Observer–controller design for three dimensional overhead cranes using time–scaling

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Abstract

In this paper we address the design of an observer–controller for a three degrees of freedom overhead crane. We consider a linear model of the crane where the length of the suspending rope is a time–varying parameter. The set of models given by frozen values of the rope length can be reduced to a single time–invariant reference model using suitable time–scalings. We construct a controller and an observer for the reference model assigning the desired closed loop eigenvalues for the both system and estimation error. The time–scaling relations can be used to derive a control law for the time–varying system that implements an implicit gain–scheduling [6]. A second gain–scheduling is used to choose suitable closed–loop eigenvalues for different values of the load and lifting/lowering operations. Using a Lyapunov–like theorem, it is also possible to find relative upper bounds for the rate of change of the varying parameter that ensure the stability of the time–varying system.

Key words: gain scheduling, linear time–varying systems, overhead cranes, pole placement.

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1 Introduction

The swinging of an object suspended from an overhead crane is an undesirable result of the crane movement and serious damage could occur during the load transport. Therefore, a satisfactory control scheme is desirable in a crane design to suppress the load swing.

Several control methodologies have been proposed in the literature [2, 4, 7, 9, 11, 5], as well as several software tools [8]. However, in quite all these cases an unrealistic assumption has been done. In fact, the cranes commonly considered in previous works are planar, i.e., it is assumed that the movement of the load lies within a plane. On the contrary, in this paper we deal with a three degrees of freedom overhead crane and we propose the design of an observer–controller that aims to minimize the load swinging, while moving it to the desired position as fast as possible.

We first develop a non–linear model of the overhead crane which takes into account simultaneous travel and transverse motions. Then, under appropriate simplifying assumptions (namely, small angles, constant rope velocity, force applied by the rope equal to the weight of the load and no external force acting on the load) a linear time–varying model of the crane is obtained, where the time–varying parameter is the length of the rope that sustains the load. The linearized model has order eight and its dynamic can be described as two decoupled fourth–order systems.

The main advantage of this model is that it enables us to use a simple and efficient synthesis procedure to design both the controller and the observer, while ensuring a high modeling power.

The controller design is realized by first considering the set of frozen models given by different constant values of the rope length. Using two suitable time–scalings, one for each sub–system, all these models can be reduced to a single time–invariant reference model that does not depend on the value of the rope length. Then, the pole placement technique enables us to design a satisfactory controller for the reference model. Finally, by inverting the time–scalings, these constant feedback gains give the corresponding time–varying gains that implement an implicit gain–scheduling.

Note that a similar approach has been already adopted in [4] where the control problem for the time–invariant reference model was posed as an LQR. However, pole assignment seems a more natural way of computing the controller for the following reasons. Firstly, pole placement allows one to directly assign the damping coefficients of the poles of the reference model that — by a property of the time–scaling — can be shown to be the same of the damping coefficients of the poles of all frozen models. Secondly, we are able to derive a closed form expression of the controller gains as a function of the desired closed loop poles, that assume the role of design parameters. Thirdly, we observed that finding by trial and error ”good” poles — both in terms of performance and of stability — was easier than tuning the coefficients of the weighting matrices used in [4] to compute the LQR controller.

The physical realization of such a gain–scheduling controller requires the knowledge of all state variables (center of mass position and velocity, load displacement with respect to the vertical and its rate of change), of the rope length and of the load weight. In a first case we assume that only the trolley position and the rope length can be measured by appropriate sensors as discussed by several authors [8, 16, 17]. In a second case, we assume that the load angle can be measured as well [8, 16, 17]. In both cases, we show how a time–varying observer can be designed via gain–scheduling and pole–placement to provide an estimate of the unknown state vector.

In this paper we introduce a further improvement with respect to previous works [4, 11] where a gain–scheduling approach has been adopted: both in the observer and in the controller case, a double gain–scheduling has been introduced. It consists of a variation of the desired eigenvalues of the reference stationary system depending on the load mass and on the lowering/lifting movement.

There are two important aspects in the approach we propose. First of all, we use the same framework to design both observer and controller. Secondly, the state–feedback gains and the observer gains are expressed in a parametrized form, as a symbolic function of the desired closed–loop dynamics (i.e., the eigenvalues of the reference closed–loop system and observer), rope length, rope velocity, trolley and load mass. As these parameters vary, the gains need not be recomputed by reapplying the whole design procedure but can simply be obtained by function evaluation.

We have also studied the stability of the closed–loop system with gain–scheduling. Recent works [12, 14] present several methodologies that can be used to find upper bounds on the rate of change of the varying parameter to ensure stability of a given parameter–varying system. These methodologies give sufficient conditions that are usually very conservative, in the sense that they often require rates of change of the varying parameter so small as to be practically meaningless. These upper bounds, in fact, depend heavily on the procedure used to determine them and are usually far from the real bounds of the system. In this paper we use the general methodology of [4], based on a Lyapunov–like theorem [13] and show that in the applicative case examined this approach gives sufficiently large bounds on the rope velocity to ensure stability of the time–varying system in all nominal operating conditions. Since this procedure is founded on the results of numerical simulations, we also propose a test to ensure that the computational error cannot invalidate the stability results.

The paper is structured as follows. Section 2 presents the time–varying model of the crane and discusses the time–scalings that can be used to reduce the set of frozen models to a single time–invariant model. Section 3 shows how a gain–scheduling control scheme can be derived to design a time–varying controller. In Section 4 two different time–varying observers are designed, depending on the available variables. In Section 5 a detailed stability analysis has been derived. Finally, in Section 6 the results of several numerical simulations, carried out on the non–linear model of the crane, show that the proposed approach gives acceptable performance while ensuring the stability of the system. The derivation of the simplified equations for the three–dimensional (3D) overhead crane is reported in the Appendix.

2 Linear time–varying model and time–scaling

A 3D overhead crane is constituted by a bridge and a trolley: the trolley moves on the bridge rails and contains the motor and all the other mechanisms necessary for the movement of the load; the bridge moves in the orthogonal direction thanks to appropriate wheels located on the end truck. In this paper we will consider a 3D overhead crane, whose model is sketched in Fig. 1. The following notation is used:

- m_T, m_B are the mass of the trolley and that of the bridge, respectively;
- $m_C = m_T + m_B$ is total mass of the crane;
- m_L is the mass of the load;
- L is the length of the suspending rope;
- x_T, z_T denote the displacement of the trolley with respect to (wrt) a fixed coordinate system;
- x_L, z_L denote the displacement of the load wrt a fixed coordinate system;
- $x_C = (m_T x_T + m_L x_L)/(m_T + m_L)$, $z_C = (m_C z_T + m_L z_L)/(m_C + m_L)$ denote the displacement of the center of gravity of the overall system wrt a fixed coordinate system;
- φ is the angle between the suspending rope and the vertical;

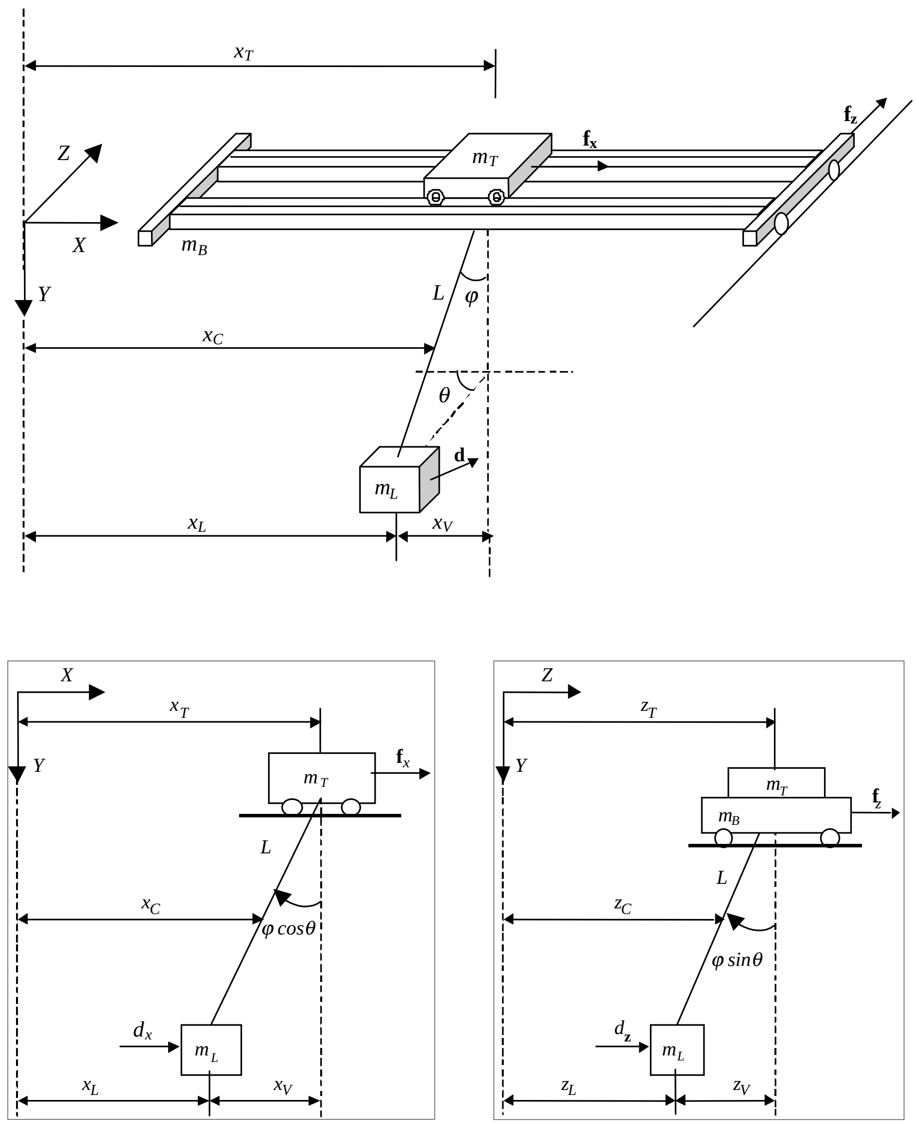

Figure 1: *Model of the 3D crane.*

- θ is the angle between the oscillation plane of the load and the XY plane, taken as positive when clockwise;
- $x_V = x_T - x_L = L\, sin\varphi\, cos\theta$, $z_V = z_T - z_L = L\, cos\varphi\, sin\theta$ denote the displacement of the load wrt the vertical;
- f_x and f_z are the control forces applied to the trolley and to the bridge, respectively;
- d_x and d_z are the wind forces acting on the load in the x and z direction;
- g is the gravitation constant.

If the load is heavy enough, it is possible to consider the suspending rope as a rigid rod. Under the assumptions reported in the Appendix (namely, small angles, force applied by the rope equal to the weight of the load and no disturbance acting on the system) we obtain the linearized model described by equation (58) (see Appendix for a derivation). Choosing the following state variables:

$$\begin{aligned} x_1(t) &= x_V(t), \quad x_2(t) = x_C(t) \\ x_3(t) &= \dot{x}_V(t), \quad x_4(t) = \dot{x}_C(t) \\ x_5(t) &= z_V(t), \quad x_6(t) = z_C(t) \\ x_7(t) &= \dot{z}_V(t), \quad x_8(t) = \dot{z}_C(t) \end{aligned} \tag{1}$$

and denoting

$$\begin{aligned} \omega_x(t) \equiv \omega_x(L(t)) &= \left(\frac{g(m_T + m_L)}{m_T L(t)}\right)^{0.5}, \\ \omega_z(t) \equiv \omega_z(L(t)) &= \left(\frac{g(m_C + m_L)}{m_C L(t)}\right)^{0.5}, \end{aligned} \tag{2}$$

we get from (58) the following state variable equation:

$$\begin{cases} \dot{\boldsymbol{x}}_t = \boldsymbol{A}_t \boldsymbol{x}_t + \boldsymbol{B}_t \boldsymbol{u}_t \\ \boldsymbol{y}_t = \boldsymbol{C}_t \boldsymbol{x}_t \end{cases} \tag{3}$$

with

$$\boldsymbol{x}_t = \begin{bmatrix} x_1(t) \\ \vdots \\ x_8(t) \end{bmatrix}, \quad \boldsymbol{u}_t = \begin{bmatrix} f_x(t) \\ f_z(t) \end{bmatrix},$$

$$
\boldsymbol{A}_t = \left[\begin{array}{cccc|cccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\omega_x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\omega_z^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

$$
\boldsymbol{B}_t = \left[\begin{array}{c|c}
0 & 0 \\
0 & 0 \\
1/m_T & 0 \\
1/(m_T + m_L) & 0 \\
\hline
0 & 0 \\
0 & 0 \\
0 & 1/m_C \\
0 & 1/(m_C + m_L)
\end{array}\right].
$$

In this paper we examine two different cases. In the first one we assume as only measurable variables the trolley displacement coordinate x_T and z_T. In the second one, we assume that both the trolley position and the load position with respect to the vertical (x_V and z_V) are measurable.

Then, in the first case of interest we write

$$
\boldsymbol{y}_t \equiv \boldsymbol{y}'_t = \left[\begin{array}{c} x_T(t) \\ z_T(t) \end{array}\right], \quad \boldsymbol{C}_t \equiv \boldsymbol{C}'_t = \left[\begin{array}{c|c}
m_L/(m_T + m_L) & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
0 & m_L/(m_C + m_L) \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{array}\right]^T, \tag{4}
$$

while in the second case we write

$$\boldsymbol{y}_t \equiv \boldsymbol{y}''_t = \begin{bmatrix} x_V(t) \\ x_T(t) \\ z_V(t) \\ z_T(t) \end{bmatrix},$$

$$\boldsymbol{C}_t \equiv \boldsymbol{C}''_t = \left[\begin{array}{cc|cc} 1 & m_L/(m_T+m_L) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & m_L/(m_C+m_L) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]^T. \tag{5}$$

The subscript t has been introduced to recall that the variables are functions of time. The model given by (3) is time–varying because both ω_x and ω_z are functions of $L(t)$. If we consider a given constant value of both ω_x and ω_z, i.e., if we consider the system (3) for a frozen value of L, we can consider the following transformations:

$$\tau_x = \omega_x t, \tag{6}$$

$$\tau_z = \omega_z t. \tag{7}$$

These transformations define a time–scaling that enable us to rewrite (1) as:

$$\begin{aligned}
x_1(t) &= x_V(t) = x_V(\tau_x) = x_1(\tau_x) \\
x_2(t) &= x_C(t) = x_C(\tau_x) = x_2(\tau_x) \\
x_3(t) &= \frac{dx_V(t)}{dt} = \frac{dx_V(\tau_x(t))}{dt} = \omega_x \frac{dx_V(\tau_x)}{d\tau_x} = \omega_x x_3(\tau_x) \\
x_4(t) &= \frac{dx_C(t)}{dt} = \frac{dx_C(\tau_x(t))}{dt} = \omega_x \frac{dx_C(\tau_x)}{d\tau_x} = \omega_x x_4(\tau_x) \\
x_5(t) &= z_V(t) = z_V(\tau_z) = x_5(\tau_z) \\
x_6(t) &= z_C(t) = z_C(\tau_z) = x_6(\tau_z) \\
x_7(t) &= \frac{dz_V(t)}{dt} = \frac{dz_V(\tau_z(t))}{dt} = \omega_z \frac{dz_V(\tau_z)}{d\tau_z} = \omega_z x_7(\tau_z) \\
x_8(t) &= \frac{dz_C(t)}{dt} = \frac{dz_C(\tau_z(t))}{dt} = \omega_z \frac{dz_C(\tau_z)}{d\tau_z} = \omega_z x_8(\tau_z).
\end{aligned} \tag{8}$$

According to (8), variables x_C and x_V (z_C and z_V) can be taken as functions of t or τ_x (τ_z), while their derivatives are changed by the time–scaling. We can write (8) as

$$\boldsymbol{x}_t = \boldsymbol{N}\boldsymbol{x}_\tau \tag{9}$$

where

$$\begin{aligned}\boldsymbol{N} &= \operatorname{diag}\{\ 1,\ \ 1,\ \ \omega_x,\ \ \omega_x,\ \ 1,\ \ 1,\ \ \omega_z,\ \ \omega_z\ \ \} \\ \boldsymbol{x}_\tau &= \begin{bmatrix} x_1(\tau_x) & x_2(\tau_x) & x_3(\tau_x) & x_4(\tau_x) & x_5(\tau_z) & x_6(\tau_z) & x_7(\tau_z) & x_8(\tau_z) \end{bmatrix}^T.\end{aligned} \tag{10}$$

According to (8), we may also write

$$\dot{\boldsymbol{x}}_t = \boldsymbol{\Omega}\boldsymbol{N}\dot{\boldsymbol{x}}_\tau \tag{11}$$

where $\dot{\boldsymbol{x}}_\tau$ is the derivative of $\boldsymbol{x}_\tau$ wrt τ_x for the first four components and wrt τ_z for the remaining ones. It has been assumed

$$\boldsymbol{\Omega} = \operatorname{diag}\{\ \omega_x,\ \ \omega_x,\ \ \omega_x,\ \ \omega_x,\ \ \omega_z,\ \ \omega_z,\ \ \omega_z,\ \ \omega_z\ \ \}. \tag{12}$$

Using (9) and (11), it is possible to rewrite the system (3) as

$$\begin{cases} \dot{\boldsymbol{x}}_\tau = \boldsymbol{A}_\tau \boldsymbol{x}_\tau + \boldsymbol{B}_\tau \boldsymbol{u}_\tau \\ \boldsymbol{y}_\tau = \boldsymbol{C}_\tau \boldsymbol{x}_\tau \end{cases} \tag{13}$$

with

$$\boldsymbol{u}_\tau = \begin{bmatrix} \frac{1}{\omega_x^2} & 0 \\ 0 & \frac{1}{\omega_z^2} \end{bmatrix} \boldsymbol{u}_t = \boldsymbol{N}_u^{-1}\boldsymbol{u}_t = \begin{bmatrix} \frac{f_x}{\omega_x^2} \\ \frac{f_z}{\omega_z^2} \end{bmatrix}, \tag{14}$$

$$\boldsymbol{A}_\tau = \boldsymbol{N}^{-1}\boldsymbol{\Omega}^{-1}\boldsymbol{A}_t\boldsymbol{N} = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{15}$$

$$\boldsymbol{B}_\tau = \boldsymbol{N}^{-1}\boldsymbol{\Omega}^{-1}\boldsymbol{B}_t\boldsymbol{N}_u = \boldsymbol{B}_t. \tag{16}$$

Finally, the output vector and the output matrices are defined as

$$\boldsymbol{y}_\tau \equiv \boldsymbol{y}'_\tau = \begin{bmatrix} x_T(\tau_x) \\ z_T(\tau_z) \end{bmatrix}, \quad \boldsymbol{C}_\tau \equiv \boldsymbol{C}'_\tau = \boldsymbol{C}'_t\boldsymbol{N} = \boldsymbol{C}'_t, \tag{17}$$

and

$$\boldsymbol{y}_\tau \equiv \boldsymbol{y}''_\tau = \begin{bmatrix} x_V(\tau_x) \\ x_T(\tau_x) \\ z_V(\tau_z) \\ z_T(\tau_z) \end{bmatrix}, \quad \boldsymbol{C}_\tau \equiv \boldsymbol{C}''_\tau = \boldsymbol{C}''_t \boldsymbol{N} = \boldsymbol{C}''_t, \tag{18}$$

in the first and in the second case, respectively.

The representation given by (13) is time–invariant and does not depend on the frozen value of L in (3).

As shown in [4], it is possible to express the relationship between the eigenvalues and eigenvectors of the matrices $\boldsymbol{A}_t$ and $\boldsymbol{A}_\tau$. Let $\boldsymbol{\Delta}_t$ ($\boldsymbol{\Delta}_\tau$) be the diagonal matrix of the eigenvalues of $\boldsymbol{A}_t$ ($\boldsymbol{A}_\tau$); then from (15) we obtain

$$\boldsymbol{\Delta}_t = \boldsymbol{\Omega}\boldsymbol{\Delta}_\tau \tag{19}$$

while between the eigenvector matrices $\boldsymbol{V}_t$ and $\boldsymbol{V}_\tau$

$$\boldsymbol{V}_t = \boldsymbol{N}\boldsymbol{V}_\tau \tag{20}$$

holds, i.e., given a matrix of eigenvectors $\boldsymbol{V}_\tau$ for $\boldsymbol{A}_\tau$ we can compute one of the possible matrices of eigenvectors $\boldsymbol{V}_t$ for $\boldsymbol{A}_t$.

Now, let us rewrite matrices $\boldsymbol{A}_t$ and $\boldsymbol{A}_\tau$ as

$$\boldsymbol{A}_t = \begin{bmatrix} \boldsymbol{A}_{t,x} & \boldsymbol{0}_{4,4} \\ \boldsymbol{0}_{4,4} & \boldsymbol{A}_{t,z} \end{bmatrix}$$

and

$$\boldsymbol{A}_\tau = \begin{bmatrix} \boldsymbol{A}_{\tau,x} & \boldsymbol{0}_{4,4} \\ \boldsymbol{0}_{4,4} & \boldsymbol{A}_{\tau,z} \end{bmatrix}$$

respectively, where $\boldsymbol{0}_{p,q}$ denotes the $p \times q$ matrix of null entries. We can observe that matrices $\boldsymbol{A}_{\tau,x}$ and $\boldsymbol{A}_{\tau,z}$ have the same set of eigenvalues:

$$\{\ 0,\quad 0,\quad j,\quad -j\ \}.$$

Thus, from equation (19) matrices $\boldsymbol{A}_{t,x}$ and $\boldsymbol{A}_{t,z}$ have, respectively, the following set of eigenvalues:

$$\{\ 0,\quad 0,\quad j\omega_x,\quad -j\omega_x\ \}, \qquad \{\ 0,\quad 0,\quad j\omega_z,\quad -j\omega_z\ \},$$

i.e., the frozen system (3) has undamped oscillations of frequency ω_x and ω_z. This also enables us to state that the variables τ_x and τ_z defined by (6) and (7) are the time measured using as unit $1/\omega_x = T_x/2\pi$ and $1/\omega_z = T_z/2\pi$), respectively, where T_x and T_z are the periods of the undamped oscillations of system (3).

3 Controller design

Let us consider a linear and time–invariant system of the form (13). If the couple $(\boldsymbol{A}_\tau, \boldsymbol{B}_\tau)$ is controllable [3], then a regulator can be designed by imposing the closed loop poles to system (13), finding a control law of the form

$$\boldsymbol{u}_\tau = -\boldsymbol{K}_\tau \boldsymbol{x}_\tau \tag{21}$$

where $\boldsymbol{K}_\tau$ is a constant matrix and does not depend on the value of L. The above equation can be transformed, using (9) and (14), into a corresponding law for the frozen system (3) that gives:

$$\boldsymbol{u}_t = -\boldsymbol{K}_t \boldsymbol{x}_t \tag{22}$$

where

$$\boldsymbol{K}_t = \boldsymbol{N}_u \boldsymbol{K}_\tau \boldsymbol{N}^{-1}. \tag{23}$$

The feedback laws (21) and (22) lead to closed loop systems whose characteristic matrices are:

$$\bar{\boldsymbol{A}}_\tau = \boldsymbol{A}_\tau - \boldsymbol{B}_\tau \boldsymbol{K}_\tau \qquad \bar{\boldsymbol{A}}_t = \boldsymbol{A}_t - \boldsymbol{B}_t \boldsymbol{K}_t. \tag{24}$$

Note that also the above matrices can be rewritten as

$$\bar{\boldsymbol{A}}_t = \begin{bmatrix} \bar{\boldsymbol{A}}_{t,x} & \boldsymbol{0}_{4,4} \\ \boldsymbol{0}_{4,4} & \bar{\boldsymbol{A}}_{t,z} \end{bmatrix}$$

and

$$\bar{\boldsymbol{A}}_\tau = \begin{bmatrix} \bar{\boldsymbol{A}}_{\tau,x} & \boldsymbol{0}_{4,4} \\ \boldsymbol{0}_{4,4} & \bar{\boldsymbol{A}}_{\tau,z} \end{bmatrix}.$$

Equations (15), (19) and (20), written for the open loop systems, still hold for the corresponding closed loop systems. The poles of the frozen closed loop system in t depend on the value of L, and thus on ω_x and ω_z, but they have the same damping factor for all values of L.

For a stationary system it is easy to find a feedback control law by imposing the closed loop eigenvalues following the procedure presented in [3]. Let us denote as

$$s^4 + a_{x,3}s^3 + a_{x,2}s^2 + a_{x,1}s + a_{x,0} \tag{25}$$

$$s^4 + a_{z,3}s^3 + a_{z,2}s^2 + a_{z,1}s + a_{z,0} \tag{26}$$

the open loop characteristic polynomials relative to matrices $A_{\tau,x}$ and $A_{\tau,z}$, respectively. Then, let

$$s^4 + p_{x,3}s^3 + p_{x,2}s^2 + p_{x,1}s + p_{x,0} \tag{27}$$

$$s^4 + p_{z,3}s^3 + p_{z,2}s^2 + p_{z,1}s + p_{z,0} \tag{28}$$

be the desired closed loop characteristic polynomials relative to matrices $\bar{\boldsymbol{A}}_{\tau,x}$ and $\bar{\boldsymbol{A}}_{\tau,z}$, respectively. Therefore, the time–invariant control law is [3]:

$$\boldsymbol{K}_\tau = \begin{bmatrix} \boldsymbol{K}_{\tau,x} & \boldsymbol{0}_{1,4} \\ \boldsymbol{0}_{1,4} & \boldsymbol{K}_{\tau,z} \end{bmatrix} \boldsymbol{P}_c^{-1} \tag{29}$$

where

$$\boldsymbol{K}_{\tau,x} = \begin{bmatrix} p_{x,0} - a_{x,0} & p_{x,1} - a_{x,1} & p_{x,2} - a_{x,2} & p_{x,3} - a_{x,3} \end{bmatrix},$$

$$\boldsymbol{K}_{\tau,z} = \begin{bmatrix} p_{z,0} - a_{z,0} & p_{z,1} - a_{z,1} & p_{z,2} - a_{z,2} & p_{z,3} - a_{z,3} \end{bmatrix},$$

$$\boldsymbol{P}_c = \begin{bmatrix} (\boldsymbol{A}_{\tau,x}^3 + a_{x,3}\boldsymbol{A}_{\tau,x}^2 + a_{x,2}\boldsymbol{A}_{\tau,x} + a_{x,1}\boldsymbol{I})\boldsymbol{B}_{\tau,x} \\ (\boldsymbol{A}_{\tau,x}^2 + a_{x,3}\boldsymbol{A}_{\tau,x} + a_{x,2}\boldsymbol{I})\boldsymbol{B}_{\tau,x} \\ (\boldsymbol{A}_{\tau,x} + a_{x,3}\boldsymbol{I})\boldsymbol{B}_{\tau,x} \\ \boldsymbol{B}_{\tau,x} \\ (\boldsymbol{A}_{\tau,z}^3 + a_{z,3}\boldsymbol{A}_{\tau,z}^2 + a_{z,2}\boldsymbol{A}_{\tau,z} + a_{z,1}\boldsymbol{I})\boldsymbol{B}_{\tau,z} \\ (\boldsymbol{A}_{\tau,z}^2 + a_{z,3}\boldsymbol{A}_{\tau,z} + a_{z,2}\boldsymbol{I})\boldsymbol{B}_{\tau,z} \\ (\boldsymbol{A}_{\tau,z} + a_{z,3}\boldsymbol{I})\boldsymbol{B}_{\tau,z} \\ \boldsymbol{B}_{\tau,z} \end{bmatrix}^T,$$

and $\boldsymbol{B}_{\tau,x}$ and $\boldsymbol{B}_{\tau,z}$ are the two non–null sub–matrices of $\boldsymbol{B}_\tau$, i.e.,

$$\boldsymbol{B}_\tau = \begin{bmatrix} \boldsymbol{B}_{\tau,x} & \boldsymbol{0}_{4,1} \\ \boldsymbol{0}_{4,1} & \boldsymbol{B}_{\tau,z} \end{bmatrix}.$$

Note that $\boldsymbol{P}_c$ is an equivalence transformation that brings the initial system into a controllable canonical form [3].

Using equation (23), we get the time–varying control law:

$$\boldsymbol{K}_t = \begin{bmatrix} \boldsymbol{K}_{t,x} & \boldsymbol{0}_{1,4} \\ \boldsymbol{0}_{1,4} & \boldsymbol{K}_{t,z} \end{bmatrix} \tag{30}$$

where

$$\boldsymbol{K}_{t,x} = \begin{matrix} [(p_{x,2} - p_{x,0} - 1)m_T\omega_x^2 & p_{x,0}(m_T + m_L)\omega_x^2 \cdots \\ p_{x,3} - p_{x,1})m_T\omega_x & p_{x,1}(m_T + m_L)\omega_x] \end{matrix}$$

and

$$\boldsymbol{K}_{t,z} = \begin{matrix} [(p_{z,2} - p_{z,0} - 1)m_C\omega_z^2 & p_{z,0}(m_C + m_L)\omega_z^2 \cdots \\ p_{z,3} - p_{z,1})m_C\omega_z & p_{z,1}(m_C + m_L)\omega_z] \end{matrix}$$

To conclude, we remark that the time–varying system (3) is controllable for all values of the load mass. In fact, let $\boldsymbol{D}_c(t)$ be the lexicographic basis of its controllability matrix, as defined in [3]. Then one can easily prove that

$$det\boldsymbol{D}_c(t) = \frac{g^4}{m_T^4 m_C^4 L^4(t)}.$$

This quantity does not depend on the value of the load mass and is always greater than zero: this ensures controllability [3].

4 Observer design

Let us consider again a linear time–invariant system of the form (13). If the couple $(\boldsymbol{A}_\tau, \boldsymbol{C}_\tau)$ is observable [3], then it is possible to construct a Luenberger observer for system (13) by finding the matrix $\boldsymbol{G}_\tau$ which imposes the desired closed loop poles to the reference error system:

$$\dot{\boldsymbol{e}}_\tau = (\boldsymbol{A}_\tau - \boldsymbol{G}_\tau \boldsymbol{C}_\tau)\boldsymbol{e}_\tau = \boldsymbol{E}_\tau \boldsymbol{e}_\tau \tag{31}$$

where

$$\boldsymbol{e}_\tau = \boldsymbol{x}_\tau - \hat{\boldsymbol{x}}_\tau$$

is the reference state estimate. If we denote $\hat{\boldsymbol{x}}_t$ the frozen system estimate and

$$\boldsymbol{e}_t = \boldsymbol{x}_t - \hat{\boldsymbol{x}}_t$$

the corresponding error, it is easy to observe that:

$$\boldsymbol{e}_t = \boldsymbol{N}\boldsymbol{e}_\tau \tag{32}$$

and

$$\dot{\boldsymbol{e}}_t = (\boldsymbol{A}_t - \boldsymbol{G}_t \boldsymbol{C}_t)\boldsymbol{e}_t = \boldsymbol{E}_t \boldsymbol{e}_t = \boldsymbol{\Omega}\boldsymbol{N}\dot{\boldsymbol{e}}_\tau \tag{33}$$

where

$$\boldsymbol{G}_t = \boldsymbol{\Omega}\boldsymbol{N}\boldsymbol{G}_\tau. \tag{34}$$

The assignment of the eigenvalues is done as in the controller case, by first transforming the time–invariant system into an observable canonical form.

4.1 First observer

Now, let us consider the first case, i.e., let us assume that the only trolley position is measurable. The equivalence transformation is:

$$\boldsymbol{P}'_o = \begin{bmatrix} \boldsymbol{C}'_{\tau,x} \\ \boldsymbol{C}'_{\tau,x}(\boldsymbol{A}_{\tau,x} + a_{x,3}\boldsymbol{I}) \\ \boldsymbol{C}'_{\tau,x}(\boldsymbol{A}^2_{\tau,x} + a_{x,3}\boldsymbol{A}_{\tau,x} + a_{x,2}\boldsymbol{I}) \\ \boldsymbol{C}'_{\tau,x}(\boldsymbol{A}^3_{\tau,x} + a_{x,3}\boldsymbol{A}^2_{\tau,x} + a_{x,2}\boldsymbol{A}_{\tau,x} + a_{x,1}\boldsymbol{I}) \\ \boldsymbol{C}'_{\tau,z} \\ \boldsymbol{C}'_{\tau,z}(\boldsymbol{A}_{\tau,z} + a_{z,3}\boldsymbol{I}) \\ \boldsymbol{C}'_{\tau,z}(\boldsymbol{A}^2_{\tau,z} + a_{z,3}\boldsymbol{A}_{\tau,z} + a_{z,2}\boldsymbol{I}) \\ \boldsymbol{C}'_{\tau,z}(\boldsymbol{A}^3_{\tau,z} + a_{z,3}\boldsymbol{A}^2_{\tau,z} + a_{z,2}\boldsymbol{A}_{\tau,z} + a_{z,1}\boldsymbol{I}) \end{bmatrix} \tag{35}$$

where $\boldsymbol{C}'_{\tau,x}$ and $\boldsymbol{C}'_{\tau,z}$ are the two non–null sub–matrices of $\boldsymbol{C}'_\tau$, i.e.,

$$\boldsymbol{C}'_\tau = \begin{bmatrix} \boldsymbol{C}'_{\tau,x} & \boldsymbol{0}_{1,4} \\ \boldsymbol{0}_{1,4} & \boldsymbol{C}'_{\tau,z} \end{bmatrix},$$

and the coefficients $a_{x,i}$, $a_{z,i}$ are defined as above. Furthermore, let us rewrite $\boldsymbol{E}_\tau$ as

$$\boldsymbol{E}_\tau \equiv \boldsymbol{E}'_\tau = \begin{bmatrix} \boldsymbol{E}'_{\tau,x} & \boldsymbol{0}_{4,4} \\ \boldsymbol{0}_{4,4} & \boldsymbol{E}'_{\tau,z} \end{bmatrix}$$

and let

$$s^4 + q_{x,3}s^3 + q_{x,2}s^2 + q_{x,1}s + q_{x,0}, \tag{36}$$

$$s^4 + q_{z,3}s^3 + q_{z,2}s^2 + q_{z,1}s + q_{z,0} \tag{37}$$

be the closed loop characteristic polynomials associated to $\boldsymbol{E}'_{\tau,x}$ and $\boldsymbol{E}'_{\tau,z}$, respectively. In such a way, we get:

$$\boldsymbol{G}'_\tau = (\boldsymbol{P}'_o)^{-1} \begin{bmatrix} \boldsymbol{G}'_{\tau,x} & \boldsymbol{0}_{1,4} \\ \boldsymbol{0}_{1,4} & \boldsymbol{G}'_{\tau,z} \end{bmatrix}^T \tag{38}$$

where

$$\boldsymbol{G}'_{\tau,x} = \begin{bmatrix} q_{x,0} - a_{x,0} & q_{x,1} - a_{x,1} & q_{x,2} - a_{x,2} & q_{x,3} - a_{x,3} \end{bmatrix},$$

$$\boldsymbol{G}'_{\tau,z} = \begin{bmatrix} q_{z,0} - a_{z,0} & q_{z,1} - a_{z,1} & q_{z,2} - a_{z,2} & q_{z,3} - a_{z,3} \end{bmatrix}.$$

According to equation (34) the time–varying matrix defining the observer dynamic is:

$$\boldsymbol{G}'_t = \begin{bmatrix} \boldsymbol{G}'_{t,x} & \boldsymbol{0}_{1,4} \\ \boldsymbol{0}_{1,4} & \boldsymbol{G}'_{t,z} \end{bmatrix}^T \tag{39}$$

where

$$\boldsymbol{G}'_{t,x} = \left[\begin{array}{cccc} (q_{x,3} - q_{x,1})\tilde{m}_T\omega_x & q_{x,1}\omega_x & (q_{x,2} - q_{x,0} - 1)\tilde{m}_T\omega_x^2 & q_{x,0}\omega_x^2 \end{array} \right],$$

$$\boldsymbol{G}'_{t,z} = \left[\begin{array}{cccc} (q_{z,3} - q_{z,1})\tilde{m}_C\omega_z & q_{z,1}\omega_z & (q_{z,2} - q_{z,0} - 1)\tilde{m}_C\omega_z^2 & q_{z,0}\omega_z^2 \end{array} \right].$$

and

$$\tilde{m}_T = \frac{m_T + m_L}{m_L}, \qquad \tilde{m}_C = \frac{m_C + m_L}{m_L}.$$

Thus, by choosing appropriate poles for the closed loop dynamic of the stationary error system, i.e., of the coefficients $q_{x,i}$ and $q_{z,i}$, a satisfactory reconstruction of the system's state can be obtained with the only knowledge of the trolley position.

Now, let us note that the time–varying system (3) with $\boldsymbol{C}_t = \boldsymbol{C}'_t$ becomes unobservable when $m_L \to 0$. In fact, if we denote as $\boldsymbol{D}'_o(t)$ the lexicographic basis of its observability matrix as defined in [3], we have

$$det\boldsymbol{D}'_o(t) = \frac{g^4 m_L^2}{m_T^2 m_C^2 L^4(t)}$$

that approaches zero when $m_L << m_T m_C$.

We propose two different solutions to this problem. The first one consists in the assumption that only small errors in the initial state estimate may occur. This is a realistic assumption that can be easily satisfied for small values of the load. In fact, it is sufficient to suppose the initial position of the centre of gravity coincident with the trolley position (that is a measurable variable). On the contrary, the alternative discussed in the following subsection, consists in the introduction of a new sensor to measure the rope angles φ and θ (or equivalently the load displacement wrt to the vertical, i.e., x_V and z_V) [8, 16, 17].

4.2 Second Observer

Now, let us consider the second case of interest, i.e., let us assume that also the load position wrt the vertical is measurable. Even in this case the time–varying observer can be designed by assigning the closed–loop eigenvalues to the time–invariant error system. To this purpose, being $\boldsymbol{C}''_\tau$ a 4×8 order matrix, let us decompose the open loop characteristic polynomial of matrix $\boldsymbol{A}_\tau$ as the product of the following four two–degrees polynomials:

$$s^2 + a_{x,11}s + a_{x,10}$$
$$s^2 + a_{x,21}s + a_{x,20}$$
$$s^2 + a_{z,11}s + a_{z,10}$$
$$s^2 + a_{z,21}s + a_{z,20}.$$

Analogously, let us decompose the characteristic polynomial of the closed–loop error system as:

$$
\begin{gathered}
s^2 + q_{x,11}s + q_{x,10} \\
s^2 + q_{x,21}s + q_{x,20} \\
s^2 + q_{z,11}s + q_{z,10} \\
s^2 + q_{z,21}s + q_{z,20}.
\end{gathered}
$$

In this case, the equivalence transformation is:

$$
\boldsymbol{P}''_o = \begin{bmatrix} \boldsymbol{C}''_{\tau,1} \\ \boldsymbol{C}''_{\tau,1}(\boldsymbol{A}_\tau + a_{x,11}\boldsymbol{I}) \\ \boldsymbol{C}''_{\tau,2} \\ \boldsymbol{C}''_{\tau,2}(\boldsymbol{A}_\tau + a_{x,21}\boldsymbol{I}) \\ \boldsymbol{C}''_{\tau,3} \\ \boldsymbol{C}''_{\tau,3}(\boldsymbol{A}_\tau + a_{z,11}\boldsymbol{I}) \\ \boldsymbol{C}''_{\tau,4} \\ \boldsymbol{C}''_{\tau,4}(\boldsymbol{A}_\tau + a_{z,21}\boldsymbol{I}) \end{bmatrix}
$$

where $\boldsymbol{C}''_{\tau,i}$ denotes the i–th row of $\boldsymbol{C}''_\tau$.

Thus, matrix $\boldsymbol{G}''_\tau$, i.e., the time–invariant observer matrix is defined as [3]:

$$
\boldsymbol{G}''_\tau = (\boldsymbol{P}''_o)^{-1} \begin{bmatrix} \boldsymbol{G}''_{\tau,x} & \boldsymbol{0}_{2,4} \\ \boldsymbol{0}_{2,4} & \boldsymbol{G}''_{\tau,z} \end{bmatrix}^T \tag{40}
$$

where

$$
\boldsymbol{G}''_{\tau,x} = \begin{bmatrix} (q_{x,11} - a_{x,11}) & (q_{x,10} - a_{x,10}) & 0 & 0 \\ 0 & 0 & (q_{x,21} - a_{x,21}) & (q_{x,20} - a_{x,20}) \end{bmatrix}, \tag{41}
$$

$$
\boldsymbol{G}''_{\tau,z} = \begin{bmatrix} (q_{z,11} - a_{z,11}) & (q_{z,10} - a_{z,10}) & 0 & 0 \\ 0 & 0 & (q_{z,21} - a_{z,21}) & (q_{z,20} - a_{z,20}) \end{bmatrix}. \tag{42}
$$

Finally, the time–varying matrix describing the observer dynamic is:

$$
\boldsymbol{G}''_t = \begin{bmatrix} \boldsymbol{G}''_{t,x} & \boldsymbol{0}_{2,4} \\ \boldsymbol{0}_{2,4} & \boldsymbol{G}''_{t,z} \end{bmatrix}^T \tag{43}
$$

where

$$
\boldsymbol{G}''_{t,x} = \begin{bmatrix} q_{x,11}\omega_x & -\dfrac{q_{x,11}\omega_x}{\tilde{m}_T} & (q_{x,10} - 1)\omega_x^2 & -\dfrac{(q_{x,10} - 1)\omega_x^2}{\tilde{m}_T} \\ 0 & q_{x,21}\omega_x & 0 & q_{x,20}\omega_x^2 \end{bmatrix}, \tag{44}
$$

$$
\boldsymbol{G}''_{t,z} = \begin{bmatrix} q_{z,11}\omega_x & -\dfrac{\omega_z q_{z,11}}{\tilde{m}_C} & (q_{z,10}-1)\omega_z^2 & -\dfrac{(q_{z,10}-1)\omega_z^2}{\tilde{m}_C} \\ 0 & q_{z,21}\omega_z & 0 & q_{z,21}\omega_z \end{bmatrix}. \tag{45}
$$

In such a way, by choosing appropriate poles for the closed loop dynamic of the stationary error system, i.e., of the coefficients $q_{x,ij}$ and $q_{z,ij}$, a satisfactory reconstruction of the system's state can be obtained with the knowledge of the trolley position and of the load position wrt the vertical.

Note that in this case the time–varying system (3) is observable [3] for all values of the load mass. In fact, the lexicographic basis of the observability matrix has unitary determinant when the output matrix is $\boldsymbol{C}_t = \boldsymbol{C}''_t$.

5 Stability analysis

The matrices $\bar{\boldsymbol{A}}_\tau = \boldsymbol{A}_\tau - \boldsymbol{B}_\tau \boldsymbol{K}_\tau$, $\bar{\boldsymbol{E}}'_\tau = \boldsymbol{A}_\tau - \boldsymbol{G}'_\tau \boldsymbol{C}'_\tau$ and $\bar{\boldsymbol{E}}''_\tau = \boldsymbol{A}_\tau - \boldsymbol{G}''_\tau \boldsymbol{C}''_\tau$ have eigenvalues with negative real parts for all values of $L(t)$. However, this is not enough to ensure stability of the time–varying closed loop model unless the rate of change of the time–varying parameter $L(t)$ is sufficiently slow.

We propose to apply as in [4] a Lyapunov–like theorem reported in [13], to determine upper bounds for the rate of change of $L(t)$ that ensure stability.

Theorem 1 (Shamma [13]). *Given the time–varying system:*

$$
\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{46}
$$

where $\boldsymbol{A}(t)$ is bounded and globally Lipschitz continuous, let there exist matrices $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$, symmetric and positive definite, such that:

1. *$\boldsymbol{P}(t)$ is continuously differentiable for all $t \geq 0$;*

2. *there exist constants α_1, α_2 and $\alpha_3 > 0$ such that, for all $t \geq 0$:*

   - $\alpha_1 \leq \lambda_{min}\{\boldsymbol{P}(t)\} \leq \lambda_{max}\{\boldsymbol{P}(t)\} \leq \alpha_2$
   - $\lambda_{min}\{\boldsymbol{Q}(t) - \dot{\boldsymbol{P}}(t)\} \geq \alpha_3$

3. *$\boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\boldsymbol{P}(t) = -\boldsymbol{Q}(t) \qquad \forall t \geq 0$*

where λ_{min} (λ_{max}) denotes the smaller (resp., larger) eigenvalue.

Under these conditions, the linear system (46) is exponentially stable. ■

Now, let us consider the controller design. Let $\bar{\boldsymbol{\Delta}}_\tau$ and $\bar{\boldsymbol{V}}_\tau$ be the eigenvalue and eigenvector matrices for $\bar{\boldsymbol{A}}_\tau^T$. Then, using the transpose of equation (15), it is possible to show that

$$\bar{\boldsymbol{\Delta}}_t = \boldsymbol{\Omega}\bar{\boldsymbol{\Delta}}_\tau \tag{47}$$

and

$$\bar{\boldsymbol{V}}_t = \boldsymbol{N}^{-1}\bar{\boldsymbol{V}}_\tau \tag{48}$$

are eigenvalue and eigenvector matrices for $\bar{\boldsymbol{A}}_t^T$. We have chosen matrix $\boldsymbol{P}(t)$ in Theorem 1 as

$$\boldsymbol{P}(t) = \bar{\boldsymbol{V}}_t\bar{\boldsymbol{V}}_t^H = \boldsymbol{N}^{-1}\bar{\boldsymbol{V}}_\tau\bar{\boldsymbol{V}}_\tau^H\boldsymbol{N}^{-1} \tag{49}$$

where ^{H} denotes the complex conjugate transpose. Thus it is easy to compute analytically matrices $\boldsymbol{Q}(t)$ and $\dot{\boldsymbol{P}}(t)$.

Exactly the same choices can be done for the error closed loop system with matrix $\bar{\boldsymbol{E}}_t'$ ($\bar{\boldsymbol{E}}_t''$). We denote as $\boldsymbol{P}'_{ob}(t)$ and $\boldsymbol{Q}'_{ob}(t)$ ($\boldsymbol{P}''_{ob}(t)$ and $\boldsymbol{Q}''_{ob}(t)$) the corresponding matrices.

The procedure outlined above, requires the computation of the minimal eigenvalue of the symmetrical matrices $(\boldsymbol{Q} - \dot{\boldsymbol{P}})$ and $(\boldsymbol{Q}'_{ob} - \dot{\boldsymbol{P}}'_{ob})$ (and $(\boldsymbol{Q}''_{ob} - \dot{\boldsymbol{P}}''_{ob})$). This is usually done numerically and it may be the case that this number is very close to zero. Thus one may worry that the sign of this quantity be incorrect because of numerical errors. The following proposition may be used to validate the approach.

Proposition 2. *Let $\boldsymbol{M} \in \mathbb{R}^{m\times m}$ be a symmetric matrix with eigenvalues λ_i and eigenvectors $\boldsymbol{v}_i$, and let $\hat{\lambda}_i$ and $\hat{\boldsymbol{v}}_i$, be the corresponding estimates ($\boldsymbol{v}_i$ and $\hat{\boldsymbol{v}}_i$ have unitary euclidean norm).*

Let us consider the intervals $\mathcal{I}_i = [\hat{\lambda}_i - \beta_i, \hat{\lambda}_i + \beta_i]$, where $\beta_i = \|\boldsymbol{M}\hat{\boldsymbol{v}}_i - \hat{\lambda}_i\hat{\boldsymbol{v}}_i\|_2$. If $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ for all $i \neq j$, then $\lambda_i \in \mathcal{I}_i$ for all i.

Proof. Follows from the fact that if $\boldsymbol{M}$ is a symmetric real matrix its eigenvalues are real, and its eigenvectors are orthogonal. Thus the relation [15]

$$\min_i |\lambda_i - \hat{\lambda}| \leq \|\boldsymbol{M}\boldsymbol{w} - \hat{\lambda}\boldsymbol{w}\|_2$$

holds $\forall\hat{\lambda} \in \mathbb{R}$ and $\forall\boldsymbol{w} \in \mathbb{R}^m$ with $\|\boldsymbol{w}\|_2 = 1$. □

6 An applicative example

In this section we show how the above procedure can be applied to a real overhead crane. We consider a model produced by Munck Cranes Inc., Ontario–Canada whose load capacity varies

Load mass (Kg)	**20÷500**	**501÷15000**	**15001÷50000**
Lifting	{ -0.995 ± 0.577j, -0.357 ± 0.216 j, -1.267 ± 0.637 j, -0.375 ± 0.196 j }	{ -0.189 ± 0.098 j, -0.636 ± 0.302 j, -0.270 ± 0.154 j, -0.798 ± 0.418 j }	{ -0.298 ± 0.151 j, -0.100 ± 0.057 j, -0.552 ± 0.273 j, -0.177 ± 0.100 j }
Lowering	{ -0.997 ± 0.891 j, -2.369 ± 0.411 j, -0.997 ± 0.891 j, -2.369 ± 0.411 j }	{ -1.903 ± 0.875 j, -0.335 ± 0.181 j, -1.621 ± 0.867 j, -0.398 ± 0.247 j }	{ -1.108 ± 0.464 j, -0.147 ± 0.072 j, -1.123 ± 0.519 j, -0.201 ± 0.107 j }

Table 1: *The selected sets of eigenvalues for* $\bar{\boldsymbol{A}}_\tau$ *when the controller is used in conjunction with the first observer.*

from 1 to 50 ton. In particular, in this paper we consider an overhead crane whose trolley mass is $m_T = 4037\,\text{Kg}$ and whose bridge mass is $m_B = 4112\,\text{Kg}$. We assume the length of the suspending rope to be: $L(t) \in [L_{min}, L_{max}]$, where $L_{min} = 2\,\text{m}$ and $L_{max} = 10\,\text{m}$. To deduce the controller and observer gain matrices we assumed that the rope length has a constant derivative $|\dot{L}(t)| = 0.5\,\text{m/s}$. Clearly this is not true during a real movement. Therefore during numerical simulations, we have removed this assumption and we have imposed an acceleration of $\pm 0.5\,\text{m/s}^2$ at the beginning and at the end of the hoisting and lowering movement, while in the central part of the movement the velocity is constant and equal to $\pm 0.5\,\text{m/s}$.

During the simulations, we have also removed the assumption of linearity thus we used the nonlinear model given in the appendix. The wind force acting on the load is taken into account as well. All numerical simulations have been carried out with the SIMULINK toolbox of MATLAB.

In previous works the authors used the gain–scheduling technique to derive a satisfactory control law for a given planar crane [4, 11]. In those works, even in the second one where also an observer has been designed, a single set of eigenvalues for the controller and a single one for the observer has been used. In this paper, we make a different choice motivated by the greater complexity of the system at hand. In particular, we divided the whole range of possible values of the load mass in three different intervals and we further distinguished among lowering and lifting movement. Then, we associated to each range a different set of eigenvalues for the reference stationary system and the error system. In this way we introduced a double gain–scheduling, thus producing a significant improvement in the performance of the controlled

Load mass (Kg)	**20÷500**	**501÷15000**	**15001÷50000**
Lifting	{ -1.385 ± 0.611j, -0.391 ± 0.173j, -1.250 ± 0.654j, -0.449 ± 0.235j }	{ -0.382 ± 0.188j, -0.144 ± 0.061j, -0.591 ± 0.311j, -0.237 ± 0.124j }	{ -0.761 ± 0.379j, -0.315 ± 0.157j, -0.973 ± 0.477j, -0.397 ± 0.195j }
Lowering	{ -1.204 ± 1.195j, -0.472 ± 0.461j, -1.203 ± 1.204j, -0.476 ± 0.495j }	{ -1.417 ± 0.681j, -0.360 ± 0.173 j, -1.850 ± 0.926 j, -0.505 ± 0.253 j }	{ -0.887 ± 0.341j, -0.156 ± 0.066 j, -0.891 ± 0.484 j, -0.257 ± 0.140 j }

Table 2: *The selected sets of eigenvalues for $\bar{\boldsymbol{A}}_\tau$ when the controller is used in conjunction with the second observer.*

system. Note that, from an applicative point of view, this does not introduce any amount in the cost of realization of the system, being the load mass assumed known [17] during each operation.

The $p_{x,i}$ and $p_{z,i}$ coefficients, i.e., the design parameters of the controller, are derived choosing the sets of eigenvalues for $\bar{\boldsymbol{A}}_\tau$ and are reported in Tab. 1–2. Tab. 1 contains the eigenvalues of $\bar{\boldsymbol{A}}_\tau$ when the first observer is used in conjunction with the controller, while the eigenvalues of $\bar{\boldsymbol{A}}_\tau$, when the controller is used in conjunction with the second observer, are reported in Tab. 2.

Analogously, the sets of eigenvalues selected for matrices $\bar{\boldsymbol{E}}'_\tau$ and $\bar{\boldsymbol{E}}''_\tau$ respectively, are summarized in Tab. 3–4.

Note that in all cases, the eigenvalues, obtained as the result of a trial and error procedure, have been reported in the mentioned tables with the following order: we have first written the eigenvalues relative to matrix $\bar{\boldsymbol{A}}_{\tau,x}$ ($\bar{\boldsymbol{E}}'_{\tau,x}$, $\bar{\boldsymbol{E}}''_{\tau,x}$), then those relative to $\bar{\boldsymbol{A}}_{\tau,z}$ ($\bar{\boldsymbol{E}}'_{\tau,z}$, $\bar{\boldsymbol{E}}''_{\tau,z}$).

Now, let us present the results of three different simulation test cases.

6.1 Simulation 1

In the first simulation, we considered a load mass equal to the maximum load capacity, i.e., equal to 50 ton. We assumed that the only measurable variables are x_T and z_T, thus the first asymptotic observer was used for the state reconstruction. The simulation was performed for a lifting movement from $L_o = 10\,\mathrm{m}$ to $L_f = 2\,\mathrm{m}$. The corresponding set of eigenvalues used for the determination of the design parameters can be easily argued from Tab. 1–3. In this

Load mass (Kg)	**20÷500**	**501÷15000**	**15001÷50000**
Lifting	{ -2.247 ± 0.1331j, -0.842 ± 0.495 j, -2.247 ± 0.1331j, -0.842 ± 0.495 j }	{ -1.943 ± 0.834cj, -0.704 ± 0.302 j, -2.331 ± 0.964 j, -0.806 ± 0.333 j }	{ -3.030 ± 0.873 j, -1.467 ± 0.423 j, -3.064 ± 0.905 j, -1.513 ± 0.447 j }
Lowering	{ -3.400 ± 0.010 j, -1.500 ± 0.010 j, -3.400 ± 0.010 j, -1.500 ± 0.010 j }	{ -2.867 ± 0.667 j, -0.434 ± 0.093 j, -2.889 ± 0.783 j, -0.658 ± 0.166 j }	{ -4.881 ± 0.610 j, -0.442 ± 0.074 j, -3.897 ± 0.596 j, -0.680 ± 0.133 j }

Table 3: *The selected sets of eigenvalues for* $\bar{\boldsymbol{E}}'_{\tau}$.

simulation we assumed that no external disturbance was acting on the load. The initial state of the crane was $x_V(0) = z_V(0) = 1.5\,\text{m}$, $x_C(0) = z_C(0) = -5\,\text{m}$, $\dot{x}_V(0) = \dot{x}_C(0) = \dot{z}_V(0) = \dot{z}_C(0) = 0\,\text{m/s}$, while the initial state of the observer was $\hat{x}_V(0) = 1\,\text{m}$, $\hat{x}_C(0) = -4.5\,\text{m}$, $\hat{z}_V(0) = 2\,\text{m}$, $\hat{z}_C(0) = -5.5\,\text{m}$, $\hat{\dot{x}}_V(0) = \hat{\dot{x}}_C(0) = \hat{\dot{z}}_V(0) = \hat{\dot{z}}_C(0) = 0\,\text{m/s}$.

In Fig. 2 the results of the first simulation are reported. Figure (a) shows the displacement of the load wrt to a fixed coordinate system. Figure (b) shows the displacement of the load wrt the vertical and enables us to conclude that quite no oscillation occurs during the load movement. In (c) we reported the estimation error on the first and on the second component of the state (the following notation has been used: $e_V = x_V - \hat{x}_V$, $e_C = x_C - \hat{x}_C$). The corresponding errors for the fifth and sixth state component have not been reported here for brevity's sake, having a similar behaviour. Finally, in (d) the curves representative of the control forces are shown.

6.2 Simulation 2

In the second simulation, we considered a limit case of a movement with no load, i.e., $m_L = 20\,\text{Kg}$. Note that $m_L = 0\,\text{Kg}$ is not a significant value. In fact, even if no load has to be transferred, an hook is suspended to the rope as well and its mass can be realistically assumed equal to 20 Kg. We assumed that both the trolley position and the load position wrt the vertical are measured, i.e., we implemented the second observer. The simulation was performed for a lowering movement from $L_o = 2\,\text{m}$ to $L_f = 10\,\text{m}$. The corresponding set of eigenvalues used for the determination of the design parameters can be easily argued from

Load mass (Kg)	**20÷500**	**501÷15000**	**15001÷50000**
Lifting	{ -4.495 ± 0.201j, -3.896 ± 0.174 j, -2.497 ± 0.112 j, -2.597 ± 0.116 j }	{ -2.797 ± 0.125 j, -1.298 ± 0.059 j, -2.797 ± 0.125 j, -1.398 ± 0.063 j }	{ -1.998 ± 0.090 j, -1.198 ± 0.054 j, -2.197 ± 0.098 j, -1.198 ± 0.054 j }
Lowering	{ -12.492 ± 0.335 j, -4.096 ± 0.139 j, -12.492 ± 0.335 j, -4.096 ± 0.139j }	{ -8.899 ± 0.127 j, -4.499 ± 0.064 j, -8.899 ± 0.127 j, -4.499 ± 0.064 j }	{ -10.499 ± 0.042 j, -10.499 ± 0.078 j, -9.499 ± 0.042 j, -9.499 ± 0.064 j }

Table 4: *The selected sets of eigenvalues for* $\bar{\boldsymbol{E}}''_{\tau}$.

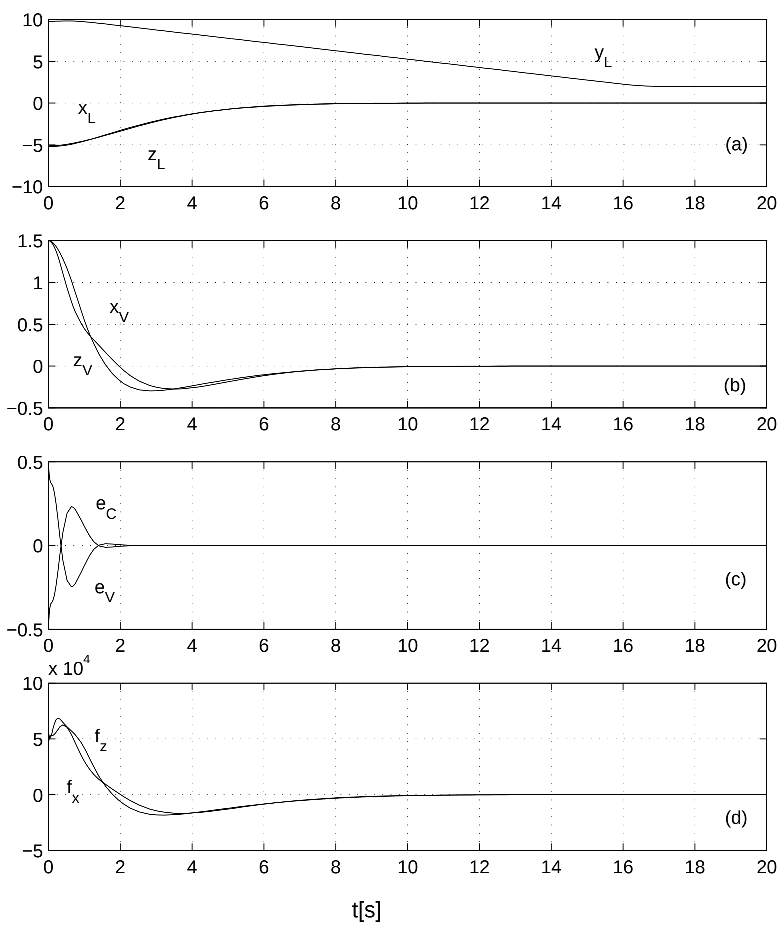

Figure 2: *Results of Simulation 1.*

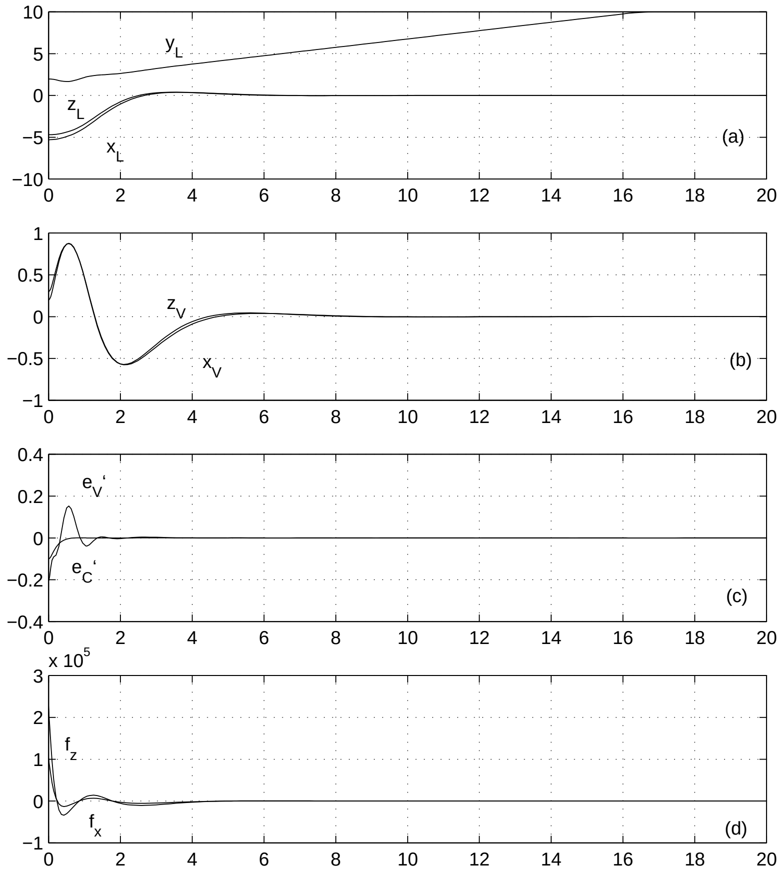

Figure 3: *Results of Simulation 2.*

Tab. 2–4. In this simulation we assumed again that no external disturbance was acting on the load. The initial state of the crane was $x_V = 0.3\,\mathrm{m}$, $x_C = -5\,\mathrm{m}$, $z_V = 0.2\,\mathrm{m}$, $z_C - 4.5\,\mathrm{m}$, $\dot{x}_V(0) = \dot{x}_C(0) = \dot{z}_V(0) = \dot{z}_C(0) = 0\,\mathrm{m/s}$, while the initial state of the observer was $\hat{x}_V = 0.3\,\mathrm{m}$, $\hat{x}_C = -5\,\mathrm{m}$, $\hat{z}_V = 0.2\,\mathrm{m}$, $\hat{z}_C - 4.5\,\mathrm{m}$, $\hat{\dot{x}}_V(0) = 0.2\,\mathrm{m/s}$, $\hat{\dot{x}}_C(0) = 0.1\,\mathrm{m/s}$, $\hat{\dot{z}}_V(0) = -0.2\,\mathrm{m/s}$, $\hat{\dot{z}}_C(0) = -0.1\,\mathrm{m/s}$.

The results of this second simulation are reported in Fig. 3 where the physical meaning of the variables is the same as those reported in Fig. 2, with the only exception of figure (c) where the estimation error of the third and of fourth state components have been reported (the following notation has been used: $e'_V = \dot{x}_V - \hat{\dot{x}}_v$, $e'_C = \dot{x}_C - \hat{\dot{x}}_C$). In fact, in this case, i.e., when the second observer is used, we introduced no error on the initial estimation of x_V, x_C, z_V and z_C, being these variables assumed known by definition of the output matrix.

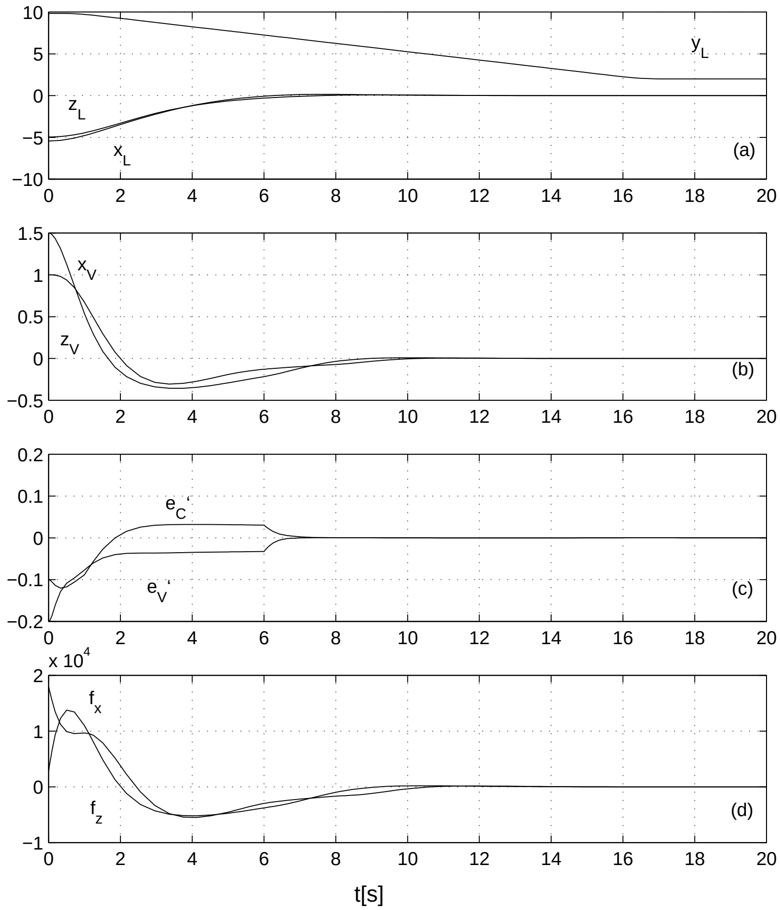

Figure 4: *Results of Simulation 3.*

6.3 Simulation 3

In this simulation we considered an intermediate value of the load mass, i.e., $m_L = 10000\,\mathrm{Kg}$. As in the previous simulation, we assumed to reconstruct the state with the second observer. We have also taken into account the wind effect acting on both the X and the Z direction. In particular we assumed that $d_x = 1000\,\mathrm{N}$ during the time interval $[1\,\mathrm{s}, 6\,\mathrm{s}]$ and $d_z = 500\,\mathrm{N}$ during the time interval $[5\,\mathrm{s}, 8\,\mathrm{s}]$. These values seem to be reasonable considering the wind modellization reported in standard engineering handbooks. A lifting movement from $L_o = 10\,\mathrm{m}$ to $L_f = 2\,\mathrm{m}$ has been taken into account. The initial state of the crane and that of the observer was the same as those already considered in the previous simulation, as well as the same significant have the plots in Fig. 4.

6.4 Stability analysis

The stability analysis presented in Section 5 requires the computation of $\lambda_{min}\{\boldsymbol{Q} - \dot{\boldsymbol{P}}\}$ (as a function of L) for different values of $\dot{L}$.

Fig. 5.a shows the plot of $\lambda_{min}\{\boldsymbol{Q} - \dot{\boldsymbol{P}}\}$ versus L for different values of $\dot{L}$ and for $m_L = 50000\,\mathrm{Kg}$. In this case we assumed that the eigenvalues of matrix $\bar{\boldsymbol{A}}_\tau$ are those reported in Tab. 2. However, the same conclusions can be drawn when considering those reported in Tab. 1.

In each diagram, only the extreme values of $\dot{L}$ have been reported for clearness. However, the other values can be easily argued considering that $\dot{L}$ varies of a quantity equal to 0.1 m/s from one curve to the adjacent one.

According to theorem 1, the upper bound on $|\dot{L}|$ is the value corresponding to the first curve that, as $|\dot{L}|$ is increased, goes to negative values. As can be seen from Fig. 5.a, relative to lifting operations, this happens for $|\dot{L}| > 1.4\,\mathrm{m/s}$. The same conclusion can be derived in the case of a lowering movement (the corresponding figures are not reported). Hence it can be concluded that the time–varying system with matrix $\bar{\boldsymbol{A}}_t$ is stable if $|\dot{L}| < 1.4\,\mathrm{m/s}$, that is to say it is always stable in nominal conditions.

The same discussion has been done for the error closed loop system when both the observers are considered. In this section we limit to present the results of the stability analysis in the case of the second observer. Analogous conclusions can be drawn in the other case.

In Fig. 5.c we reported the curves corresponding to those in Fig. 5.a where $\boldsymbol{Q}''_{ob}(t)$ and $\boldsymbol{P}''_{ob}(t)$ are determined in the same manner as $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$. The same conclusion can be derived in the case of a lowering movement (the corresponding figures are not reported here). Note that stability of the observer is guaranteed for any velocity of practical interest. This is due to our choice for the sets of observer eigenvalues, that are much more stable than those of the controller.

The analogous curves with $m_L = 20\,\mathrm{Kg}$ and relative to a lowering operation are shown in Fig. 5.b and Fig. 5.d. Similar curves can be drawn in the case of a lifting movement. Even in this case stability is proved for all operating conditions of interest. The same reasoning, as well as the same conclusions, can be repeated for all intermediate values of the load mass.

Note that in reality what is plotted in the previous figures is not λ_{min} but its estimate $\hat{\lambda}_{min}$ computed with a numerical procedure. Even if all computed values $\hat{\lambda}_{min}$ are close to zero, Proposition 2 can be used to ensure that all λ_{min} are positive. In fact, in all cases the estimated eigenvector $\hat{\boldsymbol{w}}$ associated to the estimated eigenvalue $\hat{\lambda}$ was such that the values of

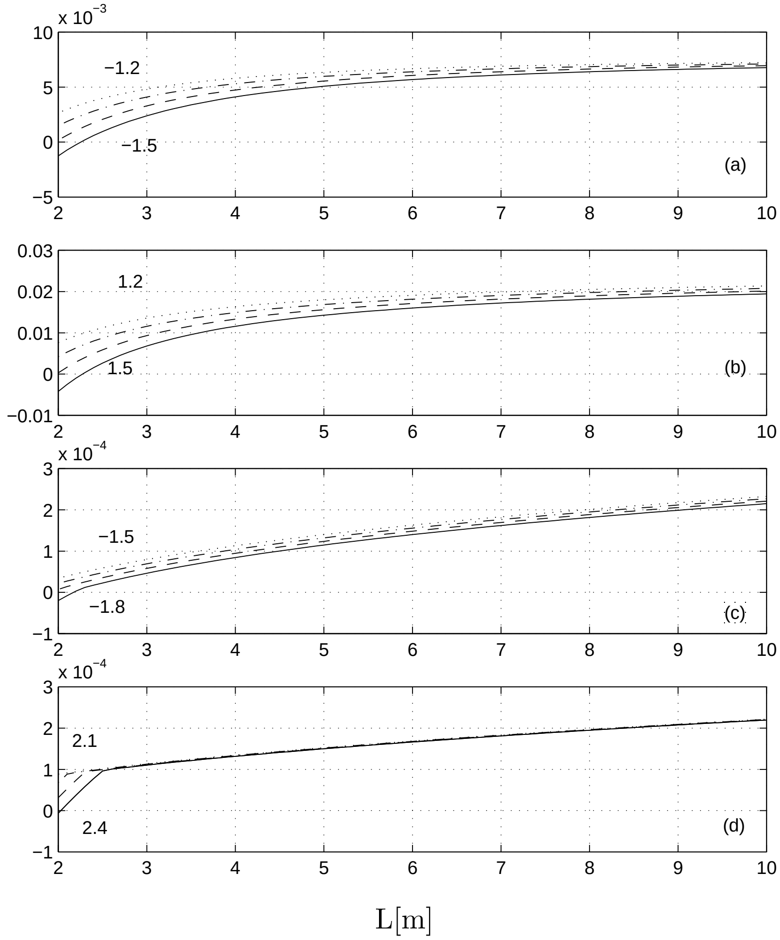

Figure 5: *The results of stability analysis: (a)–(b) plot of* $\lambda_{min}\{\boldsymbol{Q} - \dot{\boldsymbol{P}}\}$ *with* $m_L = 42500Kg$ *and* $m_L = 20Kg$*, respectively; (c)–(d) plot of* $\lambda_{min}\{\boldsymbol{Q}''_{ob} - \dot{\boldsymbol{P}}''_{ob}\}$ *with* $m_L = 42500Kg$ *and* $m_L = 20Kg$*, respectively.*

the norms $\|\{\boldsymbol{Q} - \dot{\boldsymbol{P}}\}\hat{\boldsymbol{w}} - \hat{\lambda}\hat{\boldsymbol{w}}\|_2$ and $\|\{\boldsymbol{Q}''_{ob} - \dot{\boldsymbol{P}}''_{ob}\}\hat{\boldsymbol{w}} - \hat{\lambda}\hat{\boldsymbol{w}}\|_2$ are $\approxeq 10^{-14}$, while all estimated eigenvalues are spaced much further apart.

7 Conclusions

In this paper we presented a general methodology for controlling three–dimensional overhead cranes. This work is an extension of previous ones where the authors limit to consider planar cranes.

Time–scaling relations have been used to reduce the original time–varying system to a stationary one. The observer–controller design for the reference system has been carried out via pole–placement. Then, the time–scalings inversion enabled us to derive in a parametric form the time–varying gains for both the observer and the controller. Note that in this paper we implemented a double gain–scheduling, being the eigenvalues of the closed–loop and of the reference error system dependent on the load mass and on the lowering/lifting movement.

The stability of the time–varying system has also been studied. Using a Lyapunov–like theorem it was possible to find upper bounds for the rate of change of the varying parameter (the length of the suspending rope) that ensure the stability of a given crane in all its possible operating conditions: maximum load and no load, lifting and lowering movements.

Appendix

The dynamics of the system in Fig. 1 are described by the following equations (obtained by the translational equilibrium of the three masses):

$$\begin{cases} m_T\ddot{x}_T = f_x - F sin\varphi\, cos\theta \\ m_L\ddot{x}_L = F sin\varphi\, cos\theta + d_x \\ m_C\ddot{z}_T = u_Z - F sin\varphi\, sin\theta \\ m_L\ddot{z}_L = F sin\varphi\, sin\theta + d_z \\ m_L\ddot{y}_L = m_L g - F cos\varphi \end{cases} \tag{50}$$

where F is the force in the direction of the rope (not shown in Fig. 1); the position of the load wrt the fixed coordinate system can be written as:

$$\begin{cases} x_L = x_T - L sin\varphi\, cos\theta \\ z_L = z_T - L sin\varphi\, sin\theta \\ y_L = L cos\varphi. \end{cases} \tag{51}$$

With the coordinate transformations

$$x_C = \frac{m_T x_T + m_L x_L}{m_T + m_L}, \qquad z_C = \frac{m_C z_T + m_L z_L}{m_C + m_L}, \tag{52}$$

$$x_V = L sin\varphi\, cos\theta, \qquad z_V = L cos\varphi\, sin\theta, \tag{53}$$

the first four equations of (50) can be rewritten as (we assume $m_L > 0$):

$$\begin{cases} \ddot{x}_V + \dfrac{F(\boldsymbol{\varphi}, \boldsymbol{L})}{L}\left(\dfrac{1}{m_T} + \dfrac{1}{m_L}\right) x_V = \dfrac{f_x}{m_T} - \dfrac{d_x}{m_L}, \\ \\ \ddot{x}_C = \dfrac{f_x + d_x}{m_T + m_L}, \\ \\ \ddot{z}_V + \dfrac{F(\boldsymbol{\varphi}, \boldsymbol{L})}{L}\left(\dfrac{1}{m_C} + \dfrac{1}{m_L}\right) z_V = \dfrac{f_z}{m_C} - \dfrac{d_z}{m_L}, \\ \\ \ddot{z}_C = \dfrac{f_z + d_z}{m_C + m_L}, \end{cases} \tag{54}$$

where the rope force $F(\boldsymbol{\varphi}, \boldsymbol{L})$ is a function of $\boldsymbol{\varphi} : (\varphi, \dot{\varphi}, \ddot{\varphi})$ and $\boldsymbol{L} : (L, \dot{L}, \ddot{L})$ as can be determined by twice differentiating the last equation in (51) and substituting into the fifth equation of (50):

$$F = \frac{m_L(g - \ddot{L}cos\varphi + 2\dot{L}\dot{\varphi}sin\varphi + L\dot{\varphi}^2 cos\varphi + L\ddot{\varphi}sin\varphi)}{cos\varphi}. \tag{55}$$

Equations (54) and (55) describe the full nonlinear model of the crane that is used during the simulations. To design a controller/observer a linear model is derived.

Linearizing the force F around the equilibrium point $\varphi^* : (\varphi = 0, \dot{\varphi} = 0, \ddot{\varphi} = 0)$ is equivalent to setting

$$\begin{array}{lll} sin\varphi = \varphi, & cos\varphi = 1, & \dot{\varphi}sin\varphi = 0, \\ \dot{\varphi}^2 = 0, & \ddot{\varphi}sin\varphi = 0, & \end{array} \tag{56}$$

and assuming $\ddot{L}(t) = 0$, equation (55) yields

$$F(\boldsymbol{\varphi}^*, \ddot{L}(t) = 0) = m_L g \tag{57}$$

i.e., the force along the rope is equal to the weight of the load. Substituting this value of F into equation (54) and assuming no disturb is acting on the system we obtain the reference linearized model:

$$
\begin{cases}
\ddot{x}_V + \dfrac{g(m_T+m_L)}{m_T L} x_V = \dfrac{f_x}{m_T}, \\[2ex]
\ddot{x}_C = \dfrac{f_x}{m_T+m_L}, \\[2ex]
\ddot{z}_V + \dfrac{g(m_C+m_L}{m_C L} z_V = \dfrac{f_z}{m_C}, \\[2ex]
\ddot{z}_C = \dfrac{f_z}{m_C+m_L}.
\end{cases}
\tag{58}
$$

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