

# Hybrid Petri net modeling of inventory management systems

Roberto Furcas, Alessandro Giua, Aldo Piccaluga, Carla Seatzu

DIEE, Università di Cagliari, Piazza d'Armi, 09123 Cagliari, Italy

email: {giua,seatzu}@diee.unica.it, tel: +39 070 675.58.92, fax: +39 070 675.59.00

## Abstract

This paper shows how First-Order Hybrid Petri nets, a class of nets that combine fluid and discrete event dynamics, can be used to describe inventory control systems. We describe different management policies based on fixed order quantities or periodic review, and show how the total cost can be easily evaluated by adding appropriate net structures. We finally show how using standard hybrid Petri net tools it is possible to optimize some of the systems parameter via simulation.

To appear as:

R. Furcas, A. Giua, A. Piccaluga, C. Seatzu, "Hybrid Petri net modeling of inventory management systems", *European Journal of Automation APII-JESA*, Special Issue on *Hybrid Dynamical Systems*, 2001 (to appear).

## 1. Introduction

Inventory control is needed in every organization that holds stocks of some kind, because inventories represent a major investment which should be managed efficiently. A typical manufacturing company holds 20% of its production as stock, and this has annual costs of around 25% of value. If stocks are not controlled properly the costs can become excessive and reduce an organization's ability to compete. Efficient inventory control then becomes a real factor in an organization's long-term survival.

*Inventory Management Systems* (IMS) [9] are discrete event dynamic systems whose number of reachable states is typically very large, thus the analysis and optimization of these systems require large amount of computational efforts and problems of realistic scale quickly become analytically and computationally untractable.

To cope with this problem, *fluid models* which are continuous-dynamics approximations of discrete event systems, may be successfully developed and applied to the inventory management domain. This has several advantages. Firstly, there is the possibility of considerable increase in computational efficiency, because the simulation of fluid models can often be done much more efficiently. Secondly, fluid approximations provide an aggregated formulation to deal with complex systems, thus reducing the dimension of the state space. Thirdly, the design parameters in fluid models are continuous (e.g., storage areas and order sizes), hence there is the possibility of using gradient information to speed up optimization and perform sensitivity analysis.

It should be noted that in general different fluid approximations are necessary to describe the same IMS, depending on its discrete state: new orders required or only demand satisfaction, storage full or empty, and so on. Thus, the resulting model can be better described as an *hybrid model*, where different dynamics are associated to each discrete state.

In this paper we use *First-Order Hybrid Petri Nets* (FOHPN), a class of nets that combines fluid and discrete event dynamics, to describe inventory control systems. This hybrid model was originally presented in [4] and adds to the formalism described by David and Alla [1, 2] linear algebraic tools for the analysis and control of the model. In this preliminary paper many of the features of FOHPN are not used because we are mainly interested in simulation. Thus we were able to use the simulator SYRPHICO [6] developed for the model of [1, 2] with suitable modifications, as described in Section 5.

The choice of FOHPN as a modeling tool for IMS presents many advantages [5]. Hybrid Petri nets are both a graphical and mathematical modeling tool able to simulate the dynamic concurrent activities of IMS. They enable a modular representation of an IMS, i.e., if a system is composed by many subsystems interacting among them, it is possible to model all subsystems with sub-nets and put them together to get the model of the whole IMS. Finally, as shown in [4], the FOHPN model is amenable to sensitivity analysis, i.e., it can be used to obtain information about the degrees of freedom that can be exploited when making performance optimization or optimal design of the system parameters configuration. This feature, that is not used in this work, will be the object of future research.

In this paper we show how some independent demand IMS can be modeled with FOHPN. In particular, we associate to each management policy a different net (a module). We consider *fixed order quantity systems* (FOQS) and *periodic review systems* (PRS) and also show how costs relative to the different management policies can be easily evaluated by adding appropriate net structures to the corresponding IMS module. Finally, we present a numerical example to demonstrate how all these modules can be put together and then implemented by means of an appropriate simulator [6]. The numerical example also shows how FOHPN can be used via simulation as an efficient tool for the solution of some numerical optimization problems for which there exist no closed form solution.

## 2. First-order hybrid Petri nets

We recall the Petri net formalism used in this paper following [4]. A First-Order Hybrid Petri Net (FOHPN) is a structure  $N = (P, T, Pre, Post, \mathcal{D}, \mathcal{C})$ .

The set of *places*  $P = P_d \cup P_c$  is partitioned into a set of *discrete* places  $P_d$  (represented as circles) and a set of *continuous* places  $P_c$  (represented as double circles).

The set of *transitions*  $T = T_d \cup T_c$  is partitioned into a set of discrete transitions  $T_d$  and a set of continuous transitions  $T_c$  (represented as double boxes). The set  $T_d = T_I \cup T_D \cup T_E$  is further partitioned into a set of *immediate* transitions  $T_I$  (represented as bars), a set of *deterministic timed* transitions  $T_D$  (represented as black boxes), and a set of *exponentially distributed timed* transitions  $T_E$  (represented as white boxes).

The *pre-* and *post-incidence functions* that specify the arcs are (here  $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ ):

$Pre, Post : P_d \times T \rightarrow \mathbb{N}, P_c \times T \rightarrow \mathbb{R}_0^+$ . We require (*well-formed nets*) that for all  $t \in T_c$  and for all  $p \in P_d$ ,  $Pre(p, t) = Post(p, t)$ .

The function  $\mathcal{D} : T_D \cup T_E \rightarrow \mathbb{R}^+$  specifies the delay  $d$  associated to deterministic discrete transitions and the firing rate  $\lambda$  associated to exponentially distributed transitions.

For any continuous transition  $t_j \in T_c$  we let  $\mathcal{C}(t_j) = (V'_j, V_j)$ , with  $V'_j \leq V_j$ . Here  $V'_j$  represents the *minimum firing speed* (mfs) and  $V_j$  represents the *maximum firing speed* (MFS). In the following, unless explicitly specified, the mfs of a continuous transition will be  $V'_j = 0$ .

The *incidence matrix* of the net is defined as  $\mathbf{C}(p, t) = Post(p, t) - Pre(p, t)$ . The restriction of  $\mathbf{C}$  to  $P_X$  and  $T_Y$  ( $X, Y \in \{c, d\}$ ) is denoted  $\mathbf{C}_{XY}$ . Note that by the well-formedness hypothesis  $\mathbf{C}_{dc} = 0$ . We denote the preset (postset) of transition  $t$  as  $\bullet t$  ( $t\bullet$ ) and its restriction to continuous or discrete places as  ${}^{(d)}t = \bullet t \cap P_d$  or  ${}^{(c)}t = \bullet t \cap P_c$ .

A *marking*  $\mathbf{m} : P_d \rightarrow \mathbb{N}, P_c \rightarrow \mathbb{R}_0^+$  is a function that assigns to each discrete place a non-negative number of tokens, represented by black dots and assigns to each continuous place a fluid volume;  $m_i$  denotes the marking of place  $p_i$ . The value of a marking at time  $\tau$  is denoted  $\mathbf{m}(\tau)$ . A *FOHPN system*  $\langle N, \mathbf{m}(\tau_0) \rangle$  is a FOHPN  $N$  with an initial marking  $\mathbf{m}(\tau_0)$ .

The enabling of a discrete transition depends on the marking of all its input places, both discrete and continuous. A discrete transition  $t$  is *enabled* at  $\mathbf{m}(\tau)$  if for all  $p_i \in \bullet t$ ,  $m_i(\tau) \geq Pre(p_i, t)$  and it may fire yielding

$$\mathbf{m}(\tau) = \mathbf{m}(\tau^-) + \begin{bmatrix} \mathbf{C}_{cd} \\ \mathbf{C}_{dd} \end{bmatrix} \boldsymbol{\sigma}_t$$

where  $\boldsymbol{\sigma}_t : T_d \rightarrow \mathbb{N}$  is such that  $\sigma_t(t) = 1$  and  $\sigma_t(t') = 0$  if  $t \neq t'$ .

A continuous transition is enabled only by the marking of its input discrete places. The marking of its input continuous places, however, is used to distinguish between strongly and weakly enabling. Let  $\langle N, \mathbf{m} \rangle$  be a FOHPN system. A continuous transition  $t$  is *enabled* at  $\mathbf{m}$  if for all  $p_i \in {}^{(d)}t$ ,  $m_i \geq Pre(p_i, t)$ . We say that an enabled transition  $t \in T_c$  is: *strongly enabled* at  $\mathbf{m}$  if for all places  $p_i \in {}^{(c)}t$ ,  $m_i > 0$ ; *weakly enabled* at  $\mathbf{m}$  if for some  $p_i \in {}^{(c)}t$ ,  $m_i = 0$ .

The enabling state of a continuous transition  $t_i$  defines its admissible *instantaneous firing speed*  $v_i$ . If  $t_i$  is not enabled then  $v_i = 0$ . If  $t_i$  is strongly enabled, then it may fire with any

firing speed  $v_i \in [V'_i, V_i]$ . Finally, if  $t_i$  is weakly enabled, then it may fire with any firing speed  $v_i \in [V'_i, \bar{V}_i]$ , where  $\bar{V}_i \leq V_i$  depends on the amount of fluid entering the empty input continuous place(s) of  $t_i$ . In fact, the transition cannot remove more fluid from any empty input continuous place  $\bar{p}$  than the quantity entered in  $\bar{p}$  by other transitions.

The *instantaneous firing speed* (IFS) at time  $\tau$  of a transition  $t_j \in T_c$  is denoted  $v_j(\tau)$  and  $\mathbf{v}(\tau) = [v_1(\tau), \dots, v_{n_c}(\tau)]^T$  is the IFS vector at time  $\tau$  ( $n_c$  is the number of continuous places).

We use linear inequalities to characterize the set of *all* admissible firing speed vectors  $\mathcal{S}$ . Each IFS vector  $\mathbf{v} \in \mathcal{S}$  represents a particular mode of operation of the system described by the net, and among all possible modes of operation, the system operator may choose the best according to a given objective. In all the examples considered in this paper we implicitly assume that the performance index to be optimized is the sum of the firing speeds of continuous transitions. This implies that, whenever a continuous transition is strongly enabled, then it fires at its maximum firing speed.

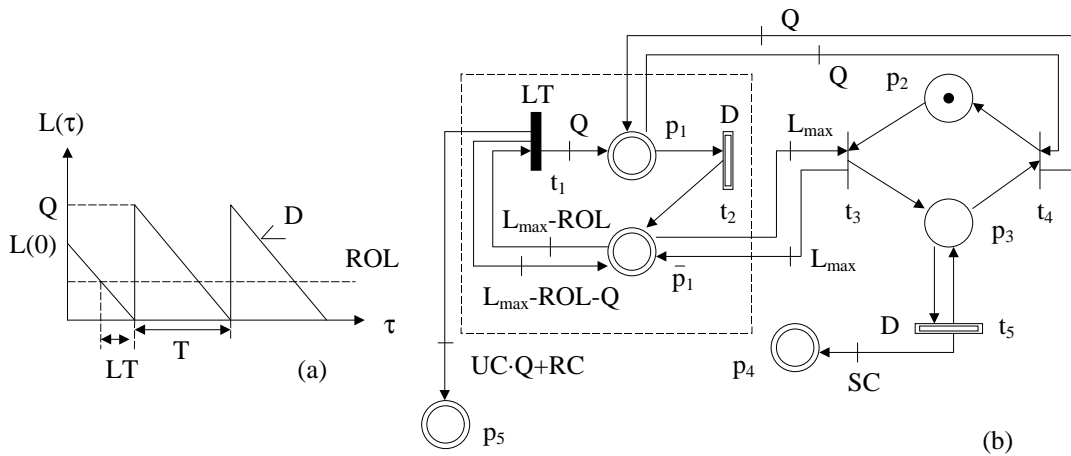
As  $\mathbf{m}$  changes the IFS vector may vary as well. In particular it changes at the occurrence of the following *macro-events*: (a) a discrete transition fires, thus changing the discrete marking and enabling/disabling a continuous transition; (b) a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak.

Let  $\tau_k$  and  $\tau_{k+1}$  be the occurrence times of two consecutive macro-events of this kind; we assume that within the interval of time  $[\tau_k, \tau_{k+1})$  the IFS vector is constant and we denote it  $\mathbf{v}(\tau_k)$ . Then the continuous behavior of a FOHPN for  $\tau \in [\tau_k, \tau_{k+1})$  is described by

$$\begin{cases} \mathbf{m}^c(\tau) &= \mathbf{m}^c(\tau_k) + \mathbf{C}_{cc}\mathbf{v}(\tau_k)(\tau - \tau_k) \\ \mathbf{m}^d(\tau) &= \mathbf{m}^d(\tau_k). \end{cases} \quad (1)$$

### 3. Modeling inventory control systems with FOHPN

Independent demand systems can use either fixed order quantities or periodic reviews [9]. Fixed order quantity systems place an order of fixed size whenever stock falls to a certain level. Such systems need continuous monitoring of stock levels and are better suited to low, irregular demand for relatively expensive items. Periodic review systems place orders of varying size at regular intervals to raise the stock level to a specified value. The operating cost of this system is lower and it is better suited to high, regular demand



**Figure 1:** *Fixed order quantity system with finite lead time: regular pattern (a) and net model (b).*

of low-value items.

In this section we present the most important models of IMS characterized by a fixed order quantity, namely FOQS with finite lead time, finite replenishment rate and back order; finally we discuss the case of periodic review systems.

### 3.1. FOQS with finite lead time

Let us consider a fixed order quantity system with a finite lead time and a fixed reorder level. Under the assumption of continuous and constant demand, the stock level of an item varies with a typical pattern shown in figure 1.a where the following notation has been used:  $L(\tau)$  is the *stock level* at the generic time instant  $\tau$ ;  $Q$  is the fixed *order quantity*;  $LT$  is the *lead time*, i.e., the delay between placing an order and receiving the goods in stock;  $T$  is the *cycle time*, i.e., the time between two consecutive replenishment;  $D$  is the *demand*, i.e., it denotes the number of units to be supplied from stock in a given time period (it coincides with the constant slope, taken as positive, of the curves in figure 1.a).

In fixed order quantity systems new orders take place whenever the stock level falls to the *reorder level*  $ROL$ .

In figure 1.b the FOHPN model for this kind of systems is presented within the dashed rectangle. The marking  $m_1$  of continuous place  $p_1$  represents the stock level while the marking  $\bar{m}_1$  of complementary place  $\bar{p}_1$  represents the available space in the storage area.

By construction, at each time instant  $\tau$  it holds:  $m_1 + \bar{m}_1 = L_{max}$ , where  $L_{max}$  represents the maximum capacity of the storage area that is in general much greater than the order quantity  $Q$ .

When  $m_1 > 0$ , transition  $t_2$  may fire at its maximum firing speed  $D$ , thus reducing the marking of  $p_1$  with a constant slope  $D$ . As soon as the marking of  $p_1$  falls to the reorder level  $ROL$  (i.e., the marking of  $\bar{p}_1$  goes over  $L_{max} - ROL$ ) discrete transition  $t_1$  is enabled and fires after  $LT$  time units. When  $t_1$  fires the ordered quantity  $Q$  is received in the stock, thus this firing produces an increasing of  $Q$  units in  $p_1$  and a decreasing of the same quantity in the complementary place  $\bar{p}_1$ .

### *Management costs evaluation*

Now, let us discuss how an appropriate FOHPN module can be used to compute the total costs related to the IMS module previously described.

The total cost  $C$  during an interval of time  $T_h$  may be calculated as follows:

$$C = UC \cdot Q \cdot N + RC \cdot N + HC \cdot L_{av} \cdot T_h + SC \cdot N_{sc} \quad (2)$$

where the following notation has been used:  $UC$  is the *unit cost*, i.e., the price charged by the suppliers for one unit of the item, or the total cost to the organization of acquiring one unit;  $N$  is the number of orders;  $RC$  is the *reorder cost*, i.e., the cost of placing a routine order for the item and might include allowances for drawing up an order, computer time, correspondence, telephone costs, receiving, use of equipment, expediting delivery, quantity checks, and so on;  $HC$  is the *holding cost*, i.e., the cost of holding one unit of an item in stock for one period of time;  $L_{av}$  is the *average stock level*;  $T_h$  is the *time held*;  $SC$  is the *shortage cost*: if the demand for an item cannot be met because stocks have been exhausted, there is usually some associated shortage cost;  $N_{sc}$  is the number of lost sales.

Thus, the total cost is given by the sum of four different costs components: the *unit cost component*  $UC \cdot Q \cdot N$ , the *reorder cost component*  $RC \cdot N$ , the *holding cost component*  $HC \cdot L_{av} \cdot T_h$ , and the *shortage cost component*  $SC \cdot N_{sc}$ .

Note that the fixed order quantity policy assumes that in nominal conditions (when the demand is constant) the system is designed so that  $D = Q/T = ROL/LT$  holds, and no shortage occurs. On the contrary, in the case of stochastic demand, shortage may occur producing a significant increase in the total cost.

In figure 1.b we show how all the above costs can be computed via an appropriate

FOHPN. The net within the dashed rectangle models the fixed order quantity system and has been already discussed. The content of continuous place  $p_5$  is equal to the sum of the unit cost component and the reorder cost component. In fact, the firing of timed transition  $t_1$  corresponds to a new order and the weight of the arc from  $t_1$  to  $p_5$  is equal to  $UC \cdot Q + RC$ .

The holding cost component can be immediately calculated by the knowledge of the marking in  $p_1$ , being  $L_{av} = 1/T_h \int_0^{T_h} m_1(\tau) d\tau$ . We assume that this measure is directly available when simulating the net with a hybrid Petri net tool.

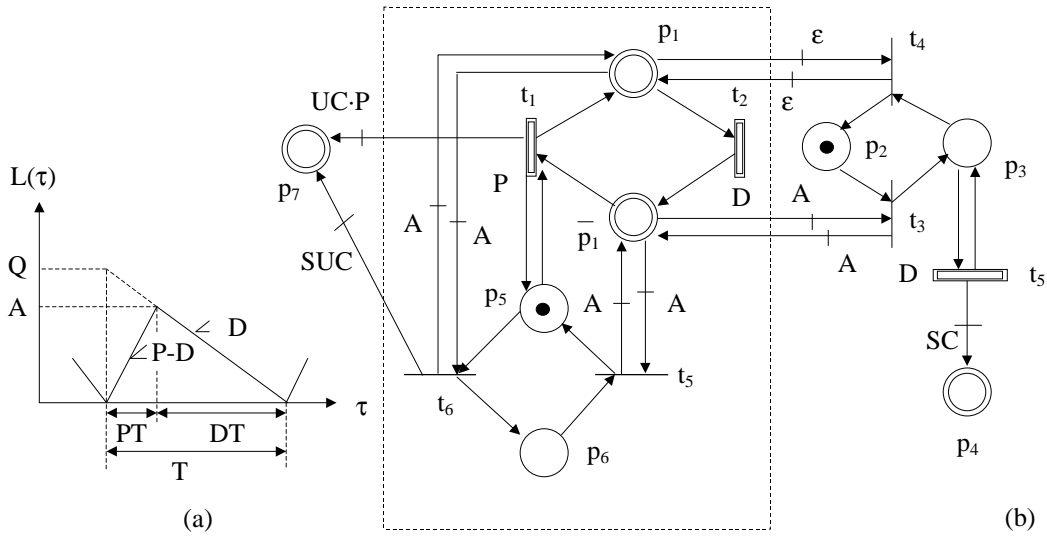
The rest of the FOHPN has been added so as to compute the shortage cost component, that is null unless unforeseen demand occurs. In normal operating condition place  $p_2$  is marked as shown in figure 1.b. A shortage occurs when place  $p_1$  becomes empty and the marking of its complementary place  $\bar{p}_1$  reaches the value  $L_{max}$ , thus enabling the firing of immediate transition  $t_3$  that moves the token from  $p_2$  to  $p_3$ . While  $p_3$  is marked, transition  $t_5$  fires at its maximum firing speed  $D$ , thus increasing the marking of continuous place  $p_4$ , that represents at each instant  $\tau$  the shortage cost. As soon as a new order arrives,  $m_1 = Q$  and immediate transition  $t_4$  fires, re-establishing the normal operating condition.

### 3.2. *FOQS with finite replenishment rates*

In the previous subsection we dealt with a typical situation met by wholesalers: a large delivery of an item instantaneously raises the stock level and then the demand reduces it. Now, let us consider the stock of finished goods at the end of a production line. If the rate of production is greater than demand, goods will accumulate at a finite rate while the line is operating. This gives a situation where the instantaneous replenishment is replaced by a finite replenishment rate. A similar pattern is met with stocks of work in progress between two machines: the first machine builds up stock at a finite rate while the second machine creates demand to reduce it.

If the rate of production is greater than demand, the stock level rises at a rate which is the difference between production and demand. If we call the rate of production  $P$ , stocks will build up at a rate  $P - D$ , as shown in figure 2.a. Stock will continue to accumulate as long as production continues. After some time,  $PT$ , a decision is made to stop production. Then, stock is used to meet demand and declines at a rate  $D$ . After some further time,  $DT$ , all stock has been used and production must start again. Thus, a decision must be





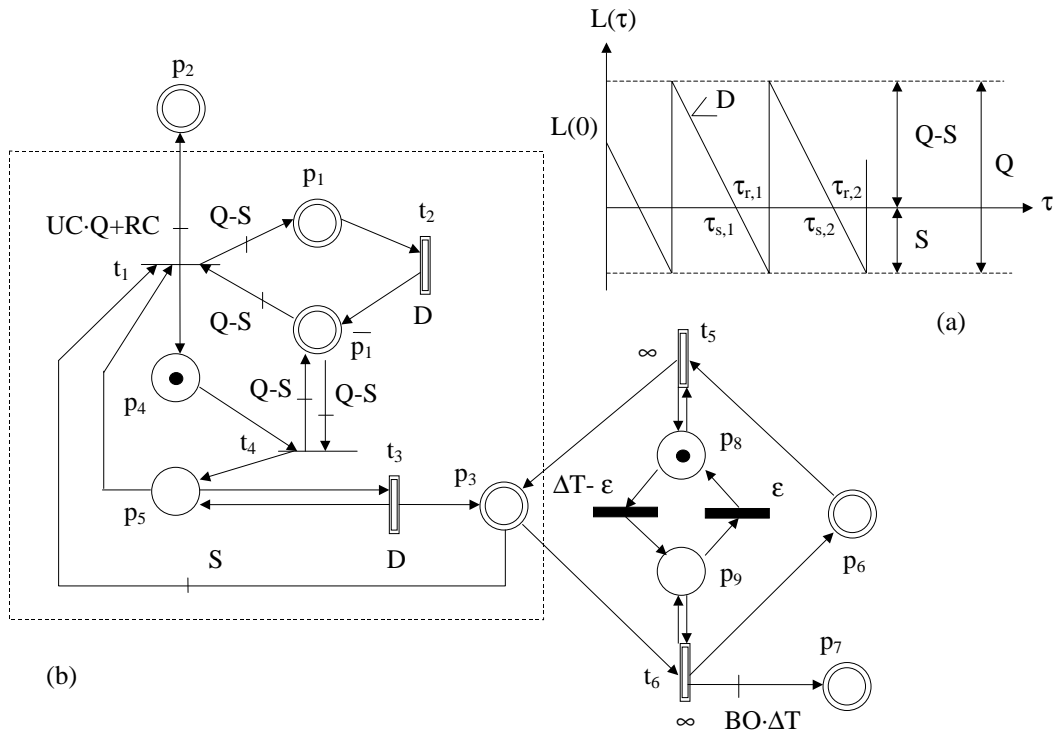
**Figure 2:** *Fixed order quantity system with finite replenishment rate: regular pattern (a) and net model (b).*

made at some point to stop production of this item and transfer facilities to making other items [9].

The resulting variation in stock level is shown in figure 2.a where  $A = (P - D) \cdot PT$ , and  $L$ ,  $Q$ ,  $T$  have the same physical meaning as in the previous subsection.

In figure 2.b the FOHPN model for this kind of systems is shown within the dashed rectangle. The marking of continuous place  $p_1$  denotes the stock level, while  $\bar{p}_1$  is its complementary place and at each time instant  $m_1 + \bar{m}_1 = A$ . When the discrete place  $p_5$  is marked, as in figure 2.b, continuous transitions  $t_1$  and  $t_2$  may fire at their maximum firing speeds,  $P$  and  $D$  respectively. Assuming  $P > D$ , the fluid content of continuous place  $p_1$  increases with a constant slope equal to  $P - D$ , while the fluid content of  $\bar{p}_1$  decreases at the same constant slope. As soon as  $m_1 = A$ , transition  $t_6$  fires thus moving the token from  $p_5$  to  $p_6$ . This produces the disabling of transition  $t_1$ , and the stock level starts decreasing at a constant slope equal to the demand  $D$ . Such a decreasing proceeds until  $p_1$  gets empty, and the content of its complementary place  $\bar{p}_1$  is equal to  $A$ , thus producing the firing of transition  $t_5$ . Then the cycle repeats unaltered.

### *Management costs evaluation*



**Figure 3:** Fixed order quantity system with back-orders: regular pattern (a) and net model (b).

The total cost  $C$  during an interval of time  $T_h$  may be calculated as:

$$C = UC \cdot P \cdot \Delta T_P + SUC \cdot N_{SU} + HC \cdot L_{av} \cdot T_h + SC \cdot N_{sc} \quad (3)$$

where the notation is the same as in the previous subsection, apart from the following variables:  $\Delta T_P$  is the length of the time interval during which production occurs,  $SUC$  is the *set-up cost* required each time production is started,  $N_{SU}$  is the *number of set-ups*, i.e., of production cycles.

Thus the total cost is given by the sum of four cost components: the *unit cost component*  $UC \cdot Q \cdot \Delta T_P$ , the *set-up cost component*  $SUC \cdot N_{SU}$ , the *holding-cost component*  $HC \cdot L_{av} \cdot T_h$ , and the *shortage cost component*  $SC \cdot N_{SC}$ .

As in the previous case, the total cost can be computed via an appropriate FOHPN module shown in figure 2.b. The content of continuous place  $p_7$  is equal to the sum of the unit cost component and the set-up cost component. In fact, the continuous transition  $t_1$  fires during all time intervals of production, while discrete transition  $t_6$  fires whenever a new production cycle occurs.

The computation of both the holding cost component and the shortage cost component is almost identical to that seen in the previous subsection. There are just two minor changes: the weights of the arcs in the self loops  $p_1, t_4$ , and  $\bar{p}_1, t_3$  are equal to  $\varepsilon \approx 0$  and  $A$ , respectively. In fact, shortages occur as soon as  $\bar{m}_1 = A$ , i.e.,  $m_1 = 0$ , and, due to the finite replenishment rate, stop as soon as  $m_1 > 0$ .

### 3.3. FOQS with back-orders

The models described so far assumed that all demand must be met in nominal condition (constant demand). The implication is that shortages are very expensive and must be avoided. There are, however, situations where planned shortages are beneficial, and an obvious example of this occurs when the cost of keeping an item in stock is higher than the gross profit made from selling it. When there is customer demand for an item which cannot be met immediately there are shortages, and each customer has a choice: he can wait for an item to come into stock, in which case it is met by a back-order, or he can withdraw his order and go to another supplier, in which case there are lost sales [9]. We shall look at the first case.

Under the assumption of continuous and constant demand, the stock level of an item

varies with a typical pattern shown in figure 3.a. where back-orders are shown as negative stock levels.

Let  $L(0)$  be the initial stock level. At time  $\tau = L(0)/D$ , the stock gets empty and the following demand cannot be met, thus producing shortages. When shortages reach a certain value  $S$ , a new order  $Q$  is immediately supplied:  $S$  units are used to satisfy the unmet demand, while  $Q - S$  units are stored in the stock. And the process repeats periodically.

In figure 3.b the FOHPN model for this kind of systems is shown within dashed lines. As in the previous cases, the fluid content of continuous place  $p_1$  represents the stock level, while place  $\bar{p}_1$  is its complementary place and at each time instant  $m_1 + \bar{m}_1 = Q - S$ . As soon as  $p_1$  gets empty, i.e.,  $\bar{m}_1 = Q - S$ , the immediate transition  $t_4$  fires thus marking discrete place  $p_5$ . This enables the firing of continuous transition  $t_3$  with a firing speed that is equal to the demand  $D$ . As soon as  $p_3$ , whose fluid content represents the amount of unmet demand, is equal to  $S$ , transition  $t_1$  fires and both  $p_3$  and  $p_5$  get empty, while an amount of fluid equal to  $Q - S$  is supplied to continuous place  $p_1$ . At this point, the demand can be satisfied and the process repeats cyclically.

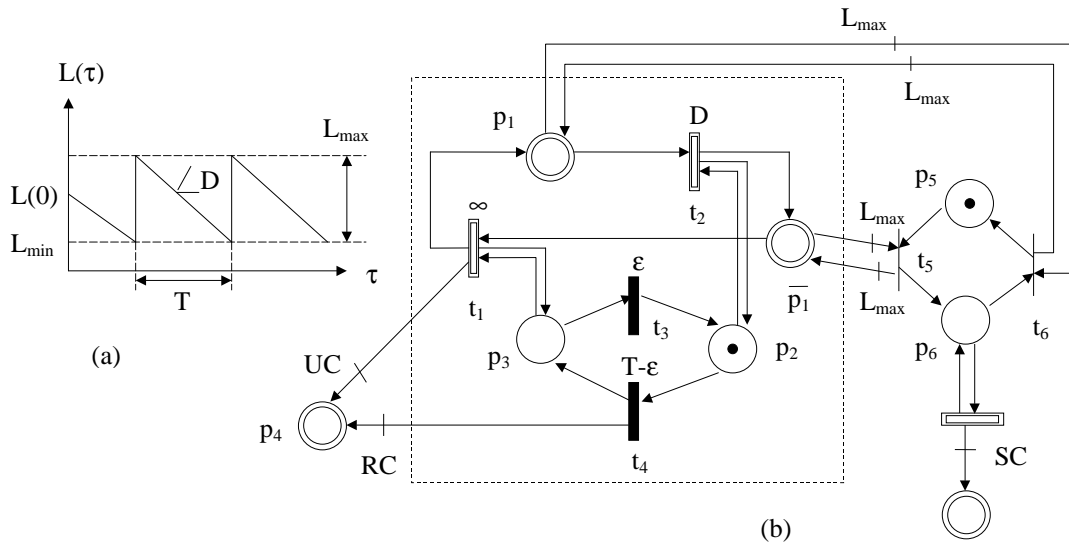
#### *Management costs evaluation*

In this case the total cost  $C$  is defined as:

$$C = UC \cdot Q \cdot N + RC \cdot N + HC \cdot L_{av} \cdot T_h - BO \sum_{i=0}^N \int_{\tau_{s,i}}^{\tau_{r,i}} L(\tau) d\tau \quad (4)$$

where the first three terms are the same as in equation (2), and the last term represents the back-order cost component. Here  $BO$  is a cost for unit of product and for unit of time (the time the customer has waited). Such a cost only increases during all the time intervals in which the stock level  $L(\tau)$  is negative, i.e., during the time intervals  $(\tau_{s,i}, \tau_{r,i})$ , for all  $i \in \mathbb{N}$ , where  $\tau_{s,i}$  and  $\tau_{r,i}$  represent the beginning and the end of the shortage in the  $i$ -th period of simulation (see figure 3).

Even in this case the total cost can be easily computed by an appropriate FOHPN module, as shown in figure 3.b. The content of continuous place  $p_2$  represents the sum of both the unit cost component and the reorder level cost component, while the holding cost component can be computed by evaluating the average marking of place  $p_1$ , as already discussed in subsection 3.1. Finally, in order to compute the back-order cost, we have introduced the FOHPN module in the right, outside the dashed rectangle. It provides a



**Figure 4:** *Periodic review system: regular pattern (a) and net model (b).*

tool for integrating in  $p_7$  the fluid content of continuous place  $p_3$ , that is exactly coincident with the back-order level. Thus, the marking of  $p_7$  will be:

$$m_7 = BO \cdot \Delta T \cdot \sum_{k=1}^n m_3(k \cdot \Delta T) \quad \text{where } n = T_h / \Delta T$$

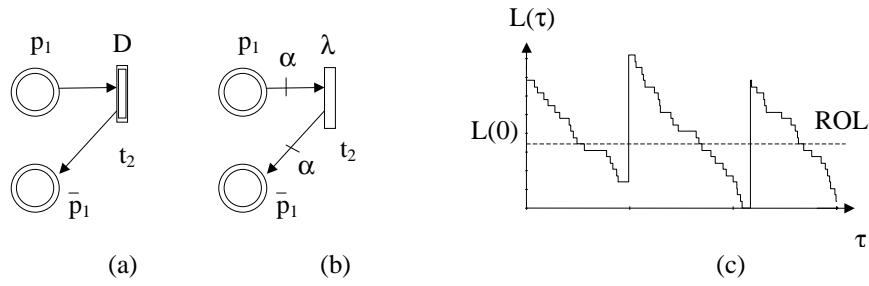
and  $\Delta T$  is the length of an appropriate finite step of integration.

### 3.4. Periodic review systems

Periodic review systems place orders of varying size at regular intervals to raise the stock level to a specified value. The operating cost of these systems is lower and it is better suited to high, regular demand of low-value items. The regular pattern of stock level is shown in figure 4.a.

The physical meaning of variables is the same as in the previous cases apart  $L_{min}$  denoting the minimum stock level during each time period  $T$ . Note that while in the case of FOQS what triggers a new order is the fact that level of the stock reaches a given value (ROL), in this case the new order is triggered by the fact that a time period  $T$  has elapsed.

The corresponding FOHPN model is shown in figure 4.b, where as in the previous cases the marking of continuous place  $p_1$  denotes the stock level, while the continuous transition



**Figure 5:** *Model of a constant demand (a). Model of a stochastic demand (b). Stock level of the IMS in figure 1 with stochastic demand (c).*

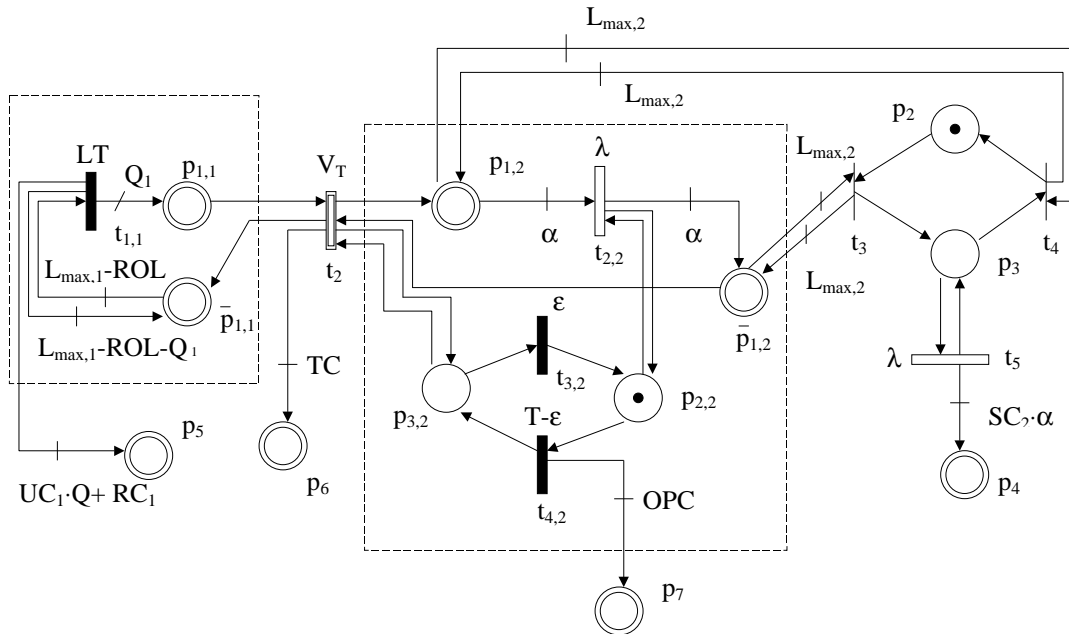
$t_2$  models the constant demand  $D$ . Place  $\bar{p}_1$  is the complementary place of  $p_1$  and at each time instant  $\tau$  the sum of their markings is equal to  $L_{max}$ . When discrete place  $p_3$  is marked, continuous transition  $t_1$  is enabled and it fires at an infinite speed transferring all the content of place  $\bar{p}_1$  in place  $p_1$ . Obviously, the firing of  $t_1$  with an infinite firing speed is only a simplifying assumption, whose aim is that of underlining that the time required for the transferring of goods may be neglected with respect to the time period  $T$ . Finally, the delay of discrete transition  $t_3$  is  $\varepsilon \cong 0$ , i.e., it fires after a very short time period, thus enabling the timed transition  $t_4$ , whose time delay is equal to  $T - \varepsilon$ . And the process still proceeds unaltered.

#### *Management costs evaluation*

Also in the case of periodic review systems the total cost  $C$  during an interval of time  $T_h$  may be calculated by a FOHPN module similar to that used in figure 1. However, in the module used for periodic review systems and shown in figure 4.b, the unitary cost component depends on the continuous firing of transition  $t_1$  whose firing represents the replenishment of the stock.

## 4. Stochastic demand

In the previous sections we have assumed that the demand is continuous and constant. Obviously, this is only a simplifying assumption that does not find its counterpart in real applications. In order to provide more realist models, FOHPN can be used to simulate exponentially distributed stochastic demand. In such a case, all continuous transitions that represent the demand have to be replaced by exponentially distributed timed transitions



**Figure 6:** *A numerical example: a double–areas warehouse.*

with an appropriate value of the average firing rate.

As an example, let us consider the subnet in figure 5.a, where  $p_1$  models the stock (its marking represents the stock level),  $\bar{p}_1$  is its complementary place (its marking represents the available space), and transition  $t_2$  models the constant demand. To model a stochastic demand, we replace continuous transition  $t_2$  by an exponentially distributed timed transition, as in figure 5.b. Moreover, to emphasize the stochastic effect of the demand, we assume that the arcs between  $p_1$ ,  $\bar{p}_1$  and the stochastic transition  $t_2$  have weight  $\alpha$  while the firing rate of transition  $t_2$  is  $\lambda = D/\alpha$  so that the expected value of the stochastic demand is equal to  $D$ .

Figure 5.c shows a possible pattern of stock level of a FOQS with finite lead time in the case of stochastic demand. Note that in such a case shortage may occur, i.e., there may exist some intervals of time with a null stock level.

## 5. A numerical optimization problem

In the literature many optimization problems concerning IMS have been studied. Bhaskaran and Malmorg [3], Malmorg [8], Malmorg and Balachandran [7] deal with optimization

problems of double–areas warehouse sizing. In [3] the authors dealt with the problem of relative sizing the two areas for minimizing total costs; [8] has the purpose of formulating an integrated evaluation model which allows distribution system designers to estimate costs associated with inventory carrying, reordering and expediting, item replenishment and retrieval, shortage in the item retrieval area, and storage space. These estimates are based on management policies that address inventory management, space allocation, and storage layout. A limited computational experiment has been used to demonstrate how the model could be applied to investigate alternative policy scenarios. Although the model does not lend itself to analytical optimization methods, it does provide a means for exploring alternative facility design and operating plan combinations. In [7] the authors formulated a model to describe the cost effects of alternative warehouse layouts in a dual address order picking environment.

In this subsection we consider a double–areas warehouse. The first area  $\mathcal{M}_1$  is denoted as the *storage area*. As soon as products are finished they are put into  $\mathcal{M}_1$  with no precise criterion. We assume that this area is managed with a fixed order quantity policy with finite lead time, instantaneous replenishment rate and no shortage. The second area  $\mathcal{M}_2$ , denoted as *pick–area*, collects all products that are ready to be sold. Here products are ordered according to specific criteria, and much higher holding costs are needed. We assume that this area is managed with a periodic review policy.

As it can be seen in figure 6, the FOHPN of this system can be obtained by simply connecting in series the FOHPN modules discussed in the previous subsections where constant deterministic demand has been replaced by an exponentially distributed timed demand. The two FOHPN elementary modules are connected through transition  $t_2$  that is assumed to fire with a maximum firing speed  $V_T$ .

The model used for costs computation is quite similar to that already introduced in the previous section, apart from two new cost components due to transfer of goods. Two new continuous places  $p_6$  and  $p_7$  have been added, whose fluid content denote respectively the *transfer cost component* and the *order–picking cost component*. The first one is proportional to the amount of goods supplied, while the second one is proportional to the number of orders supplied. Therefore, in this case the total cost is given by:

$$\begin{aligned}
C = & UC_1 \cdot Q_1 \cdot N_1 + RC_1 \cdot N_1 + HC_1 \cdot L_{av,1} \cdot T_h + \\
& HC_2 \cdot L_{av,2} \cdot T_h + SC_2 \cdot N_{sc,2} + TC \cdot N_{1,2} + OPC \cdot N_2
\end{aligned} \tag{5}$$

where subscript has been used to distinguish among variables in  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Moreover,



we denoted as  $N_{1,2}$  the number of units supplied from  $\mathcal{M}_1$  to  $\mathcal{M}_2$  in a given time period.

Let us assume that all parameters relative to the storage-area  $\mathcal{M}_1$  are fixed. We want to determine via simulation the size  $L_{max,2}$  of the storage capacity of the pick-area, so that it becomes empty at the end of each time period  $T_2 = L_{max,2}/D$ , while minimizing the total cost  $C$ . Here  $L_{min,2} = 0$ .

Our approach requires the implementation of the net in figure 6 for many different values of  $L_{max,2}$  and thus many different values of  $T_2$ .

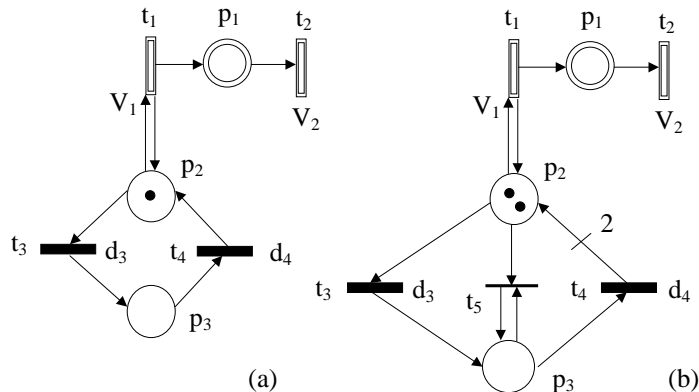
The following numerical values have been used:  $L_{max,1} = 2000$  units,  $Q_1 = 500$  units,  $ROL = 400$  units,  $LT = 10$  days,  $UC_1 = 24000$  £ a unit,  $RC_1 = 125000$  £ an order,  $HC_1 = 15000$  £ a unit a day,  $HC_2 = 90000$  £ a unit a day,  $SC_2 = 200000$  £ a unit,  $TC = 2000$  £ a unit,  $OPC = 200000$  £ an order,  $V_T = 10^6$  units a day,  $L_1(0) = 1550$  units,  $L_2(0) = L_{max,2}$  units,  $\varepsilon = 1$  hour,  $T_h = 4$  years.

We use a stochastic demand (transitions  $t_{2,2}$  and  $t_5$ ) with exponential distribution. We assume the average demand is  $D = 20$  units/day and we associate to the two stochastic transitions: firing rate  $\lambda = 2$  1/days and arc weights  $\alpha = 10$  units.

The results of the simulations are shown in figure 8 where the total cost with respect to  $L_{max,2}$  is plotted. The optimal value computed is  $L_{max,2}^* = 200$  and the corresponding total cost is  $C^* = 8.5 \cdot 10^8$  £. Such a behaviour can be easily interpreted: when  $L_{max,2} \rightarrow 0$ , we need a large number of order to satisfy the demand, thus greatly increasing the order-picking cost component; when  $L_{max,2} \rightarrow \infty$  the average stock level in  $\mathcal{M}_2$  increases, thus producing excessive the holding costs. Note that in quite all cases, the stochastic demand produces shortage.

Let us observe that the time held  $T_h$ , i.e., the interval of simulation, has been chosen much greater than the nominal cycle times characteristic of the IMS  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , so as to ensure that the computed costs, and the optimum as well, would not significantly depend on the initial chosen conditions.

In order to simulate the above net we have used the software SYRPHICO [6] that provides an accurate and fast simulation tool for the hybrid Petri net model of Alla and David [1, 2]. Note however, that while SYRPHICO uses a *token reservation* policy, in FOHPN a *concurrent enabling* policy is adopted (as described in Section 2). Although concurrent enabling is more general than token reservation, in these examples it was



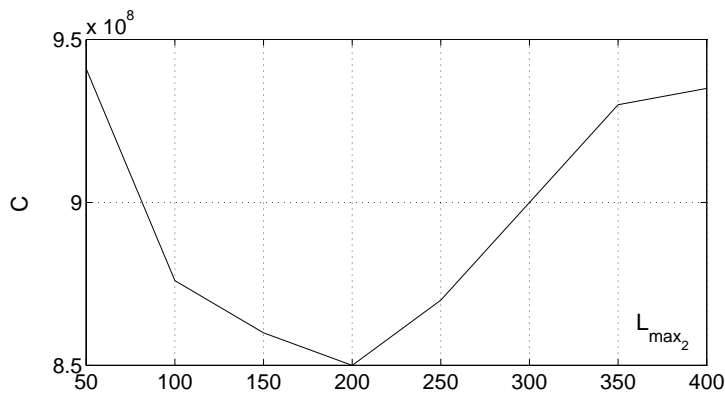
**Figure 7:** A FOHPN (a) and its equivalent representation implemented with *Sirphyco* (b).

possible to adapt all the FOHPN models here described so as to implement them with SYRPHICO. As an example, let us consider the simple net in figure 7.a. In accordance to the concurrent enabling policy, when  $p_2$  is marked both  $t_1$  and  $t_3$  are enabled. Thus, the continuous transition  $t_1$  may fire at its maximum firing speed  $V_1$ , until a time  $d_3$  has elapsed, when discrete transition  $t_3$  fires. On the contrary, under the token reservation policy the firing of  $t_1$  is never enabled, because the token in  $p_2$  is reserved for the firing of discrete transition  $t_3$ .

The desired behavior of the net in figure 7.a can be simulated with SYRPHICO, converting it into the equivalent hybrid Petri net in figure 7.b. Here, one token in  $p_2$  is reserved to transition  $t_3$ , while the second one enables the firing of continuous transition  $t_1$ . When a time  $d_3$  has elapsed,  $t_3$  fires removing one token from  $p_2$  and enabling the firing of immediate transition  $t_5$  that removes the remaining token in  $p_2$ . After a time delay  $d_4$ ,  $t_4$  fires and two tokens are put again into discrete place  $p_2$ , and the process repeats cyclically.

## 6. Conclusions

In this paper we dealt with the problem of modeling and control IMS using FOHPN. The choice of FOHPN as a modeling tool presents many advantages: they can be used as a visual-communication aid; they enables us to set up equations, algebraic equations, and other mathematical models governing the behaviour of systems; with the addition



**Figure 8:** *The results of the numerical optimization problem.*

of tokens, FOHPN are able to simulate the dynamic concurrent activities of IMS; finally, they enable a modular representation of an IMS, thus enabling us to deal even with very large dimension systems.

In this paper we considered fixed order quantity systems and periodic review systems and showed how costs relative to different management policies can be easily evaluated by adding appropriate FOHPN modules to the corresponding IMS module.

## References

- [1] ALLA H., DAVID R., “Continuous and Hybrid Petri Nets,” *Journal of Circuits, Systems, and Computers*, Vol. 8, No. 1, february 1998. p. 159-88.
- [2] ALLA H., DAVID R., “A modeling and analysis tool for discrete events systems: continuous Petri Net,” *Performance Evaluation*, vol. 33, no. 3, 1998, p. 175-99.
- [3] BHASKARAN K., MALMBORG C.J., “Economic tradeoffs in sizing warehouse reserve storage area,” *Applied Math. Modelling*, vol. 14, july 1990, p. 381-5.
- [4] BALDUZZI F., GIUA A., MENGA G., “Modelling and control with First-Order Hybrid Petri Nets,” *IEEE Trans. on Robotics and Automation*, vol. 16, no. 4, august 2000, p. 382-399.
- [5] GIUA A., FURCAS R., PICCALUGA A., SEATZU C., “Modelling and control of inventory management policies using First-Order Hybrid Petri Nets,” *Proc. 4th Int. Conf. on Automation of Mixed Processes: Hybrid Dynamic Systems*, september 2000, p. 273-279.

- [6] GUERRE-CHALEY G., “Conception et réalisation d’un simulateur de réseaux de Petri continus et hybrides pour l’évaluation de performances des systèmes discrets et/ou continus,” *CNAM dissertation, Conservatoire National des Arts et Métiers*, (Grenoble, France), avril 1997.
- [7] MALMORG C.J., BALACHANDRAN S., “A model based evaluation of a commonly used rule of thumb for warehouse layout,” *Applied Math. Modelling*, vol. 10, april 1986, p. 133-8.
- [8] MALMORG C.J., “An integrated storage system evaluation model,” *Applied Math. Modelling*, vol. 20, may 1996, p. 359-70.
- [9] WATERS C.D.J., “Inventory Control and Management,” *Ed. John Wiley and Sons*, 1992.