time, $t_T(\theta) := t_U(\theta)$. Furthermore,*

$$\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta) \;=\; \alpha \Delta v_U(\theta, t_U(\theta))\frac{dt_U}{d\theta}(\theta), \tag{12}$$

where α has the following value. (i). If $t_U(\theta)$ lies in the interior of an empty period at p, then $\alpha = 1$. (ii). If the triggering event at U is the termination of an empty period at p, then

$$\alpha \;=\; \frac{v_T(\theta, t_T(\theta)^+) - v_U(\theta, t_U(\theta)^-)}{v_U(\theta, t_U(\theta)^+) - v_U(\theta, t_U(\theta)^-)}, \tag{13}$$

and $0 \leq \alpha \leq 1$.

Proof. Since $m_p(t_U(\theta)) = 0$, Equation (1) implies that the event at U triggers an event at T at the same time. (i). By (1), it is obvious that $\alpha = 1$ since $v_T(\theta, t) = v_U(\theta, t)$ in some neighborhood of $t_U(\theta)$. (ii). Since $t_T(\theta) = t_U(\theta)$ is the end-point of an empty period at p, $v_T(\theta, t_T(\theta)^-) = v_U(\theta, t_T(\theta)^-)$, and hence Equations (12)-(13) follows from (1). Furthermore, since $t_T(\theta)$ is the end-point of an empty period at p, it follows that (a) the triggering event at U raises $v_U(\theta), \cdot)$ at $t = t_T(\theta)$, which causes $v_T(\theta, \cdot)$ to stay flat or rise there as well, implying that $v_T(\theta, t_T(\theta)^-) \leq v_T(\theta, t_T(\theta)^+)$; and (b) $v_U(\theta, t_U(\theta)^+) - v_T(\theta, t_T(\theta)^+) \geq 0$. Recalling that $v_T(\theta, t_T(\theta)^-) = v_U(\theta, t_U(\theta)^-)$, this implies that $0 \leq \alpha \leq 1$. $\square$

Notice that in the above discussion, the situation where $\alpha \notin \{0, 1\}$ arises only when the event at T, be it exogenous or induced, results in the termination of an empty period at $p \in in(T)$, and hence the event is type-2 endogenous as well. Such events will be discussed later. We also point out that while empty periods provide a common mechanism for pairs of triggering-induced events, there are other ways such event-pairs can arise in controlled networks. We will consider such events later for threshold-based flow control, where an endogenous event at a transition T triggers a change in the flow rate at another transition. The endogenous event at T means that the workload at a place $p \in in(T)$ reaches a certain threshold. This requires a computation of the term $\frac{dt_T}{d\theta}(\theta)$, which is the subject of the following result concerning type-1 endogenous events.

Proposition 3.7 *Consider a type-1 endogenous event at a place p with respect to a function $\gamma(\theta)$, and let $T := out(p)$. Define $\xi_p(\theta) := \max\{\tau < t_T(\theta) : m_p(\theta, \tau) = 0\}$; if no such τ exists, define $\xi_p(\theta) = 0$. For $U := in(p)$, let $\tau_{\ell,U}(\theta)$, $\ell = 1, \ldots, L$ denote the jump-times of the function $v_U(\theta, \cdot)$ in the interval $(\xi_p(\theta), t_T(\theta))$, and let $\tau_{m,T}(\theta)$, $m = 1, \ldots, M$ denote the jump times of the function $v_T(\theta, \cdot)$ in the same interval.*

(i). The following relation holds:

$$\begin{aligned}\frac{dt_T}{d\theta}(\theta) \;=\; & \frac{1}{\Big(v_U(\theta, t_T(\theta)) - v_T(\theta, t_T(\theta)^-)\Big)} \times \Big[\frac{d\gamma}{d\theta}(\theta) \\ & - \int_{\xi_p(\theta)}^{t_T(\theta)} \Big(\frac{\partial v_U}{\partial \theta}(\theta, \tau) - \frac{\partial v_T}{\partial \theta}(\theta, \tau)\Big) d\tau - \sum_{\ell=1}^{L} \Delta v_U(\theta, \tau_{\ell,U}(\theta))\frac{d\tau_{\ell,U}}{d\theta}(\theta) \\ & + \sum_{m=1}^{M} \Delta v_T(\theta, \tau_{m,T}(\theta))\frac{d\tau_{m,T}}{d\theta}(\theta) + \Big(v_U(\theta, \xi_p(\theta)^+) - v_T(\theta, \xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta)\Big].\end{aligned} \tag{14}$$

(ii). In the special case where $\gamma(\theta) = 0$, namely the endogenous event in question is the start of an empty period at p,

$$\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta) \;=\; \int_{\xi_p(\theta)}^{t_T(\theta)} \Big(\frac{\partial v_U}{\partial\theta}(\theta,\tau) - \frac{\partial v_T}{\partial\theta}(\theta,\tau)\Big)d\tau$$
$$+\sum_{\ell=1}^{L}\Delta v_U(\theta,\tau_{\ell,U}(\theta))\frac{d\tau_{\ell,U}}{d\theta}(\theta) - \sum_{m=1}^{M}\Delta v_T(\theta,\tau_{m,T}(\theta))\frac{d\tau_{m,T}}{d\theta}(\theta)$$
$$-\Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta). \tag{15}$$

Proof. (i). Since place p is nonempty throughout the interval $(\xi_p(\theta), t_T(\theta))$, Equation (2) implies that

$$m_p(\theta, t_T(\theta)) \;=\; \int_{\xi_p(\theta)}^{t_T(\theta)} \Big(v_U(\theta,\tau) - v_T(\theta,\tau)\Big)d\tau. \tag{16}$$

Moreover, $m_p(\theta, t_T(\theta)) = \gamma(\theta)$. Plugging this in (16) and taking derivatives with respect to θ we obtain,

$$\frac{d\gamma}{d\theta}(\theta) \;=\; \Big(v_U(\theta, t_T(\theta)) - v_T(\theta, t_T(\theta)^-)\Big)\frac{dt_T}{d\theta}(\theta)$$
$$+\int_{\xi_p(\theta)}^{t_T(\theta)} \Big(\frac{\partial v_U}{\partial\theta}(\theta,\tau) - \frac{\partial v_T}{\partial\theta}(\theta,\tau)\Big)d\tau + \sum_{\ell=1}^{L}\Delta v_U(\theta,\tau_{\ell,U}(\theta))\frac{d\tau_{\ell,U}}{d\theta}(\theta) -$$
$$\sum_{m=1}^{M}\Delta v_T(\theta,\tau_{m,T}(\theta))\frac{d\tau_{m,T}}{d\theta}(\theta) - \Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_T}{d\theta}(\theta), \tag{17}$$

where we recall that $v_U(\theta,\cdot)$ is continuous at $t = t_T(\theta)$ by Assumption 3.3(1). Now Equation (14) follows from (17) after some algebra.

(ii). Since the endogenous event in question is the start of an empty period at p we have that $\gamma(\theta) = 0$ and $\frac{d\gamma}{d\theta}(\theta) = 0$ as well. Moreover, we have that $v_U(\theta, t_T(\theta)^+) = v_T(\theta, t_T(\theta)^+)$, and hence $v_U(\theta, t_T(\theta)) - v_T(\theta, t_T(\theta)^-) = -\Delta v_T(\theta, t_T(\theta))$. Plugging this in (14), Equation (15) follows. □

Equation (15) provides a recursive structure for the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ in time, as can be seen from the sum-terms in its RHS. The integral term there typically can be computed easily. However, the last term requires further attention. This term, corresponding to the end of an empty period at place p at the time $\xi_p(\theta)$, depends on the way the empty period is terminated. It can be the result of a continuous change in the function $v_U(\theta,t) - v_T(\theta,t)$ about $t = \xi_p(\theta)$, or a discontinuity of that function at the point $t = \xi_p(\theta)$. In the latter case we have a type-2 endogenous event at p, and the term $\big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\big)\frac{d\xi_p}{d\theta}(\theta)$ depends on the specific event. The following result specifies this term accordingly.

Proposition 3.8 *Consider a transition T and a place $p \in in(T)$, and let $U := in(p)$. Suppose that $\xi_p(\theta)$ is the end-time of an empty period at p. The term $\Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta)$ is computable according to the following situations: (i). The functions $v_U(\theta,t)$ and $v_T(\theta,t)$ are continuous about $t = \xi_p(\theta)$. Then $\Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta) = 0$. (ii). The end of the empty period at p is due to an exogenous event at T. Then*

$$\Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta) \;=\; \alpha_1\Delta V_T(\theta,\xi_p(\theta))\frac{d\xi_p}{d\theta}(\theta) \tag{18}$$

with

$$\alpha_1 \;:=\; \frac{v_U(\theta,\xi_p(\theta)^+) - V_T(\theta,\xi_p(\theta)^+)}{V_T(\theta,\xi_p(\theta)^-) - V_T(\theta,\xi_p(\theta)^+)}, \tag{19}$$

and α_1 satisfies the inequalities $0 \leq \alpha_1 \leq 1$. (iii). The end of the empty period at p is an induced event that is triggered by an event at U where $v_U(\theta,\cdot)$ is discontinuous at $\xi_p(\theta)$. Then

$$\Big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\Big)\frac{d\xi_p}{d\theta}(\theta) \;=\; \alpha_2 \Delta v_U(\theta,\xi_p(\theta))\frac{d\xi_p}{d\theta}(\theta), \tag{20}$$

with

$$\alpha_2 \;:=\; -\frac{v_T(\theta,\xi_p(\theta)^+) - v_U(\theta,\xi_p(\theta)^+)}{v_U(\theta,\xi_p(\theta)^-) - v_U(\theta,\xi_p(\theta)^+)}, \tag{21}$$

and α_2 satisfies the inequalities $-1 \leq \alpha_2 \leq 0$.

Proof. Part (i) is obvious since $v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+) = v_U(\theta,\xi_p(\theta)) - v_T(\theta,\xi_p(\theta))$, and $v_U(\theta,\xi_p(\theta)) - v_T(\theta,\xi_p(\theta)) = 0$ because an empty period ends at time $t = \xi_p(\theta)$.

(ii). Equations (18) and (19) are obtained by dividing and multiplying the term $\big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\big)\frac{d\xi_p}{d\theta}(\theta)$ by the term $V_T(\theta,\xi_p(\theta)^-) - V_T(\theta,\xi_p(\theta)^+)$. Next, by Assumption 3.3(3), the function $v_U(\theta,t)$ is continuous at time $t = \xi_p(\theta)$, and hence $v_U(\theta,\xi_p(\theta)^-) = v_U(\theta,\xi_p(\theta)) = v_U(\theta,\xi_p(\theta)^+)$. Therefore α_1 is the term α in Equation (11), and by Proposition 3.5(iii), $0 \leq \alpha_1 \leq 1$.

(iii). Equations (20) and (21) are obtained by dividing and multiplying the term $\big(v_U(\theta,\xi_p(\theta)^+) - v_T(\theta,\xi_p(\theta)^+)\big)\frac{d\xi_p}{d\theta}(\theta)$ by the term $v_U(\theta,\xi_p(\theta)^-) - v_U(\theta,\xi_p(\theta)^+)$. Next, a bit of algebra shows that $1+\alpha_2$ is the term α in Equation (13). By Proposition 3.6(ii), $0 \leq \alpha \leq 1$, and therefore, $-1 \leq \alpha_2 \leq 0$. □

Propositions 3.5-3.8 describe the dynamic evolution of the terms $\Delta v_T(\theta,t_T(\theta))\frac{dt_T}{d\theta}(\theta)$, which is recursive in both time and network flows. This description is complete for uncontrolled networks, however, it must be further specified for the case of controlled networks. Of course no single paper can cover all possible types of network controls, but we discuss here a class of controls that are of interest to us and may arise in applications in transportation and manufacturing.

This class consist of threshold flow control, where the maximum firing rate at a given transition, T, is modulated by the workload volume at a certain place, p. The threshold at p can be dependent of θ and hence is denoted by $\gamma_p(\theta)$. Given two random functions $V_{T,1}(\theta,t)$ and $V_{T,2}(\theta,t)$, that are continuously differentiable in (θ,t), the control is defined via the relation $V_T(\theta,t) = V_{T,1}(\theta,t)$ or $V_T(\theta,t) = V_{T,2}(\theta,t)$, depending on whether $m_p(\theta,t) < \gamma_p(\theta)$ or $m_p(\theta,t) > \gamma_p(\theta)$. In order to avoid chattering we may have to specify a third function, $V_{T,3}(\theta,t)$, such that $V_T(\theta,t) = V_{T,3}(\theta,t)$ whenever $m_p(\theta,t) = \gamma_p(\theta)$; an example will clearly illustrate this point in the sequel. Thus, the control has the following form,

$$V_T(\theta,t) \;=\; \begin{cases} V_{T,1}(\theta,t), & \text{if } m_p(\theta,t) < \gamma_p(\theta), \\ V_{T,2}(\theta,t), & \text{if } m_p(\theta,t) > \gamma_p(\theta), \\ V_{T,3}(\theta,t), & \text{if } m_p(\theta,t) = \gamma_p(\theta). \end{cases} \tag{22}$$

This kind of control results in induced events at T that are triggered by the endogenous events at p, comprised of $m_p(\theta,t)$ becoming equal to, or ceasing to be $\gamma_p(\theta)$. There is no direct relationship, similar to the ones explored by Propositions 3.5-3.8, between the term $\Delta v_T(\theta,t_T(\theta))\frac{dt_T}{d\theta}(\theta)$ and the analogous term at $U := out(p)$. Instead, we have to compute the terms $\Delta V_T(\theta,t_T(\theta))$ and $\frac{dt_T}{d\theta}(\theta)$ separately. The former term, we assumed, can be computed directly (and easily) from the sample path since $V_T(\theta,t)$ is a maximum transition flow-rate process. The letter term is computable via Equation (14) (Proposition 3.7) in the case of type-1 endogenous events, and via the various ways for handling type-2 endogenous events that are induced or exogenous, as specified in the proofs of Propositions 3.5 and 3.6. All of this will be exemplified and made clear in Section 4.

Finally, to complete the description of the IPA derivatives, we have to specify the terms $\frac{\partial v_T}{\partial \theta}(\theta,t_T(\theta))$ in Equations (6) and (9). For the maximum transition firing-rate processes, we can assume that the terms

$\frac{\partial V_T}{\partial \theta}(\theta, t_T(\theta))$ are computable directly from the sample path, be they exogenous or controlled; while the terms $\frac{\partial v_T}{\partial \theta}(\theta, t_T(\theta))$ can be computed recursively by Equation (1) in an obvious way.

We close this subsection by mentioning that the aforementioned formulae for the IPA derivatives, and especially Equations (14) and (15), can be derived from the general framework presented in [6]. However, that framework is quite abstract and hence the derivations would be tedious. Instead, the derivations here, made simple and direct by using the special structure of marked graphs, lead to a natural recursive computation that is based on the terms $\Delta v_T(\theta, t_T(\theta))\frac{dt_T}{d\theta}(\theta)$.

3.4 Unbiasedness of the IPA derivatives

The main purpose of the IPA derivatives $\frac{dL}{d\theta}(\theta)$ is to provide unbiased statistical estimators for $\frac{dJ}{d\theta}(\theta)$, and this is ensured by the equation $E\big(\frac{dL}{d\theta}(\theta)\big) = \frac{dJ}{d\theta}(\theta)$. In this case we say that the IPA derivatives are unbiased. Since $J(\theta) = E\big(L(\theta)\big)$, unbiasedness means that the operators of expectation and pointwise differentiation with respect to θ are interchangeable, namely that $E\big(\frac{dL}{d\theta}(\theta)\big) = \frac{d}{d\theta}\big(E(L(\theta))\big)$. As pointed out in [3], the unbiasedness of IPA is ensured by the following two conditions: 1). For every $\theta \in \Theta$, w.p.1 the sample derivative $\frac{dL}{d\theta}(\theta)$ exists. (2). W.p.1, the function $L(\theta)$ is Lipschitz continuous on its domain Θ, and its Lipschitz constant has a finite first moment.

The study of stochastic flow networks in relation to IPA was primarily motivated by the realization that the IPA derivatives in their setting is unbiased for a far-larger class of systems than in the setting of the traditional (discrete) queueing networks [4, 20]. However, counterexamples exist [9], and this behooves us to ascertain the unbiasedness of the IPA derivatives derived in this paper. In this regard our first result concerns uncontrolled networks, and it has a general scope. For controlled networks we do not believe that analogous results hold true without specifying particular characteristics of the controls, and such a study is beyond the scope and size of a single paper. Instead, we prove unbiasedness for the example of threshold-flow control that is considered in the next section; while having apparent general features that may broaden the scope of the analysis, we defer a more comprehensive treatment of this problem to a later publication.

Consider an uncontrolled continuous marked graph defined by Equations (1) and (2), where $\theta \in \Theta \subset R$ is a variable parameter of the maximum transitions' firing rates, $V_T(\theta, t)$, at one or more transitions T. Suppose that the initial workload (marking) $m_p(0)$ is given for every place p, and it is independent of θ. For every transition T, define the mapping $\mathcal{V}_T : \Theta \to L^1[0, t_f]$ as follows: $\mathcal{V}_T(\theta)$ is the function (of t) $V_T(\theta, t)$. We make the following assumption.

Assumption 3.9 *For every transition T, the following holds:*

1. *W.p.1, for every $\theta \in \Theta$, the function $V_T(\theta, t)$ (as a function of time) is piecewise continuous and piecewise continuously-differentiable, and its partial derivative $\frac{\partial V_T}{\partial t}(\theta, t)$ has its sign changed a finite number of times in the interval $[0, t_f]$.*

2. *W.p.1, the mapping $\mathcal{V}_T : \Theta \to L^1[0, t_f]$ is Lipschitz continuous, and the Lipschitz constant has a finite first moment.*

3. *Every elementary circuit in the network has at least one place containing a positive amount of fluid at time $t = 0$.*

Proposition 3.10 *Suppose that Assumptions 3.3 and 3.9 are satisfied. Then for every transition T and place p, the IPA derivatives $\frac{dL_T}{d\theta}(\theta)$ and $\frac{dL_p}{d\theta}(\theta)$ are unbiased.* □

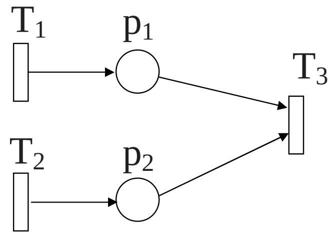

Figure 1: The Petri net system considered in Section 4.

The proof is technically involved and hence relegated to the appendix.

We close this section by mentioning that an analogous result holds true for the case where θ is a parameter of the initial marking, and its proof is similar to that of Proposition 3.10.

4 Example

Consider the Petri net shown in Figure 1, where transition T_3 acts as a join operation for the fluid flows through the source transitions T_1 and T_2.[4] This network, though simple, highlights the notions of concurrency and synchronization that are inherent in Petri nets, and therefore its analysis captures essential elements of the recursive structure of IPA presented in Section 3. To simplify the notation we will refer to the flow rates through the transitions as $V_i(\theta,t) := V_{T_i}(\theta,t)$ and $v_i(\theta,t) := v_{T_i}(\theta,t)$, $i = 1,2,3$, and to the places' workload as $m_j(\theta,t) := m_{p_j}(\theta,t)$, $j = 1,2$.

The network can adequately model the dynamics of inventories and backlogging in a single-stage manufacturing system. In this setting T_3 represents the production process, T_1 represents the process of orders of finished products, and T_2 represents the arrival process of raw material (parts) to the system. Production takes place only when there are standing orders for finished products and parts to match them, and hence there are no finished-product inventories. However, input (parts') inventories build up when there are parts that are not matched by standing orders, and similarly, backorders arise whenever there are standing product-demands without sufficient raw material to match them. The fluid contents in the places p_1 and p_2 represent the backorders and inventory levels, respectively.

In many practical situations the product-order process is unpredictable, and the plant manager is facing the challenge of balancing backorder and inventory costs by a suitable choice of a policy for scheduling the procurement and delivery of parts. Such a balance can be quantified by a performance function comprised of a weighted sum of the costs associated with backorder and inventory, respectively, during a given production horizon. When the policies are parameterized by a continuous variable θ, IPA can come into play for computing (or approximating) a value of θ that minimizes the performance function.

The forthcoming example concerns a feedback law where $\theta > 0$ is a threshold parameter of the backorder level which determines the value of $V_2(\theta,t)$.[5] Suppose that $\{V_1(t)\}$ is an exogenous processes that is independent of θ, and given two constants $V_{2,1} > 0$ and $V_{2,2} > V_{2,1}$, define $V_2(\theta,t) = V_{2,1}$ if $m_1(\theta,t) < \theta$,

[4]the material in this section excluding unbiasedness of IPA (Proposition 4.2, below), was presented in [12].

[5]It is a common practice in production management to control safety stock levels by a base threshold of the inventories themselves and not the backorder levels. However, the latter arguably can be justified in certain situations, although we are not aware of a current application. Nonetheless we consider it here as an example of our IPA framework in the setting of a controlled network. Practically both controls can be applied concurrently, and we defer the question of determining the optimal threshold values for such a combined approach to a later study.

and $V_2(\theta,t) = V_{2,2}$ as long as $m_1(\theta,t) > \theta$. We assume that the process $\{V_3(\theta,t)\}$ is deterministic and constant, namely $V_3(\theta,t) = V_3$ for a given constant $V_3 > 0$. We make the reasonable assumption that $V_{2,1} < V_3 \leq V_{2,2}$.

The purpose of such a control law is to regulate the process $\{V_2(\theta,t)\}$ by having it switch from $V_{2,1}$ to $V_{2,2}$ whenever m_1 crosses the threshold value θ in the upward direction, and vice versa if m_1 crosses θ downwards. However, this may give rise to the Zeno phenomenon when $m_1(\theta,t) = \theta$ as is evident from the following scenario: m_1 rises to θ and then it attempts to decline due to the resulting increase in V_2; consequently it tries to rise again, etc. This chattering phenomenon is due to the fact that the traffic flows are characterized by rates as opposed to the movement of discrete jobs. Such a situation can arise only whenever $m_1(\theta,t) = \theta$ while $m_2(\theta,t) = 0$ and $V_{2,1} \leq V_1(t) \leq V_3$. In this case, of course, $V_2(\theta,t) = V_1(t)$ as long as the chatter continues, and $m_1(\theta,t)$ remains equal to θ. To include this phenomenon in the model, we define $V_2(\theta,t)$ via the following threshold-control law,

$$V_2(\theta,t) \;=\; \begin{cases} V_{2,1}, & \text{if } m_1(\theta,t) < \theta \\ V_{2,2}, & \text{if } m_1(\theta,t) > \theta \\ V_1(t), & \text{if } m_1(\theta,t) = \theta, \;\; m_2(\theta,t) = 0, \\ & \text{and } V_{2,1} \leq V_1(t) \leq V_3 \\ V_{2,1}, & \text{under all other circumstances;} \end{cases} \tag{23}$$

it is readily seen that the equations (1), (2), and (23) are consistent throughout the network.

The following cost function reflects on a balance between parts' inventories and products' backlogging,

$$L(\theta) \;:=\; \int_0^{t_f} \big(C_1 m_1(\theta,t) + C_2 m_2(\theta,t)\big)dt \tag{24}$$

for given constants $C_1 > 0$, $C_2 > 0$, and $t_f > 0$. In the rest of this section we define an example, describe for it the structures of the IPA derivative $\frac{dL}{d\theta}(\theta)$, and provide simulation results for minimizing $J(\theta) := E\big(L(\theta)\big)$.

In the simulation example that we consider it is assumed that the process $\{V_1(t)\}$ is bursty and hence modeled as a sequence of impulses whose timing is a point process. Thus, this process has the form

$$V_1(t) = \sum_{n=1}^{\infty} \alpha_n \delta(t - s_n), \tag{25}$$

where $\delta(\cdot)$ is the Dirac delta function, the weighting factors $\alpha_n > 0$, $n = 1, 2, \ldots$, constitutes a stochastic process, and the time-point process s_n, $n = 1, 2, \ldots$, is a monotone-increasing random process. We mention that s_n may be larger than t_f for n large enough; in this case we define $n_f := \max\{n = 1, \ldots : n < t_f\}$ and we note that $V_1(t)$ has only n_f impulses in the interval $[0, t_f]$.

Observe that Equation (25) excludes the possibility of the third and fourth cases in Equation (23) and reduces it to the following form,

$$V_2(\theta,t) \;=\; \begin{cases} V_{2,1}, & \text{if } m_1(\theta,t) \leq \theta \\ V_{2,2}, & \text{if } m_1(\theta,t) > \theta. \end{cases} \tag{26}$$

To ensure stability of the control law defined by (26) we assume, in addition to the inequalities $V_{2,1} < V_3 \leq V_{2,2}$, that $\frac{1}{V_3}\big(V_{2,2} - V_{2,1}\big) < 1$, since this fraction-term is the loop gain.

We also mention that neither process $\{\alpha_n\}$ nor the increment-process $\{s_n - s_{n-1}\}$ need be iid, and all that we require is that with these processes, Assumption 3.3 is satisfied. This would be the case if these two processes are exogenous, and the conditional distributions of $\alpha_n|\alpha_{n-1}$ and $(s_n - s_{n-1})|s_{n-1}$ have uniformly-bounded density functions.

The IPA derivative is computable via Equations (7) and (9) in conjunction with the results in Section 3, where the corresponding events have the following forms.

- *Exogenous events.* All of the exogenous events are jumps in $V_1(t)$; these are type-1 events and hence $\frac{dt_1}{d\theta}(\theta) = 0$.
- *Type-1 endogenous events.* There are only the following three possibilities: (i) Start of an empty period at p_1; (ii) start of an empty period at p_2, and (iii) $m_1(\theta, \cdot)$ crosses θ downwards.

  Case (i): $v_1(t_3(\theta)) = 0$, and

$$v_3(\theta, t_3(\theta)^-) = \begin{cases} V_{2,1}, & \text{if } m_2(\theta, t_3(\theta)) = 0 \\ V_3, & \text{if } m_2(\theta, t_3(\theta)) > 0. \end{cases} \tag{27}$$

  The term $\Delta v_3(\theta, t_3(\theta))\frac{dt_3}{d\theta}(\theta)$ is computable by Equation (15), in whose RHS the integral term is zero and all other terms are known (having been computed) by time $t = t_3(\theta)$. Furthermore, if $m_2(\theta, t_3(\theta)) = 0$, this event triggers the end of an empty period at p_2, and in this case $v_3(\theta, t_3(\theta)^+) = 0$ and $v_2(\theta, t_3(\theta)^+) = V_{2,1}$. (These quantities may serve as the term $v_T(\theta, \xi_p(\theta)^+)$ at the end of Equation (15) for the next type-1 endogenous event at p_2.)

  Case(ii): $v_2(\theta, t_3(\theta)) = V_{2,1}$, and $v_3(\theta, t_3(\theta)^-) = V_3$. It is impossible to have $m_1(\theta, t_3(\theta)) = 0$.

  Case (iii): $v_1(\theta, t_3(\theta)) = 0$, and $v_3(\theta, t_3(\theta)^-) = V_{2,2}$. Furthermore, this event triggers an induced event at T_2, where $v_2(\theta, t_2(\theta)^-) = V_{2,2}$ and $v_2(\theta, t_2(\theta)^+) = V_{2,1}$. In the RHS of (14), the integral term is zero, and $\frac{d\gamma}{d\theta}(\theta) = 1$.
- *Type-2 endogenous events.* Only the following two situations are possible: (i) End of an empty period at p_1, and (ii) end of an empty period at p_2.

  Case (i) is the result of a jump in $V_1(\cdot)$, which is a type-1 exogenous event (as discussed earlier) and hence $\frac{dt_3}{d\theta}(\theta) = 0$.

  Case (ii) must be triggered by the start of an empty period at p_1, which is a type-1 endogenous event as discussed in the previous paragraph.
- *Induced events.* Only the following three situations are possible: (i) Jump in $v_3(\theta, \cdot)$ triggered by a jump in $v_1(\cdot)$ while p_1 is empty; (ii) jump in $v_3(\theta, \cdot)$ triggered by a jump in $v_2(\theta, \cdot)$ while p_2 is empty; and (iii) jump in $v_2(\theta, \cdot)$ induced by $m_1(\theta, \cdot)$ crossing the value of θ.

  Case (i) is of a type-1 exogenous event and hence $\frac{dt_3}{d\theta}(\theta) = 0$.

  Case (ii): The jump in $v_2(\theta, \cdot)$ must be upwards and hence $v_2(\theta, t_2(\theta)^-) = V_{2,1}$ while $v_2(\theta, t_2(\theta)^+) = V_{2,2}$, and this event is triggered by a jump in $m_1(\theta, \cdot)$ upward across θ; it is a type-1 exogenous event and hence $\frac{dt_3}{d\theta}(\theta) = 0$.

  Case (iii): An upward jump must be triggered by an upward jump up in $m_1(\theta, t)$ across θ, which is a type-1 exogenous event and hence $\frac{dt_2}{d\theta}(\theta) = 0$. On the other hand, a jump down in $V_2(\theta, \cdot)$ is triggered by a type-1 endogenous event at p_1, as described in the previous paragraph.

The unbiasedness of the IPA derivative will be derived from the following assumption. Clearly part 1 of it can be relaxed.

Assumption 4.1 *1. The conditional distributions of $\alpha_n|\alpha_{n-1}$ and $(s_n - s_{n-1})|s_{n-1}$ have probability density functions on their respective supports, and these density functions have a common upper bound.*

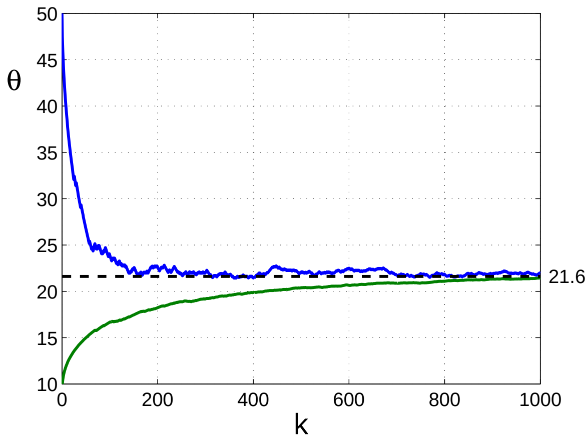

Figure 2: Results of optimization runs: θ vs. k.

2. The following inequality holds,

$$E\Big(\int_0^{t_f} V_1(t)dt\Big) < \infty. \tag{28}$$

Proposition 4.2 *Suppose that Assumption 4.1 is satisfied, and that* $\frac{1}{V_3}\big(V_{2,2} - V_{2,1} < 1$. *Then the IPA derivative* $\frac{dL}{d\theta}(\theta)$ *is unbiased.* □

For a proof, please see the appendix.

We tested IPA in conjunction with an optimization problem of minimizing $J(\theta) := E\big(L(\theta)\big)$, where $L(\theta)$ is defined by (24) with the initial marking $m_1(\theta,0) = m_2(\theta,0) = 0$. The problem has the following specifications: $V_{2,1} = 2$, $V_{2,2} = 6$, and $V_3 = 6$; s_n, $n = 1, 2, \ldots$, are equally spaced with increments of 10 time units, i.e., $s_n = 10n$, and α_n, $n = 1, 2, \ldots$, are mutually-independent random variables having the exponential distribution with the mean of 50. The final time is $t_f = 1,000$ time units, and the weighting factors in (24) are $C_1 = C_2 = 1.0$.

The optimization algorithm that we used is a stochastic-approximation method of the Robbins-Monro type [15]. In a general setting of sample-based optimization, this algorithm computes an iteration-sequence $\{\theta_k\}_{k=1}^{\infty}$ by the following formula,

$$\theta_{k+1} \;=\; \theta_k - \lambda_k h_k, \tag{29}$$

where $-h_k$ is a random descent direction and $\lambda_k > 0$ is a step size.[6] The initial iteration point θ_1 is chosen from Θ in an arbitrary fashion. It is well known (see [15]) that under broad assumptions and conditions, the iteration sequence $\{\theta_k\}_{k=1}^{\infty}$ converges with probability 1 to a local minimum of the function $J(\theta) := E\big(L(\theta)\big)$. One of the main conditions on the step-size sequence is that $\sum_{k=1}^{\infty} \lambda_k = \infty$ while $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$, and after some experimentation we chose $\lambda_k = 0.5/k^{0.6}$.

The results of two runs for 1,000 iterations each, with the respective initial-parameter values of $\theta_1 = 10$ and $\theta_1 = 50$, are shown in Figure 3, and they indicate convergence to about $\theta = 21.6$. Independent verification via repeated simulations (not shown here) supported this result by indicating that the minimum of $J(\theta)$ indeed was at or about the value obtained by the optimization algorithm.

[6] Guards must be put in place to ensure that every point θ_k remains in the set $\Theta := [0, C]$. This can be achieved by projecting the RHS of (29) onto Θ if need be, but we did not encounter this situation in the run of the algorithm.

5 Conclusions

This paper extends a recent framework for IPA in the setting of stochastic flow models to a class or marked event graphs. It considers uncontrolled networks as well as networks with threshold-based flow control. Its main results concern a characterization of the IPA derivative process in abstract terms, and an event-based recursive framework for its computation. Future research will focus on applications to manufacturing and other areas, the identification of classes of systems having special structures that admit special-purpose algorithms, and extensions of the technique to Petri nets that are not marked graphs.

6 Appendix

This section provides proofs to Proposition 3.10 and Proposition 4.2.

Proof of Proposition 3.10. Recall that $L_T(\theta)$ and $L_p(\theta)$ are defined by Equations (4) and (5), and that their IPA derivatives are given by Equations (6), (7), and (9). By Assumption 3.3, these IPA derivatives exists w.p.1 at every given $\theta \in \Theta$, and thus, it suffices to prove that the sample functions $L_T(\theta)$ and $L_p(\theta)$ have a Lipschitz constant throughout Θ having a finite first moment. This is what we next do.

Fix $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$. For a transition T, place p, and time t, we define the terms $\delta V_T(t) := V_T(\theta_2, t) - V_T(\theta_1, t)$, $\delta v_T(t) := v_T(\theta_2, t) - v_T(\theta_1, t)$, and $\delta m_p(t) := m_p(\theta_2, t) - m_p(\theta_1, t)$. By Assumption 3.9.1 and the network equations (1) and (2), we can divide the time-interval $[0, t_f)$ into a finite or infinite sequence of consecutive subintervals, $[t_i, t_{i+1})$, $i = 0, 1, \ldots$, with $t_0 := 0$ and $\cup_{i \geq 0}[t_i, t_{i+1}) = [0, t_f)$, such that for every $i \geq 0$, either one of the following four situations arises at every $t \in (t_i, t_{i+1})$: (i) $\varepsilon_T(\theta_2, t) = \varepsilon_T(\theta_1, t) = \emptyset$ (recall that $\varepsilon_T(\theta, t)$ is the set of places $p \in in(T)$ such that $m_p(\theta, t) = 0$); (ii) $\varepsilon_T(\theta_2, t) = \emptyset$, and there exists a place p such that for every $t \in (t_i, t_{i+1})$, $p \in \varepsilon_T(\theta_1, t)$; (iii) $\varepsilon_T(\theta_1, t) = \emptyset$, and there exists a place p such that for every $t \in (t_i, t_{i+1})$, $p \in \varepsilon_T(\theta_2, t)$; (iv) There exist places p_1 and p_2 (possibly, but not necessarily, $p_1 = p_2$) such that for every $t \in (t_i, t_{i+1})$, $p_1 \in \varepsilon_T(\theta_1, t)$ and $p_2 \in \varepsilon_T(\theta_2, t)$. We next analyze these four potential situations to derive recursive bounds on the terms $\int_{t_i}^{t_{i+1}} \delta v_T(t)dt$ that will serve as a basis for the argument underscoring the proof.

The following will be shown: In case (i),

$$\int_{t_i}^{t_{i+1}} \delta v_T(t)dt \ = \ \int_{t_i}^{t_{i+1}} \delta V_T(t)dt. \tag{30}$$

In case (ii), with $U := in(p)$,

$$\int_{t_i}^{t_{i+1}} \delta V_T(t)dt \leq \int_{t_i}^{t_{i+1}} \delta v_T(t)dt \leq \delta m_p(t_i) + \int_{t_i}^{t_{i+1}} \delta v_U(t)dt. \tag{31}$$

In case (iii), with $U := in(p)$,

$$\delta m_p(t_i) + \int_{t_i}^{t_{i+1}} \delta v_U(t)dt \leq \int_{t_i}^{t_{i+1}} \delta v_T(t)dt \leq \int_{t_i}^{t_{i+1}} \delta V_T(t)dt. \tag{32}$$

In case (iv), with $U_1 := in(p_1)$ and $U_2 := in(p_2)$,

$$\int_{t_i}^{t_{i+1}} \delta v_{U_2}(t)dt + \delta m_{p_2}(t_i) \leq \int_{t_i}^{t_{i+1}} \delta v_T(t)dt \leq \int_{t_i}^{t_{i+1}} \delta v_{U_1}(t)dt + \delta m_{p_1}(t_i). \tag{33}$$

The proofs are as follows. In case (i), Equation (30) is obvious from (1) since $v_T(\theta_j, t) = V_T(\theta_j, t)$ for $j = 1, 2$, and for all $t \in [t_i, t_{i+1})$.

Case (ii): Note that for every $t \in [t_i, t_{i+1})$, $v_T(\theta_2, t) = V_T(\theta_2, t)$ while $v_T(\theta_1, t) = v_U(\theta_1, t)$, and hence $\delta v_T(t) = V_T(\theta_2, t) - v_U(\theta_1, t)$. Subtracting and adding $V_T(\theta_1, t)$, we get that $\delta v_T(t) = \delta V_T(t) + V_T(\theta_1, t) - v_U(\theta_1, t)$. Since $m_p(\theta_1, t) = 0\ \forall t \in [t_i, t_{i+1})$, we have that $v_U(\theta_1, t) \leq V_T(\theta_1, t)$; hence (and by the former equation) $\delta v_T(t) \geq \delta V_T(t)$. Integrating over $t \in [t_i, t_{i+1})$, the left inequality of (31) follows.

Next, by (2),

$$m_p(\theta_2, t_{i+1}) = m_p(\theta_2, t_i) + \int_{t_i}^{t_{i+1}} \big(v_U(\theta_2, t) - V_T(\theta_2, t)\big)dt. \tag{34}$$

Also, $\delta v_T(t) = v_T(\theta_2, t) - v_T(\theta_1, t) = V_T(\theta_2, t) - v_U(\theta_1, t) = V_T(\theta_2, t) + \delta v_U(t) - v_U(\theta_2, t)$. Integrating over $t \in [t_i, t_{i+1}]$ and using (34), it follows that

$$\int_{t_i}^{t_{i+1}} \delta v_T(t)dt = \int_{t_i}^{t_{i+1}} \delta v_U(t)dt + m_p(\theta_2, t_i) - m_p(\theta_2, t_{i+1}). \tag{35}$$

But $m_p(\theta_2, t_i) = \delta m_p(t_i)$ (since $m_p(\theta_1, t_i) = 0$) and $m_p(\theta_2, t_{i+1}) \geq 0$, hence the right inequality of (31) follows.

Case (iii): The assumption that $\theta_1 \leq \theta_2$ was used to define the difference terms $\delta V_T(t)$, $\delta v_T(t)$, and $\delta m_p(t)$, but we observe that Equation (31) is true whether $\theta_1 \leq \theta_2$ or $\theta_2 \leq \theta_1$. Therefore, in case (iii), (32) follows from Equation (31) of case (ii) by swapping θ_1 with θ_2.

Case (iv): For every $t \in [t_i, t_{i+1}]$, $v_T(\theta_2, t) = v_{U_2}(\theta_2, t)$ and $v_T(\theta_1, t) = v_{U_1}(\theta_1, t)$, and therefore $\delta v_T(t) = v_{U_2}(\theta_2, t) - v_T(\theta_1, t) = \delta v_{U_2}(t) + v_{U_2}(\theta_1, t) - v_T(\theta_1, t)$. Integrating over $t \in [t_i, t_{i+1}]$ and using (2) while recalling that $U_2 = in(p_2) \in in(T)$, we get that

$$\int_{t_i}^{t_{i+1}} \delta v_T(t)dt = \int_{t_i}^{t_{i+1}} \delta v_{U_2}(t)dt + m_{p_2}(\theta_1, t_{i+1}) - m_{p_2}(\theta_1, t_i). \tag{36}$$

But $m_{p_2}(\theta_1, t_{i+1}) \geq 0$, and $m_{p_2}(\theta_2, t_i) = 0$ and hence $m_{p_2}(\theta_1, t_i) = -\delta m_{p_2}(t_i)$; consequently, and by (36), the left inequality of (33) follows. The right inequality is derivable by analogous arguments. This completes the proof of Equations (30)-(33).

We next show that for every transition T and $i = 1 = 0, 1, \ldots$, there exists a transition U such that

$$\int_0^{t_{i+1}} \delta v_T(t)dt \leq \int_0^{t_i} \delta v_U(t)dt + \int_{t_i}^{t_{i+1}} \delta V_U(t)dt. \tag{37}$$

Fix T and $i = 0, 1, \ldots$. Then either one of the following two situations must arise: (I). Either case (i) or (iii) occurs in the interval $[t_i, t_{i+1})$, and hence $\varepsilon_T(\theta_1, t) = \emptyset\ \forall t \in (t_i, t_{i+1})$. (II). Either case (ii) or (iv) occurs in the interval $[t_i, t_{i+1})$, and hence there exists a place p_1 such that for every $t \in (t_i, t_{i+1})$, $p_1 \in \varepsilon_T(\theta_1, t)$. Suppose that situation (II) arises, and let $U_1 := in(p_1)$. Now the same reasoning is applied to U_1: either situation (I) or (II) arises in the interval $[t_i, t_{i+1})$. Again if situation (II) arises, then there exists a place p_2 such that for every $t \in (t_i, t_{i+1})$, $p_2 \in \varepsilon_{U_1}(\theta_1, t)$. Continuing in this way, with the notation $U_0 := T$, we have the following: There exist sequences of transitions U_j and places p_{j+1}, $j = 0, \ldots$, such that for every j, $p_{j+1} \in \varepsilon_{U_j}(\theta_1, t)\ \forall t \in (t_i, t_{i+1})$, and $U_{j+1} = in(p_{j+1})$. By Assumption 3.9.3 these sequences must be finite, and the last transition, U_k (for some $k \geq 0$), satisfies the condition $\varepsilon_{U_k}(\theta_1, t) = \emptyset\ \forall t \in (t_i, t_{i+1})$. Note that this includes the possible case where U_k is a source transition.

Now for $j = 0, \ldots, k-1$, either one of the above cases (ii) or (iv) applies, and by the right inequality of the corresponding equation (31) or (33), we have that

$$\int_{t_i}^{t_{i+1}} \delta v_{U_j}(t)dt \leq \delta m_{p_{j+1}}(t_i) + \int_{t_i}^{t_{i+1}} \delta v_{Uj+1}(t)dt. \tag{38}$$

On the other hand, for U_k, either case (i) or (iii) applies, and correspondingly, (30) or the right inequality of (32) imply that

$$\int_{t_i}^{t_{i+1}} \delta v_{U_k}(t)dt \leq \int_{t_i}^{t_{i+1}} \delta V_{U_k}(t)dt. \tag{39}$$

Summing up (38) over $j = 0, \ldots, k-1$, and using (39), we obtain,

$$\int_{t_i}^{t_{i+1}} \delta v_{U_0}(t)dt \leq \sum_{j=0}^{k-1} \delta m_{p_{j+1}}(t_i) + \int_{t_i}^{t_{i+1}} \delta V_{U_k}(t)dt. \tag{40}$$

Observe that $p_{j+1} \in out(U_{j+1}) \cap in(U_j)$, and therefore, Equation (2) implies that $\delta m_{p_{j+1}}(t_i) - \delta m_{p_{j+1}}(0) = \int_0^{t_i} \delta v_{U_{j+1}}(t)dt - \int_0^{t_i} \delta v_{U_j}(t)dt \; \forall j = 0, \ldots, k-1$. Since by assumption θ is a parameter of the maximum transitions' firing speeds but not the initial marking, we have that $\delta m_{p_{j+1}}(0) = 0$, and hence, $\delta m_{p_{j+1}}(t_i) = \int_0^{t_i} \delta v_{U_{j+1}}(t)dt - \int_0^{t_i} \delta v_{U_j}(t)dt$. Using this in (40), it follows that

$$\int_{t_i}^{t_{i+1}} \delta v_{U_0}(t)dt \leq \int_0^{t_i} \delta v_{U_k}(t)dt - \int_0^{t_i} \delta v_{U_0}(t)dt + \int_{t_i}^{t_{i+1}} \delta V_{U_k}(t)dt. \tag{41}$$

But $U_0 = T$, and therefore (41) is (37) with $U := U_k$.

Next, recall that $[0, t_f) = \cup_{i \geq 0}[t_i, t_{i+1})$. By a repeated application of (37), for every $i \geq 0$, there exists a transition U_i such that,

$$\int_0^{t_f} \delta v_T(t)dt \leq \sum_{i \geq 0} \int_{t_i}^{t_{i+1}} \delta V_{U_i}(t)dt. \tag{42}$$

For every transition T, let K_T be the Lipschitz constant stipulated in Assumption 3.9.2, then, $\int_0^{t_f} |\delta V_T(t)|dt \leq K_T(\theta_2 - \theta_1)$. Let $\mathcal{T}$ denote the set of all transitions in the network, and define $K := \sum_{T \in \mathcal{T}} K_T$. Equation (42) implies that, for every transition T, $\int_0^{t_f} \delta v_T(t)dt \leq K(\theta_2 - \theta_1)$. Notice that the analysis yielding this inequality used the right inequalities of Equations (31)-(33); replicating the arguments using the left inequalities instead, yields the inequality $\int_0^{t_f} \delta v_T(t)dt \geq -K(\theta_2 - \theta_1)$, and this implies that

$$\left|\int_0^{t_f} \delta v_T(t)dt\right| \leq K(\theta_2 - \theta_1). \tag{43}$$

By Equations (4) and (43), K is a Lipschitz constant for $L_T(\theta)$. As for $L_p(\theta)$, Equation (2) implies that, for every time $t \in [0, t_f]$ and place p, with $U := in(p)$ and $T := out(p)$, $\delta m_P(t) = \int_0^t \big(\delta v_U(\tau) - \delta v_T(\tau)\big)d\tau$. Consequently, and by (43), $|\delta m_p(t)| \leq 2K(\theta_2 - \theta_1)$, and hence, and by (5), $2t_f K$ is a Lipschitz constant for $L_p(\theta)$. This completes the proof. □

Proof of Proposition 4.2. Recall that $L(\theta)$ is defined by (24). By Assumption 4.1.1, for every $\theta \in \Theta$, the IPA derivative $\frac{dL}{d\theta}(\theta)$ exists w.p.1. Therefore, it suffices to show that the random function $L(\theta)$ has a Lipschitz constant with a finite first moment. That is what we now do.

Fix $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$, and $\forall t \in [0, t_f]$, define the notation $\delta v_j(t) := v_j(\theta_2, t) - v_j(\theta_1, t)$, $\delta V_j(t) := V_j(\theta_2, t) - V_j(\theta_1, t)$, $j = 1, 2, 3$, and $\delta m_j(t) := m_j(\theta_2, t) - m_j(\theta_1, t)$, $j = 1, 2$. Now by (26) and the flow equations (1) and (2), tracing through the system's events reveals the following monotonicity properties: for every $t \in [0, t_f]$, $\delta m_2(t) \leq 0$, and $\delta m_1(t) \geq 0$. Next, by (25), $m_1(\theta, t)$ rises by α_n at each time $t = s_n$, and by (26), the resulting maximum contributions to $-\delta m_2(t)$ and $\delta m_1(t)$ is $\frac{V_{2,2} - V_{2,1}}{V_3}(\theta_2 - \theta_1)$. Therefore we have the following inequalities,

$$\begin{aligned} -\sum_{n=0}^{n_f} \alpha_n \frac{V_{2,2} - V_{2,1}}{V_3}(\theta_2 - \theta_1) \leq \delta m_2(t) \leq 0 \\ \leq \delta m_1(t) \leq \sum_{n=0}^{n_f} \alpha_n \frac{V_{2,2} - V_{2,1}}{V_3}(\theta_2 - \theta_1), \end{aligned} \tag{44}$$

where we recall that n_f is the number of impulses $V_1(t)$ has in the interval $[0, t_f]$. Define $K := \sum_{n=1}^{n_f} \alpha_n \frac{V_{2,2} - V_{2,1}}{V_3}$. By (44) and (24), the term $(C_1 + C_2)Kt_f$ is a Lipschitz constant for $L(\theta)$. But by Assumption 4.1.2 and the fact that $\int_0^{t_f} V_1(t)dt = \sum_{n=1}^{n_f} \alpha_n$, this Lipschitz constant has a finite first moment. □

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