Hybrid Control of Production Systems with Local Optimization*

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Abstract - *In this paper we consider First–Order Hybrid Petri Nets (FOHPN), a model that consists of continuous places holding fluid, discrete places containing a non–negative integer number of tokens, and transitions, either discrete or continuous. This Petri net model reveals to be particularly suited for modeling manufacturing systems and its control can be efficiently framed as a conflict resolution policy that aims at optimizing a given objective function. The use of linear algebra leads to sensitivity analysis that allows one to study of how changes in the structure of the model influence the optimal behavior. It further enables us to determine admissible ranges of variation for some design parameters in order to improve the optimal myopic solution.*

1. INTRODUCTION

First–Order Hybrid Petri Nets (FOHPN) are nets that consist of continuous places holding fluid, discrete places containing a non–negative integer number of tokens, and transitions, either discrete or continuous. This hybrid Petri net model has been introduced by the authors in [3, 4] and follows the formalism described by David and Alla [2, 7].

In a previous work [3] the authors have shown that FOHPNs are well suited for modeling automated manufacturing systems characterized by unreliable machines, buffers of finite capacity, general service time distributions and routing policies, where the continuous transitions model the production of the machines. Continuous firing of these transitions corresponds to a continuous production at rates determined by the current values of their instantaneous firing speeds (IFS).

In [3] the focus was on conflict resolution policies, i.e. on the computation of IFSs, seen as the decisions that a plant operator must take in oder to optimize the process. This can be done

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by solving a linear programming problem (LPP) of the form $\max_{\mathbf{v}}\{\mathbf{c}^T\mathbf{v}|\mathbf{A}\mathbf{v} \leq \mathbf{b}, \mathbf{v} \geq \mathbf{0}\}$ where the set of admissible IFS vectors $\mathbf{v}$ can be characterized by the feasible solutions of a linear constraint set $\mathcal{S}$, and the different objective functions to be maximized can be associated to different conflict resolution policies. The constraint set $\mathcal{S}$ is a function of the current marking of the net, because it is characterized by the marking of the discrete places and by the set of non–empty continuous places, i.e., it is characterized by the *macro–state* of the net. This formulation leads to a myopic procedure which generates a piecewise optimal control policy during each time interval of time in which the macro–state remains constant. As the system evolves through a sequence of macro–states upon the occurrence of the *macro–events*, the myopic procedure will be called repeatedly.

In this paper, as in [4] we adopt the optimal basis approach, i.e., the simplex method, to solve LPP. We show how it is possible to efficiently applied sensitivity analysis techniques that pertain to LPP in the FOHPN framework. Sensitivity analysis [8, 10] serves as a tool for studying of how optimal solutions vary according to changes of the given linear program in terms of the coefficients of the matrix, the right–hand side vector and the objective function coefficients.

Sensitivity analysis techniques have been proposed in this paper to obtain information about the degrees of freedom that can be exploited when making performance optimization or optimal design of the system parameters configuration. The FOHPN model of a multi–class production system has been examined in detail. Two different control problems have been considered and a local (myopic) optimal control policy has been derived by solving a sequence of linear programming problems. Finally perturbations on the maximum machine production rates, on the re–working factor and on the mix factor have been discussed along with different numerical examples.

2. BACKGROUND

We recall the Petri net formalism used in this paper following [3, 4]. For a more comprehensive introduction to place/transition Petri nets see [9]. The common notation and semantics for timed nets can be found in [1].

An *First–Order Hybrid Petri Nets* (FOHPN) is a structure $N = (P, T, Pre, Post, \mathcal{D}, \mathcal{C})$.

The set of *places* $P = P_d \cup P_c$ is partitioned into a set of *discrete* places P_d (represented as circles) and a set of *continuous* places P_c (represented as double circles). The cardinality of P, P_d and P_c is denoted n, n_d and n_c.

The set of *transitions* $T = T_d \cup T_c$ is partitioned into a set of discrete transitions T_d and a set of continuous transitions T_c (represented as double boxes). The set $T_d = T_I \cup T_D \cup T_E$ is further partitioned into a set of *immediate* transitions T_I (represented as bars), a set of *deterministic timed* transitions T_D (represented as black boxes), and a set of *exponentially distributed timed* transitions T_E (represented as white boxes).

The *pre-* and *post-incidence functions* that specify the arcs are (here $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$):

$$Pre, Post : \begin{cases} P_d \times T \to \mathbb{N} \\ P_c \times T \to \mathbb{R}_0^+ \end{cases}$$

We require (*well-formed nets*) that for all $t \in T_c$ and for all $p \in P_d$, $Pre(p,t) = Post(p,t)$.

The function $\mathcal{D} : T_t \to \mathbb{R}^+$ specifies the timing associated to timed discrete transitions. We associate to a deterministic timed transition $t_j \in T_D$ its (constant) firing delay $\delta_j = \mathcal{D}(t_j)$. We associate to an exponentially distributed timed transition $t_j \in T_E$ its average firing rate $\lambda_j = \mathcal{D}(t_j)$, i.e., the average firing delay is $\frac{1}{\lambda_j}$, where λ_j is the parameter of the corresponding exponential distribution.

The function $\mathcal{C} : T_c \to \mathbb{R}_0^+ \times \mathbb{R}_\infty^+$ specifies the firing speeds associated to continuous transitions (here $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$). For any continuous transition $t_j \in T_c$ we let $\mathcal{C}(t_j) = (V'_j, V_j)$, with $V'_j \leq V_j$. Here V'_j represents the *minimum firing speed* (mfs) and V_j represents the *maximum firing speed* (MFS).

We denote the preset (postset) of transition t as ${}^\bullet t$ ($t^\bullet$) and its restriction to continuous or discrete places as ${}^{(d)}t = {}^\bullet t \cap P_d$ or ${}^{(c)}t = {}^\bullet t \cap P_c$. Similar notation may be used for presets and postsets of places. The *incidence matrix* of the net is defined as $\mathbf{C}(p,t) = Post(p,t) - Pre(p,t)$. The restriction of $\mathbf{C}$ to P_X and T_Y ($X, Y \in \{c, d\}$) is denoted $\mathbf{C}_{XY}$. Note that by the well-formedness hypothesis $\mathbf{C}_{dc} = 0$.

A *marking*

$$\mathbf{m} : \begin{cases} P_d \to \mathbb{N} \\ P_c \to \mathbb{R}_0^+ \end{cases}$$

is a function that assigns to each discrete place a non-negative number of tokens, represented by black dots and assigns to each continuous place a fluid volume; m_i denotes the marking of place p_i. The value of a marking at time τ is denoted $\mathbf{m}(\tau)$. The restriction of $\mathbf{m}$ to P_d and P_c are denoted with $\mathbf{m}^d$ and $\mathbf{m}^c$, respectively. An *FOHPN system* $\langle N, \mathbf{m}(\tau_0)\rangle$ is an FOHPN N with an initial marking $\mathbf{m}(\tau_0)$.

The enabling of a discrete transition depends on the marking of all its input places, both discrete and continuous.

Definition 2.1. *Let $\langle N, \mathbf{m}\rangle$ be an FOHPN system. A discrete transition t is* enabled *at $\mathbf{m}$ if for all $p_i \in {}^\bullet t$, $m_i \geq Pre(p_i, t)$.* ■

A continuous transition is enabled only by the marking of its input discrete places. The marking of its input continuous places, however, is used to distinguish between strongly and weakly enabling.

Definition 2.2. *Let $\langle N, \mathbf{m}\rangle$ be an FOHPN system. A continuous transition t is* enabled *at $\mathbf{m}$ if for all $p_i \in {}^{(d)}t$, $m_i \geq Pre(p_i, t)$.*

We say that an enabled transition $t \in T_c$ is:

- strongly enabled *at $\mathbf{m}$ if for all places $p_i \in {}^{(c)}t$, $m_i > 0$;*
- weakly enabled *at $\mathbf{m}$ if for some $p_i \in {}^{(c)}t$, $m_i = 0$.*

■

3. NET DYNAMICS

In this section we define the behaviour of a net. The overall hybrid behavior that combines both time–driven and event–driven dynamics can be found in [5]. A *macro event* occurs when:

(a) either a discrete transition fires, thus changing the discrete marking or enabling/disabling a continuous transition; (b) or a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak.

The *instantaneous firing speed* (IFS) at time τ of a transition $t_j \in T_c$ is denoted $v_j(\tau)$. We can write the equation which governs the evolution in time of the marking of a place $p_i \in P_c$ as

$$\dot{m}_i(\tau) = \sum_{t_j \in T_c} C(p_i, t_j) v_j(\tau) \tag{1}$$

where $\mathbf{v}(\tau) = [v_1(\tau), \ldots, v_{n_c}(\tau)]^T$ is the IFS vector at time τ. Indeed Equation 1 holds assuming that at time τ no discrete transition is fired and that all speeds $v_j(\tau)$ are continuous in τ.

The enabling state of a continuous transition t_j defines its admissible IFS v_j.

- If t_j is not enabled then $v_j = 0$.
- If t_j is strongly enabled, then it may fire with any firing speed $v_j \in [V'_j, V_j]$.
- If t_j is weakly enabled, then it may fire with any firing speed $v_j \in [V'_j, \overline{V}_j]$, where $\overline{V}_j \leq V_j$ since t_j cannot remove more fluid from any empty input continuous place $\overline{p}$ than the quantity entered in $\overline{p}$ by other transitions.

We now characterize the set of all admissible IFS vectors.

Definition 3.1. (admissible IFS vectors) *Let $\langle N, \mathbf{m} \rangle$ be an FOHPN system. Let $T_{\mathcal{E}}(\mathbf{m}) \subset T_c$ ($T_{\mathcal{N}}(\mathbf{m}) \subset T_c$) be the subset of continuous transitions enabled (not enabled) at $\mathbf{m}$, and $P_{\mathcal{E}} = \{p_i \in P_c \mid m_i = 0\}$ be the subset of empty continuous places. Any* admissible IFS vector $\mathbf{v}$ *at* $\mathbf{m}$ *is a feasible solution of the following linear set:*

$$\begin{cases} (a) & V_j - v_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\mathbf{m}) \\ (b) & v_j - V'_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\mathbf{m}) \\ (c) & v_j = 0 & \forall t_j \in T_{\mathcal{N}}(\mathbf{m}) \\ (d) & \sum_{t_j \in T_{\mathcal{E}}} C(p, t_j) v_j \geq 0 & \forall p \in P_{\mathcal{E}}(\mathbf{m}) \end{cases} \tag{2}$$

The set of all feasible solutions is denoted $\mathcal{S}(N, \mathbf{m})$. ■

Constraints of the form (2.a), (2.b), and (2.c) follow from the firing rules of continuous transitions. Constraints of the form (2.d) follow from (1), because if a continuous place is empty then its fluid content cannot decrease. Note that the set $\mathcal{S}$ is a function of the marking of the net. Thus as $\mathbf{m}$ changes it may vary as well. In particular it changes at the occurrence of the macro–events.

Let τ_k and τ_{k+1} be the occurrence times of two consecutive macro–events; we assume that within the interval of time $[\tau_k, \tau_{k+1})$, that we call *macro–period*, the IFS vector is constant and we denote it $\mathbf{v}(\tau_k)$. Then the continuous behavior of an FOHPN for $\tau \in [\tau_k, \tau_{k+1})$ is described by:

$$\begin{cases} \mathbf{m}^c(\tau) & = \mathbf{m}^c(\tau_k) + \mathbf{C}_{cc}\mathbf{v}(\tau_k)(\tau - \tau_k) \\ \mathbf{m}^d(\tau) & = \mathbf{m}^d(\tau_k) \end{cases} \tag{3}$$

On the other hand, the firing of a discrete transition t_j at $\mathbf{m}(\tau)$ yields the marking

$$\begin{cases} \mathbf{m}^c(\tau) & = \mathbf{m}^c(\tau^-) + \mathbf{C}_{cd}\boldsymbol{\sigma}(\tau) \\ \mathbf{m}^d(\tau) & = \mathbf{m}^d(\tau^-) + \mathbf{C}_{dd}\boldsymbol{\sigma}(\tau) \end{cases} \tag{4}$$

where $\boldsymbol{\sigma}(\tau)$ is the *firing count vector* associated to the firing of the discrete transition t_j.

4. SENSITIVITY ANALYSIS FOR FOHPN

The formalism previously introduced can be used to define the concept of *conflict* in a net. We will only treat conflicts at continuous places since the computation of an admissible IFS vector is only affected by this type of conflicts. A conflict resolution policy can be obtained by solving a linear programming problem (LPP) of the form

$$\begin{array}{l} \max \mathbf{c}^T \mathbf{v} \quad \text{s.t.} \\ \mathbf{v} \in \mathcal{S}(N, \mathbf{m}) \end{array} \tag{5}$$

aimed at a *global optimization* of the system resources. Obviously, many different performance indices can be considered as the objective function in the LP formulation of the problem. All these LPP may be solved taking into account only the constraints related to enabled transitions since we know that the IFS of transitions that are not enabled are 0. Let $I_t = \{\alpha_1, \ldots, \alpha_k\}$ be the set of indices of the enabled continuous transitions and $I_p = \{\alpha_{2k+1}, \ldots, \alpha_\ell\}$ be the set of indices of the empty continuous places. Thus we can write:

$$\begin{array}{l} \max \sum_{j \in I_t} c_j \; v_j \qquad \text{s.t.} \\ \left\{ \begin{array}{lcl} v_{\alpha_1} + s_1 & = & V_{\alpha_1} \\ \ldots & & \\ v_{\alpha_k} + s_k & = & V_{\alpha_k} \\ v_{\alpha_1} - s_{k+1} & = & V'_{\alpha_1} \\ \ldots & & \\ v_{\alpha_k} - s_{2k} & = & V'_{\alpha_k} \\ \sum_{j \in I_t} \mathbf{C}(p_{\alpha_{2k+1}}, t_j) \; v_j - s_{2k+1} & = & 0 \\ \ldots & & \\ \sum_{j \in I_t} \mathbf{C}(p_{\alpha_\ell}, t_j) \; v_j - s_\ell & = & 0 \end{array} \right. \\ s_j \geq 0 \end{array} \tag{6}$$

Defining vector $\mathbf{x} = [v_{\alpha_1}, \ldots, v_{\alpha_k}, s_1, \ldots, s_\ell]^T$ we obtain the following standard form:

$$\max_{\mathbf{x}} \{\mathbf{c}^T \mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \; \mathbf{x} \geq 0\}. \tag{7}$$

Here $\mathbf{x}$ is a vector with $\ell + k$ variables, $\mathbf{A}$ is the $\ell \times (\ell + k)$ matrix constraints and we assume that $\mathbf{A}$ has full rank, $\mathbf{c}$ is the $(\ell + k)$–vector of the objective coefficients, while $\mathbf{b}$ represents the ℓ–vector of the right–hand side constants.

In this work the *simplex method* will be used to solve LPP. This is an iterative method in which at each step and in an efficient manner a new basis is computed. Each basis represents a vertex of the feasible region. We denote an optimal basic solution $\mathbf{x}^o$, the corresponding optimal basis $\mathcal{B}$ (a set of ℓ indices), and $\mathbf{A}_\mathcal{B}$ the optimal basis matrix obtained by taking only those columns of $\mathbf{A}$ whose indices are in $\mathcal{B}$. An optimal basic solution $\mathbf{x}^o$ can always be written as:

$$\mathbf{x}^o = \begin{bmatrix} \mathbf{x}_\mathcal{B} \\ \mathbf{x}_\mathcal{N} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\mathcal{B}^{-1} \; \mathbf{b} \\ \mathbf{0} \end{bmatrix}.$$

The variables with index in $\mathcal{B}$ are the basic variables while the others, whose index set is denoted $\mathcal{N}$, are called nonbasic. Note that the optimal solution may be degenerate, i.e. we have many basis associated with it. It may also be the case that more than one basic optimal solution exists.

Sensitivity analysis refers to the study of how optimal solutions change according to changes of the given linear program in terms of the coefficients of the matrix, the right–hand side and the objective function. Suppose that the LPP (7) has an optimal solution. If there is any change in the values of b_j, c_j or a_{ij} the optimal solution is likely to change in general.

In the next sections we will develop sensitivity analysis with respect to the design parameters by assuming changes in the right–hand side vector and in the matrix coefficients. Perturbations in the cost coefficients will not be considered in this work.

4.1. The perturbed model

The following perturbed LPP is treated:

$$\max_{\mathbf{x}} \{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}(\mathbf{q})\mathbf{x} = \mathbf{b}(\mathbf{q}),\ \mathbf{x} \geq 0\} \tag{8}$$

where $\mathbf{q} = [q_0, \ldots, q_p]^T$ is a vector of uncertain parameters. The nominal value is denoted $\overline{\mathbf{q}}$. For a given value of $\mathbf{q}$, the optimal solution of (8) is

$$\mathbf{x}^o(\mathbf{q}) = \begin{bmatrix} \mathbf{x}_{\mathcal{B}}(\mathbf{q}) \\ \mathbf{x}_{\mathcal{N}}(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathcal{B}}^{-1}(\mathbf{q})\ \mathbf{b}(\mathbf{q}) \\ \mathbf{0} \end{bmatrix}.$$

We compute with the simplex method an optimal solution in $\overline{\mathbf{q}}$ and the corresponding optimal basis $\mathcal{B}$. The sensitivity of the basic variables $\mathbf{x}_{\mathcal{B}}(\overline{\mathbf{q}})$ with respect to q_i can be computed, at least within a certain domain where the optimal basis does not change, by taking the partial derivatives

$$\frac{\partial \mathbf{x}_{\mathcal{B}}(\overline{\mathbf{q}})}{\partial q_i} = \mathbf{A}_{\mathcal{B}}^{-1}(\overline{\mathbf{q}}) \left(\frac{\partial \mathbf{b}(\overline{\mathbf{q}})}{\partial q_i} - \frac{\partial \mathbf{A}_{\mathcal{B}}(\overline{\mathbf{q}})}{\partial q_i} \mathbf{x}_{\mathcal{B}}(\overline{\mathbf{q}}) \right) \tag{9}$$

while the non–basic variables $\mathbf{x}_{\mathcal{N}}(\overline{\mathbf{q}})$ do not change. It is only required first order differentiability of $\mathbf{A}_{\mathcal{B}}(\overline{\mathbf{q}})$ and $\mathbf{b}(\overline{\mathbf{q}})$ with respect to q_i. For simplicity in this presentation we make the following assumptions:

1. Only one parameter q_i varies at a time, that is $\mathbf{q} = \overline{\mathbf{q}} + \lambda \mathbf{e}_i$, where $\mathbf{e}_i$ is the i–th canonical basis vector. Under this assumption the sensitivity given by (9) can be regarded as function of λ in the allowable range.

2. Matrix $\mathbf{A}$ and vector $\mathbf{b}$ are linear functions of the parameter λ. Then we can write:

$$\mathbf{A}_{\mathcal{B}}(\lambda) = \mathbf{A}_{\mathcal{B}} + \lambda \mathbf{A}_{\mathcal{B}}^*$$
$$\mathbf{b}(\lambda) = \mathbf{b} + \lambda \mathbf{b}^*$$

   where $\mathbf{A}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}(\overline{\mathbf{q}})$, $\mathbf{b} = \mathbf{b}(\overline{\mathbf{q}})$.

3. The variation of each parameter q_i influences only one column, say the j-th, of matrix $\mathbf{A}_{\mathcal{B}}(\lambda)$. Then

$$\mathbf{A}_{\mathcal{B}}(\lambda) = \mathbf{A}_{\mathcal{B}} + \lambda \mathbf{A}_{\mathcal{B}}^* = \mathbf{A}_{\mathcal{B}} + \lambda \mathbf{a}^* \mathbf{e}_j^T.$$

4.2. Perturbation of the right–hand side vector

Let us assume that the right–hand side constant vector $\mathbf{b}$ varies linearly with the parameter $\lambda \in \mathbb{R}$, that is $\mathbf{b}(\lambda) = \mathbf{b} + \lambda \mathbf{b}^*$. In the FOHPN framework, this perturbation corresponds to changes in the entries of the vector $\mathbf{V} = [V_{\alpha_1}, \ldots, V_{\alpha_k}]^T$, which denotes the MFS vector and of the vector $\mathbf{V}' = [V'_{\alpha_1}, \ldots, V'_{\alpha_k}]^T$, which denotes the mfs vector. As an example, in a manufacturing system we may want to add servers to a machine in order to increase the overall productivity of the system.

Let $\mathbf{x}^o$ be an optimal basic solution of (7) and $\mathcal{B}$ an associated optimal basis. The perturbed optimal solution $\mathbf{x}^o(\lambda)$ has basic components:

$$\mathbf{x}^o_{\mathcal{B}}(\lambda) = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}(\lambda) = \mathbf{A}_{\mathcal{B}}^{-1}(\mathbf{b} + \mathbf{b}^*\lambda) = \mathbf{x}^o_{\mathcal{B}} + \lambda \mathbf{x}^*_{\mathcal{B}} \tag{10}$$

where $\mathbf{x}^o_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b} = [\beta_1, \ldots, \beta_\ell]^T$ and $\mathbf{x}^*_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}^* = [\beta^*_1, \ldots, \beta^*_\ell]^T$. The optimal value of the objective function is

$$J(\lambda) = \mathbf{c}^T_{\mathcal{B}}\mathbf{x}^o_{\mathcal{B}}(\lambda) = \mathbf{c}^T_{\mathcal{B}}\mathbf{x}^o_{\mathcal{B}} + \lambda \mathbf{c}^T_{\mathcal{B}}\mathbf{x}^*_{\mathcal{B}} = J + \lambda J^*. \tag{11}$$

Equations (10) and (11) hold only when λ belongs to a certain interval $\Lambda_{\mathcal{B}} = [\underline{\lambda}_{\mathcal{B}}, \overline{\lambda}_{\mathcal{B}}]$ also called the allowable range, where the optimal basis $\mathcal{B}$ remains unchanged. This requires non–negativity of the basic variables, $\mathbf{x}^o_{\mathcal{B}}(\lambda) \geq \mathbf{0}$, and the bounds for the parameter λ can be computed as follows:

$$\underline{\lambda}_{\mathcal{B}} = \begin{cases} -\infty & \text{if} \quad I^+ = \emptyset \\ \max_{i \in I^+} \left\{ -\frac{\beta_i}{\beta^*_i} \right\} \end{cases} \tag{12}$$

and

$$\overline{\lambda}_{\mathcal{B}} = \begin{cases} +\infty & \text{if} \quad I^- = \emptyset \\ \min_{i \in I^-} \left\{ -\frac{\beta_i}{\beta^*_i} \right\} \end{cases} \tag{13}$$

where $I^+ = \{i \geq 1 \mid \beta^*_i > 0\}$ and $I^- = \{i \geq 1 \mid \beta^*_i < 0\}$. Since $\mathbf{A}_{\mathcal{B}}^{-1}$ is invertible, then $\mathbf{A}_{\mathcal{B}}^{-1}\ \mathbf{b}^* \neq \mathbf{0}$, i.e. either $\underline{\lambda}_{\mathcal{B}}$ or $\overline{\lambda}_{\mathcal{B}}$ must be finite.

Much attention has been devoted in the literature [8, 10] to the case in which the optimal solution $\mathbf{x}^o$ of the nominal LPP is unique. In this case $\mathbf{x}^o$ is not a degenerate solution and the unique optimal basis remains constant within the allowable range, therefore the value of the objective function is linear in λ. As λ reaches the boundary of the allowable range, a degenerate solution is found, a new basis can be computed with an allowable range that will not overlap the previous one except at the end point. As the basis changes, the derivative of the objective function with respect to the parameter λ, i.e., $\frac{\mathrm{d}J(\lambda)}{\mathrm{d}\lambda} = J^*$, may also change, thus it may not be defined only at a finite number of points whereas we can instead provide right and left values.

In the manufacturing domain this non–differentiability behavior has been already observed in tandem lines by Fu and Suri [11] when the average production rates of two machines are equal. With our approach the result is immediately generalized to more general cases. However the situation can be more complex when more than one optimal solution exists.

4.3. Perturbation of the matrix coefficients

We assume that the basis matrix $\mathbf{A}_{\mathcal{B}}$ varies linearly with the parameter $\lambda \in \mathbb{R}$, according to $\mathbf{A}_{\mathcal{B}}(\lambda) = \mathbf{A}_{\mathcal{B}} + \lambda \mathbf{A}^*_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}} + \lambda \mathbf{a}^* \mathbf{e}^T_j$, i.e., we assume that only the j–th column of $\mathbf{A}_{\mathcal{B}}$ may vary.

The results we present here also hold when a single row of $\mathbf{A}_\mathcal{B}$ varies linearly with the parameter λ. Nevertheless this case is less relevant in the context of FOHPN. Basically perturbations of the matrix coefficients $\mathbf{A}$ correspond in the FOHPN framework to variations of the arc–weights between continuous places and transitions, as it can be seen from Equation (6). Multiple variations of the coefficients along a column correspond to a redistribution of the inflow or outflow of a single continuous transition. In a manufacturing system this situation is quite common and it arises when we deal with changes of the percentage of parts that need to be reworked or with changes of the routing coefficients.

Let $\mathbf{x}^o$ be an optimal basic solution of (7) and $\mathcal{B}$ an associated optimal basis. We recall the matrix equality:

$$\mathbf{A}_\mathcal{B}^{-1}(\lambda) = \mathbf{A}_\mathcal{B}^{-1} - \frac{\mathbf{A}_\mathcal{B}^{-1}\mathbf{a}^*\mathbf{e}_j^T\mathbf{A}_\mathcal{B}^{-1}\lambda}{1+\mathbf{e}_j^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{a}^*\lambda}.$$

Then the perturbed optimal solution $\mathbf{x}^o(\lambda)$ has basic components:

$$\mathbf{x}_\mathcal{B}^o(\lambda) = \mathbf{A}_\mathcal{B}^{-1}(\lambda)\mathbf{b} = \mathbf{x}_\mathcal{B}^o - \frac{\lambda}{1+v\lambda}\mathbf{x}_\mathcal{B}^* \tag{14}$$

where $\mathbf{x}_\mathcal{B}^o = \mathbf{A}_\mathcal{B}^{-1}\mathbf{b}$, $v = \mathbf{e}_j^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{a}^*$ and $\mathbf{x}_\mathcal{B}^* = \mathbf{A}_\mathcal{B}^{-1}\mathbf{a}^*\mathbf{e}_j^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{b}$. The relative cost coefficient vector of the optimal solution $\mathbf{x}^o(\lambda)$ is:

$$\mathbf{r}(\lambda) = \left(\mathbf{c}_\mathcal{B}^T\mathbf{A}_\mathcal{B}^{-1}(\lambda)\mathbf{A}\right)^T - \mathbf{c} = \mathbf{r}^o - \frac{\lambda}{1+v\lambda}\mathbf{r}^* \tag{15}$$

where $\mathbf{r}^o = \left(\mathbf{c}_\mathcal{B}^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{A}\right)^T - \mathbf{c}$ and $\mathbf{r}^* = \left(\mathbf{c}_\mathcal{B}^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{a}^*\mathbf{e}_j^T\mathbf{A}_\mathcal{B}^{-1}\mathbf{A}\right)^T$. Finally the optimal value of the objective function is given by

$$J(\lambda) = \mathbf{c}_\mathcal{B}^T\mathbf{x}_\mathcal{B}^o(\lambda) = \mathbf{c}_\mathcal{B}^T\mathbf{x}_\mathcal{B}^o - \frac{\lambda}{1+v\lambda}\mathbf{c}_\mathcal{B}^T\mathbf{x}_\mathcal{B}^*. \tag{16}$$

Equations (14-16) hold only when the parameter λ belongs to a certain interval $\Lambda_\mathcal{B} = [\underline{\lambda}_\mathcal{B}, \overline{\lambda}_\mathcal{B}]$ wherein the optimal basis $\mathcal{B}$ remains unchanged. This requires: (1) non–singularity of the basis matrix, i.e., $1 + v\lambda > 0$, (2) non-negativity of the basic variables, $\mathbf{x}_\mathcal{B}^o(\lambda) \geq \mathbf{0}$, and (3) non-negativity of the relative cost coefficients, $\mathbf{r}(\lambda) \geq \mathbf{0}$, i.e., the optimality condition. The bounds for the parameter λ can be computed as follows. Let us define:

$$\mathbf{y} = \begin{bmatrix} 1 \\ \mathbf{x}_\mathcal{B}^o \\ \mathbf{r}^o \end{bmatrix}, \qquad \mathbf{y}^* = \begin{bmatrix} 0 \\ \mathbf{x}_\mathcal{B}^* \\ \mathbf{r}^* \end{bmatrix}$$

and let us consider the following sets of indices: $I^+ = \{i \geq 1 \mid (vy_i - y_i^*) > 0\}$ and $I^- = \{i \geq 1 \mid (vy_i - y_i^*) < 0\}$. Then we can easily find:

$$\underline{\lambda}_\mathcal{B} = \begin{cases} -\infty & \text{if} \quad I^+ = \emptyset \\ \max_{i\in I^+}\left\{-\frac{y_i}{vy_i - y_i^*}\right\} \end{cases} \tag{17}$$

and

$$\overline{\lambda}_\mathcal{B} = \begin{cases} +\infty & \text{if} \quad I^- = \emptyset \\ \min_{i\in I^-}\left\{-\frac{y_i}{vy_i - y_i^*}\right\} \end{cases} \tag{18}$$

¿From Equations (14) and (16) we observe that the optimum IFS vector and the objective function do not vary linearly with the parameter λ within the allowable interval $\Lambda_\mathcal{B} = [\underline{\lambda}_\mathcal{B}, \overline{\lambda}_\mathcal{B}]$

as it does happen if the perturbations of the matrix coefficients are made infinitesimally small. Therefore the gradient of the objective function with respect to the j-th column vector of $\mathbf{A}$, say $\mathbf{a}_j = \lambda \mathbf{a}^*$, is a non–linear function of the parameter λ. In particular, if $v \neq 0$, for each value of $\lambda \in \Lambda_{\mathcal{B}}$ such that $\lambda \neq -\frac{1}{v}$, the derivative of the objective function with respect to the parameter λ can be easily computed as

$$\frac{\mathrm{d}J(\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(1+v\lambda)^2}\mathbf{c}_{\mathcal{B}}^T \mathbf{x}_{\mathcal{B}}^*. \tag{19}$$

Note that in the case of $v = 0$ the objective function $J(\lambda)$ varies linearly with the parameter λ within the allowable interval $\Lambda_{\mathcal{B}}$.

5. EXAMPLE: A MULTI–CLASS PRODUCTION SYSTEM

In this section we consider the model of an open production system consisting of two shaping machines and an assembly machine with two classes of parts flowing through, as shown in Figure 1.

Each part is required to be processed according to its production cycle that specifies the sequence of machines it must visit and the operation performed by them. Parts of class 1 and 2, coming from external independent sources, are queued in the buffers B_1 and B_2 which are both feeding machine M_1, and then start the processing at machine M_1. Buffer B_2 has a finite capacity C_{B2} while buffer B_1 has an unlimited capacity. At the exit of machine M_1, parts of class 2 are ready to enter the assembly machine M_a while parts of class 1 flow into the buffer B_3 with finite capacity C_{B3}, then to machine M_2, where after the processing some parts may be required to be reworked on the same machine. At the exit of machines M_1 and M_2 parts of both classes are respectively collected in the buffers B_{a1} and B_{a2} with unlimited capacity and then are packed together by the assembly machine according to a specified production mix.

Since machines are unreliable, we must also take into account a certain failure model. In particular, we can consider either a time–dependent failure (TDF) model assuming that a machine fails after a given time has elapsed since the previous repair operation, or an operation–dependent failure (ODF) model assuming that a machine fails after a given production volume has been processed since the previous repair operation [3].

Although this model may seem quite simple it captures the key difficulties of common control problems arising in manufacturing systems. Problems of parts routing, admission and service rate selection have been deeply studied in recent years but the determination of an explicit solution is still an open problem even for very simple systems [6, 12].

In this paper we address the production rate selection problem as a constrained optimization control problem within the linear algebraic framework offered by FOHPN model. In this way, we can explicitly determine the optimal control policies with regard to the desired performance measure. Furthermore we can also derive the sensitivity of the system design parameters, such as maximum machine production rates, re-working factor, production mix factor, etc., with respect to the performance measures.

For this purpose, let us model the production network depicted in Figure 1 by using in a modular way some elementary manufacturing components described by basic FOHPN models as discussed in [3].

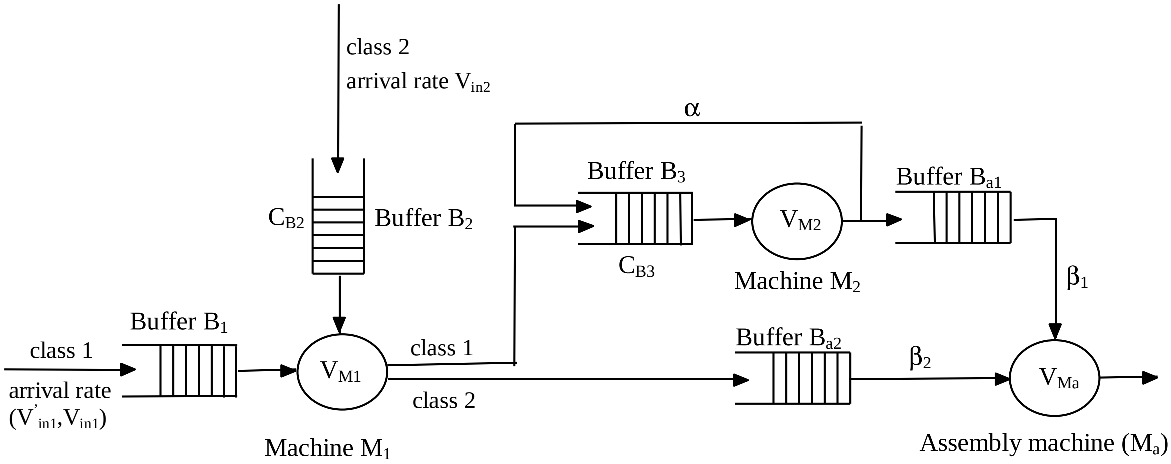

Figure 1: *A multi–class production network.*

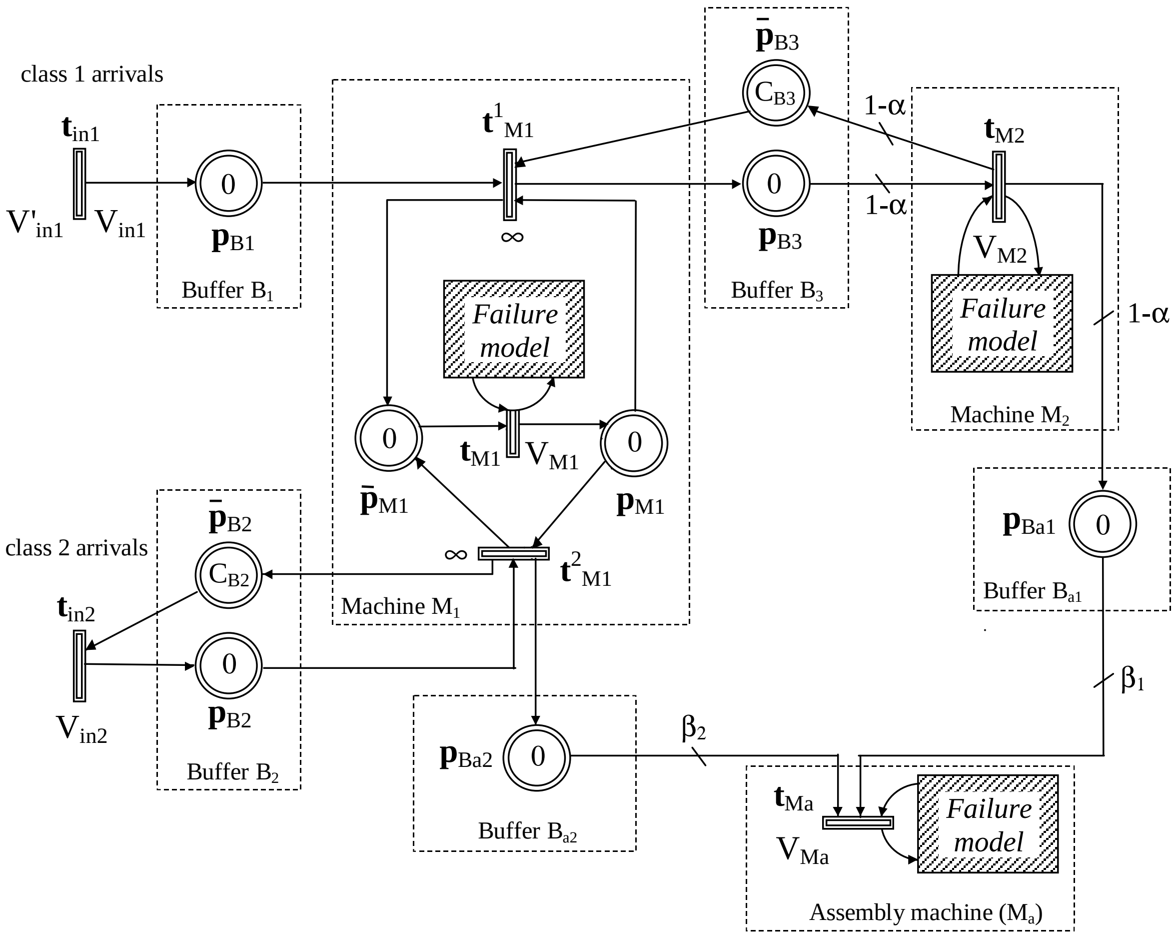

Figure 2: *FOHPN model of the production network in Figure 1.*

Initially, all buffers are assumed empty and all machines are assumed potentially operating at their maximum production rate. Parts of class 1 arrive in the system at a bounded rate $V'_{in,1} \leq v_{in,1} \leq V_{in,1}$ while parts of class 2 arrive in the system at a bounded rate $v_{in,2} \leq V_{in2}$. Parts of class 2 may be rejected upon arrival if buffer B_2 is full. Machine M_1 and M_a work on parts of both classes with maximum production rates V_{M1} and V_{Ma}, while machine M_2 works on parts of class 1 with maximum production rate V_{M2}. The FOHPN model of the production system under consideration is shown in Figure 2, where the initial marking shown assumes that all buffers are initially empty.

5.1. Numerical examples

In this section we show how to solve production control problems and make sensitivity analysis by means of the FOHPN framework. In particular we show that our method can solve via simulation intractable systems with classical methods.

We now highlight the main step followed by an FOHPN simulator, that makes use of a solver to implement linear programming and sensitivity analysis algorithms. First of all, we define the control problem that we want to solve in terms of a given performance measure that has to be optimized. Then, the solver will provide the optimal machines production rates according to the constraints defined by the current macro–state, by solving a sequence of linear programming problems, at the occurrence of the macro–events. At each step, sensitivity analysis can be done in order to make adjustment on the optimal myopic solution that represents the reference values for the machine production rates within the next macro–period.

Let N be the final macro–event, $T = \cup_{k=0}^{N-1}[\tau_k, \tau_{k+1})$ a finite time horizon, where τ_k for $k = 0, 1, 2, \ldots,$ denote the occurrence times of the macro–events. The length of the k–th macro–period is denoted $\Delta_k = (\tau_{k+1} - \tau_k)$. Let us define the instantaneous firing speed vector $\mathbf{v}(\tau_k) = [v_{in1}(\tau_k), v_{in2}(\tau_k), v^1_{M1}(\tau_k), v^2_{M1}(\tau_k), v_{M2}(\tau_k), v_{Ma}(\tau_k)]^T$ and let $J = \mathbf{c}^T\mathbf{v}(\tau_k)$ be the performance function to be optimized. Here τ_k denotes the occurrence in time of the macro–event. Finally, let $V_{in1} = 5$, $V'_{in1} = 2$, $V_{in2} = 4$, $V_{M1} = 7$, $V_{M2} = 5$, $V_{Ma} = 7$, $\alpha = 0.2$, $\beta_1 = 0.8$ and $\beta_2 = 0.2$.

5.1.1. Maximization of the machines utilization

The first problem we consider is the maximization of the machines utilization; thus the coefficients vector of the performance index is $\mathbf{c} = [0, 0, 1, 1, 1, 1]^T$. We show how to derive an optimal control policy that myopically maximizes the machines utilization over a finite time horizon and we describe the developments within the first three macro–periods: (MP1) all buffers are empty; (MP2) machine M_2 breaks down; (MP3) buffer B_3 becomes full.

(MP1). At the beginning of the first macro–period of length Δ_0, — at time τ_0 all buffers are empty and all machines are operational — we define and solve the following constrained linear optimization problem:

$$
\begin{array}{l}
\max_{\mathbf{v}(\tau_0)}\left[v_{M1}^1(\tau_0)+v_{M1}^2(\tau_0)+v_{M2}(\tau_0)+v_{Ma}(\tau_0)\right] \\
\text{s.t.} \\
\left\{\begin{array}{lll}
v_{in1}(\tau_0) & \leq & V_{in1} \\
v_{in1}(\tau_0) & \geq & V'_{in1} \\
v_{in2}(\tau_0) & \leq & V_{in2} \\
v_{M1}^1(\tau_0)+v_{M1}^2(\tau_0) & \leq & V_{M1} \\
v_{M2}(\tau_0) & \leq & V_{M2} \\
v_{Ma}(\tau_0) & \leq & V_{Ma} \\
v_{M1}^1(\tau_0) & \leq & v_{in1}(\tau_0) \\
v_{M1}^2(\tau_0) & \leq & v_{in2}(\tau_0) \\
(1-\alpha)v_{M2}(\tau_0) & \leq & v_{M1}^1(\tau_0) \\
\beta_2 v_{Ma}(\tau_0) & \leq & v_{M1}^2(\tau_0) \\
\beta_1 v_{Ma}(\tau_0) & \leq & (1-\alpha)v_{M2}(\tau_0)
\end{array}\right.
\end{array}
\tag{20}
$$

The solver provides the following optimal solution: $J^o(\tau_0) = 17$ with $\mathbf{v}^o(\tau_0) = [4, 3, 4, 3, 5, 5]^T$ which represents an optimal control policy for the machine production rates that may be adopted during the first macro–period.

In particular, throughout the interval Δ_0, B_{a2} is increasing at a rate equal to 2, while the other buffers content are constant and equal to 0. The sensitivity analysis of the right–hand side vector reveals the following results:

$$
\begin{array}{ll}
\Lambda_{\mathcal{B}}[V_{M1}] = [5, 8], & g_4 = \frac{\mathrm{d}J}{\mathrm{d}V_{M1}} = 1 \\
\Lambda_{\mathcal{B}}[V_{M2}] = [3.75, 6.25], & g_5 = \frac{\mathrm{d}J}{\mathrm{d}V_{M2}} = 2
\end{array}
$$

where $\Lambda_{\mathcal{B}}[\gamma]$ denotes the allowable range for the parameter γ and $g_i = \frac{\mathrm{d}J}{\mathrm{d}\gamma}$ is the dual price associated to the i–th constraint. This means that we can achieve a better performance if we either improve the maximum production rate of machines M_1 or M_2 by changing their values within their allowable ranges, since all other dual prices are equal to 0. In particular, if we consider a linear perturbation on the parameter V_{M2}, i.e., we set $V_{M2}(\lambda) = V_{M2} + \lambda$ for $\lambda \in [-1.25, 1.25]$, the objective function $J(\tau_0)$ and the optimal basic solution $\mathbf{v}^o(\tau_0)$ will vary linearly with the parameter λ.

As an example, let $V_{M2} = 5.1$, $(\lambda = 0.1)$. By solving the LP problem (20) we obtain $J^*(\tau_0) = 17.2$ with $\mathbf{v}^*(\tau_0) = [4.08, 2.92, 4.08, 2.92, 5.1, 5.1]^T$. We can easily derive the gradient of the optimal basic solution with respect to λ as $\nabla_\lambda \mathbf{v}^o(\tau_0) = [0.8, -0.8, 0.8, -0.8, 1, 1]$, and the derivative of the objective function with respect to λ (the dual price g_5) as $\frac{\mathrm{d}J}{\mathrm{d}\lambda} = 2$, evaluated for $\lambda = 0.1$, that will remain both constant as long as $\lambda \in [-1.25, 1.25]$.

Now suppose that at time τ_1, after an interval of time that may depend on the production volume performed by the machines, a failure occurs at machine M_2.

(MP2). At the beginning of the second macro–period of length Δ_1, — at time τ_1 machine M_2 breaks down — we define and solve the following constrained linear optimization problem, where a new constraint set has been specified, i.e., the fifth constraint is now $v_{M2}(\tau_1) = 0$.

$$
\begin{array}{l}
\max_{\mathbf{v}(\tau_1)} \left[v_{M1}^1(\tau_1) + v_{M1}^2(\tau_1) + v_{M2}(\tau_1) + v_{Ma}(\tau_1) \right] \\
\text{s.t.} \\
\left\{
\begin{array}{lcl}
v_{in1}(\tau_1) & \leq & V_{in1} \\
v_{in1}(\tau_1) & \geq & V'_{in1} \\
v_{in2}(\tau_1) & \leq & V_{in2} \\
v_{M1}^1(\tau_1) + v_{M1}^2(\tau_1) & \leq & V_{M1} \\
v_{M2}(\tau_1) & = 0 & \\
v_{Ma}(\tau_1) & \leq & V_{Ma} \\
v_{M1}^1(\tau_1) & \leq & v_{in1}(\tau_1) \\
v_{M1}^2(\tau_1) & \leq & v_{in2}(\tau_1) \\
(1-\alpha)v_{M2}(\tau_1) & \leq & v_{M1}^1(\tau_1) \\
\beta_2 v_{Ma}(\tau_1) & \leq & v_{M1}^2(\tau_1) \\
\beta_1 v_{Ma}(\tau_1) & \leq & (1-\alpha)v_{M2}(\tau_1)
\end{array}
\right.
\end{array}
\tag{21}
$$

The solver provides the following optimal solution: $J^o(\tau_1) = 7$ with $\mathbf{v}^o(\tau_1) = [3, 4, 3, 4, 0, 0]^T$ which represents an optimal control policy that may be adopted during the second macro–period. The above solution means that the failure of machine M_2 forces machine M_a to produce at a rate $v_{Ma}(\tau_1) = 0$, thus increasing the content of buffer B_{a2} at a rate equal to 4 and the content of buffer B_3 at a rate equal to 3. The sensitivity analysis of the right–hand side vector reveals the following results:

$$
\begin{array}{ll}
\Lambda_{\mathcal{B}}[V_{M1}] = [6, 9], & g_4 = \frac{\mathrm{d}J}{\mathrm{d}V_{M1}} = 1 \\
\Lambda_{\mathcal{B}}[V_{M2}] = [0, 3.75], & g_5 = \frac{\mathrm{d}J}{\mathrm{d}V_{M2}} = 2.
\end{array}
$$

This means that we can achieve a better performance if we improve the maximum production rate of machine M_1 by changing its value within its allowable range, since all other dual prices are equal to 0 and machine M_2 is down. As an example if we set $V_{M1}(\tau_1) = 8$ then we obtain the optimal solution $J^o(\tau_1) = 8$ with $\mathbf{v}^o(\tau_1) = [4, 4, 4, 4, 0, 0]^T$ corresponding to a linear increment of both objective function and basic solution. Since buffer B_3 has a finite capacity C_{B3}, it will reach its maximum level after an interval of time

$$
\Delta_1 = \frac{C_{B3} - m_{pB3}(\tau_1)}{v_{M1}^1(\tau_1) - (1-\alpha)v_{M2}(\tau_1)}.
$$

Therefore at time $\tau_2 = \tau_1 + \Delta_1$, assuming that machine M_2 will be repaired at time $\tau_3 > \tau_2$, buffer B_3 will be full.

(MP3). At the beginning of the third macro–period of length Δ_2, — at time τ_2 buffer B_3 is full — we define and solve the following constrained linear optimization problem, where a new constraint set has been specified, i.e., the ninth constraint is now $v^1_{M1}(\tau_2) \leq (1-\alpha)v_{M2}(\tau_2)$.

$$
\begin{array}{l}
\max_{\mathbf{v}(\tau_2)} \left[v^1_{M1}(\tau_2) + v^2_{M1}(\tau_2) + v_{M2}(\tau_2) + v_{Ma}(\tau_2)\right] \\
\text{s.t.} \\
\left\{
\begin{array}{lcl}
v_{in1}(\tau_2) & \leq & V_{in1} \\
v_{in1}(\tau_2) & \geq & V'_{in1} \\
v_{in2}(\tau_2) & \leq & V_{in2} \\
v^1_{M1}(\tau_2) + v^2_{M1}(\tau_2) & \leq & V_{M1} \\
v_{M2}(\tau_2) & = & 0 \\
v_{Ma}(\tau_2) & \leq & V_{Ma} \\
v^1_{M1}(\tau_2) & \leq & v_{in1}(\tau_2) \\
v^2_{M1}(\tau_2) & \leq & v_{in2}(\tau_2) \\
v^1_{M1}(\tau_2) & \leq & (1-\alpha)v_{M2}(\tau_2) \\
\beta_2 v_{Ma}(\tau_2) & \leq & v^2_{M1}(\tau_2) \\
\beta_1 v_{Ma}(\tau_2) & \leq & (1-\alpha)v_{M2}(\tau_2)
\end{array}
\right.
\end{array}
\tag{22}
$$

The solver provides the following solution: $J^o(\tau_2) = 4$ with $\mathbf{v}^o(\tau_2) = [2, 4, 0, 4, 0, 0]^T$ which represents an optimal control policy that may be adopted during the third macro–period.

Throughout the interval Δ_2 the content of buffer buffers B_1 is increasing at a rate equal to 2 and the content of buffer B_{a2} is increasing at a rate equal to 4. The sensitivity analysis of the right–hand side vector reveals the following results:

$$
\begin{array}{ll}
\Lambda_{\mathcal{B}}[V_{in2}] = [0, 8], & g_3 = \frac{\mathrm{d}J}{\mathrm{d}V_{in2}} = 1 \\
\Lambda_{\mathcal{B}}[V_{M2}] = [0, 2.5], & g_5 = \frac{\mathrm{d}J}{\mathrm{d}V_{M2}} = 2.8.
\end{array}
$$

This means that we can achieve a better performance if we make a control action on the external arrival of parts of class 2. This can be done by changing the arrival rate of parts of class 2 within its allowable range, since all other dual prices are equal to 0 and machine M_2 is down.

Therefore the optimal myopic control policy $\mathcal{V}^o = \{\mathbf{v}^o(\tau_0), \mathbf{v}^o(\tau_1), \mathbf{v}^o(\tau_2), \ldots\}$ that provides the maximization of the machines utilization is defined as follows:

$$
\begin{array}{lll}
\mathbf{v}^o(\tau_0) = [4, 3, 4, 3, 5, 5]^T, & J^o(\tau_0) = 17, & [\tau_0, \tau_1) \\
\mathbf{v}^o(\tau_1) = [3, 4, 3, 4, 0, 0]^T, & J^o(\tau_1) = 7, & [\tau_1, \tau_2) \\
\mathbf{v}^o(\tau_2) = [2, 4, 0, 4, 0, 0]^T, & J^o(\tau_2) = 4, & [\tau_2, \tau_3) \\
\quad \ldots & &
\end{array}
$$

Note that the numerical tools that can be used for solving constrained linear optimization problems, such as *LINDO*, do only provide dual prices associated to the constraints and the allowable ranges related to the perturbations made on the right–hand side vector and the objective function coefficients. The framework we propose in this paper allows designers to make more general sensitivity analysis by also considering perturbations on the matrix coefficients.

5.1.2. Perturbation of the re–working and mix factors

Let us now consider the following questions that may arise during the simulation of this production network. What if we change the re–working factor α? What if we change the mix factors β_1 and β_2? We can answer all these questions.

The control problem we consider here is the maximization of the system outflow; thus the coefficients vector of the performance index is $\mathbf{c} = [0,0,0,0,0,1]^T$. If we assume that at time τ_k all buffers are empty and all machines are operational, we define and solve the following constrained linear optimization problem:

$$\max_{\mathbf{v}(\tau_k)} \left[v_{Ma}(\tau_k)\right] \quad \text{s.t.}$$
$$\begin{cases} v_{in1}(\tau_k) & \leq & V_{in1} \\ v_{in1}(\tau_k) & \geq & V'_{in1} \\ v_{in2}(\tau_k) & \leq & V_{in2} \\ v^1_{M1}(\tau_k) + v^2_{M1}(\tau_k) & \leq & V_{M1} \\ v_{M2}(\tau_k) & \leq & V_{M2} \\ v_{Ma}(\tau_k) & \leq & V_{Ma} \\ v^1_{M1}(\tau_k) & \leq & v_{in1}(\tau_k) \\ v^2_{M1}(\tau_k) & \leq & v_{in2}(\tau_k) \\ v_{M2}(\tau_k) & \leq & v^1_{M1}(\tau_k) \\ \beta_2 v_{Ma}(\tau_k) & \leq & v^2_{M1}(\tau_k) \\ \beta_1 v_{Ma}(\tau_k) & \leq & (1-\alpha)v_{M2}(\tau_k) \end{cases} \tag{23}$$

The solver provides the following optimal solution: $J^o(\tau_k) = 4$ with $\mathbf{v}^o(\tau_k) = [4,3,4,3,4,4]^T$. By the same developments as in the previous subsection we can easily state that machine M_2 is the bottleneck in this production system. The sensitivity analysis of the right–hand side vector reveals the following results:

$$\Lambda_{\mathcal{B}}[V_{M2}] = [3,5], \quad g_5 = \frac{\mathrm{d}J}{\mathrm{d}V_{M2}} = 1.$$

This means that we can achieve a better performance if we improve the maximum production rate of machine M_2 by changing its value within its allowable range, since all other dual prices are equal to 0.

The sensitivity analysis of the matrix coefficients provides the following optimal basic solution $\mathbf{x}^o_{\mathcal{B}} = [v_{in1}, v_{in2}, v^1_{M1}, v^1_{M1}, v_{M2}, v_{Ma}, s2, , s3, s4, s5, s7]^T$ with corresponding optimal basis $\mathcal{B}$. The allowable range for the re–working factor α that appears in the matrix coefficients, is:

$$\Lambda_{\mathcal{B}}[\alpha] = [-0.4, 0.2].$$

Note that for physical reasons we should consider $\Lambda_{\mathcal{B}}(\alpha) = [0, 0.2]$, since $0 \leq \alpha \leq 1$. Let us now consider a linear perturbation on the matrix coefficient $a = (1-\alpha)$, i.e., we set $a(\lambda) = a + \lambda$. As an example, let $\alpha = 0.1$, $(\lambda = 0.1)$. By solving the LP problem (23) we obtain $J^*(\tau_k) = 4.5$ with $\mathbf{v}^*(\tau_k) = [4,3,4,3,4,4.5]^T$. This is a particular case in which the parameter v introduced in Section is equal to 0. Therefore the objective function $J(\tau_k)$ and the optimal basic solution $\mathbf{v}^o(\tau_k)$ vary linearly with parameter λ as long as $\lambda \in [0, 0.2]$. Thus, we can easily derive the gradient of the optimal basic solution with respect to λ as $\nabla_\lambda \mathbf{v}^o(\tau_0) = [0,0,0,0,0,5]$, and the derivative of the objective function with respect to λ (by applying Eq. (19)) as $\frac{\mathrm{d}J(\lambda)}{\mathrm{d}\lambda} = 5$, evaluated for $\lambda = 0.1$, that will remain both constant as long as $\lambda \in [0, 0.2]$.

Finally, let us now consider the mix factors β_1 and β_2. The sensitivity analysis provides the following allowable ranges for the mix factors β_1 and β_2 that appear in the matrix coefficients:

$$\Lambda_{\mathcal{B}}[\beta_1] = [0.5161, 0.8]$$
$$\Lambda_{\mathcal{B}}[\beta_2] = [0.2, 0.4839].$$

As an example, let $\beta_1 = 0.6$, $\beta_2 = 0.4$, $(\lambda = 0.2)$. By solving the LP problem (23) we obtain $J^*(\tau_k) = 5.3333$ with $\mathbf{v}^*(\tau_k) = [4, 3, 4, 3, 4, 5.3333]^T$. In this case we have $v = -1.25$, therefore the derivative of the objective function with respect to the mix factors is a non–linear function of the perturbation λ. However we can apply Eq. (19) and obtain the derivative of the objective function with respect to λ, evaluated for $\lambda = 0.2$, as $\frac{\mathrm{d}J(\lambda)}{\mathrm{d}\lambda} = 8.8889$.

6. CONCLUSIONS

In this paper we have considered First–Order Hybrid Petri Nets. We have shown how the control of the first–order continuous behaviour can be framed as a conflict resolution policy that aims to optimize a given objective function. Since the set of all possible behaviours of the net during a macro–state can be represented by a convex set defined by a system of linear inequalities, an IFS vector can be selected among all admissible ones by solving a linear programming problem. Sensitivity analysis enabled us to study how optimal solutions change according to changes of the linear program in terms of the coefficients of the matrix, the right–hand side and the objective function.

An FOHPN model of a multi–class production system has been considered and two different production control problems have been studied. The sensitivity of the system design parameters, such as maximum machine production rates, re–working and production mix factors, with respect to the performance measure have also been derived.

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