Cycle time Optimization of Deterministic Timed Weighted Marked Graphs*

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Abstract

Timed marked graphs, a special class of Petri nets, are extensively used to model and analyze cyclic manufacturing systems. Weighted marked graphs are convenient to model automated production systems such as robotic work cells or embedded systems. The main problem for designers is to find a trade off between minimizing the cost of the resources and maximizing the system's throughput. It is possible to apply analytical techniques for the average cycle time optimization problem of such systems. The problem consists in finding an initial marking to minimize the average cycle time (i.e., maximize the throughput) while the weighted sum of tokens in places is less than or equal to a given value.

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I. INTRODUCTION

Timed Petri nets (PNs) are well known as efficient tools for modeling discrete event systems, especially manufacturing systems. In this paper, we study a particular class of timed Petri nets called *timed weighted marked graphs* (TWMGs). The main feature of this class of nets is that each place has only one input and one output transition. Moreover, the firing delay of each transition is *deterministic*.

Timed weighted marked graphs and timed marked graphs (TMGs) find wide applications in manufacturing. They can model complex assembly lines and solve cyclic scheduling problems. Workshop operations and products are usually modeled by transitions and tokens, respectively. Between two successive transformations, semi-finished products have to be stored or moved from a workshop to another. The amount of products, also called Work In Process (WIP), that have to be stored or moved may have economical consequences. Therefore, the main problem for designers is to find a proper schedule of WIP that allows the system to reach a given productivity while the amount of WIP is the smallest.

Teruel *et al.* proposed several techniques for the analysis of WMGs in [1]. Campos *et al*. [2] developed methods to compute the average cycle time of TMGs for a given initial marking. Munier [3] proposed a method to transform a WMG into an MG under *single server* semantics hypothesis and Nakamura and Silva [4] discussed the same problem under *infinite server* semantics hypothesis. Giua *et al*. [5] dealt with the firing rate optimization of cyclic timed event graphs by token allocations and proposed a mixed integer linear programming problem (ILPP) to compute an optimal solution. However, in the literature, few works are found to deal with the optimization problem of TWMGs. Sauer [6] proposed a heuristic solution based on an iterative process to solve the marking optimization problem of TWMGs. He *et al*. [7] presented a novel heuristic method to deal with the marking optimization problem which was shown to be more effective than that of Sauer [6]. Benazouz *et al*. [8] developed an algorithm to minimize the overall buffer capacities with throughput constraint for TWMGs.

In this paper, we study the cycle time optimization problem of a TWMG, an issue that to the best of our knowledge has not been addressed in the literature. The problem consists in finding an initial marking to *minimize the average cycle time* while the weighted sum of tokens in places is less than or equal to a given value.

This problem has a practical relevance in many applications. As an example, for a manufacturing system operated with a periodic scheduling the *cycle time* is the inverse of the *throughput*. Thus if the cycle time is reduced the throughput is maximized. In addition, in a Petri net model of a manufacturing system, tokens in the net represent resources allocated to it such as machines, transportation devices, buffer slots, etc. Thus a bound on the weighted sum of tokens in the net describes a limited availability of resources or equivalently a limited budget to acquire them.

The main contributions of the present paper can be summarized as follows:

1) The cycle time optimization problem of TWMGs is originally presented.

2) Periodicity of transformation of TWMGs into TMGs is proposed and the initial marking of a TWMG is partitioned into several subsets with regard to the periodicity.

3) Transformation of the cycle time optimization problem of TWMGs into the cycle time optimization problem of TMGs is developed.

4) An ILPP combined with the results reported in [5] is presented to deal with the cycle time optimization problem of TMGs.

This paper presents some original technical results and examples of their applications. However, formal proofs of the results are omitted for lack of space and to make the paper more accessible.

This paper is structured as follows. In the following section, we briefly recall some basic concepts and the main properties. In Section III, we present the problem statement. In Section IV, we propose an analytical method for the cycle time optimization problem based on the work in [5]. In Section V, numerical examples are shown to illustrate the algorithm. Conclusions and future work are finally drawn in Section VI.

II. Background

A. Generalities

We assume that the reader is familiar with the structure, firing rules, and basic properties of PNs (see [9], [10], and [1]). In this section, we will recall the formalism used in the paper. A *place/transition* net (P/T net) is a structure $N = (P, T, \boldsymbol{Pre}, \boldsymbol{Post})$, where P is a set of n places; T is a set of m transitions; $\boldsymbol{Pre} : P \times T \rightarrow \mathbb{N}$ and $\boldsymbol{Post} : P \times T \rightarrow \mathbb{N}$ are the pre- and post-incidence functions that specify the arcs; $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the incidence matrix, where $\mathbb{N}$ is a set of non-negative integers.

A vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^T \in \mathbb{N}^{|T|}$ such that $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$ is a *T-semiflow*. A vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{N}^{|P|}$ such that $\boldsymbol{y} \neq \boldsymbol{0}$ and $\boldsymbol{y}^T \cdot \boldsymbol{C} = \boldsymbol{0}$ is a *P-semiflow*. The supports of a T-semiflow and a P-semiflow are defined by $\|\boldsymbol{x}\|$=$\{t_i \in T | x_i > 0\}$ and $\|\boldsymbol{y}\|$=$\{p_i \in P | y_i > 0\}$, respectively. A *minimal* T-semiflow (P-semiflow) is a T-semiflow $\|\boldsymbol{x}\|$ (P-semiflow $\|\boldsymbol{y}\|$) that is not a superset of the support of any other T-semiflow (P-semiflow), and its components are mutually prime.

A marking is a vector $\boldsymbol{M} : P \rightarrow \mathbb{N}$ that assigns to each place of a P/T net a non-negative integer of tokens, represented by black dots; we denote the marking of place p as $M(p)$. A P/T system or net system $\langle N, \boldsymbol{M_0} \rangle$ is a net N with an initial marking $\boldsymbol{M_0}$. A transition t is enabled at $\boldsymbol{M}$ if $\boldsymbol{M} \geq \boldsymbol{Pre}(\cdot, t)$ and an enabled transition t may fire yielding a marking $\boldsymbol{M'}$ with

$$\boldsymbol{M'} = \boldsymbol{M} + \boldsymbol{C}(\cdot, t), \tag{1}$$

where $\boldsymbol{Pre}(\cdot, t)$ (resp. $\boldsymbol{C}(\cdot, t)$) denotes the column of the matrix $\boldsymbol{Pre}$ (resp. $\boldsymbol{C}$) associated with transition t.

A P/T net is said to be *ordinary* when all of its arc weights are $1's$. A *marked graph* (also called an event graph) is an ordinary Petri net such that it satisfies the condition $|^{\bullet}p| = |p^{\bullet}| = 1$. A *weighted marked graph* (also called a weighted event graph) is a net that also satisfies this condition but may not be ordinary, i.e., the weight associated with each arc is a non-negative integer number.

A net is *strongly connected* if there exists a directed path from any node in $P \cup T$ to every other node. Let us define an *elementary circuit* γ (or elementary cycle) of a net as a directed path that goes from one node back to the same node without passing twice on the same node. In a strongly connected net, it is easy to show that each node belongs to an elementary circuit, and thus the name cyclic nets are also used to denote this class.

A deterministic timed P/T net is a pair $N^\delta = (N, \delta)$, where $N = (P, T, \boldsymbol{Pre}, \boldsymbol{Post})$ is a standard P/T net, and $\delta : T \to \mathbb{N}$, called firing delay, assigns a non-negative integer fixed firing duration to each transitions. A transition with a firing delay equal to 0 is said to be immediate. We consider a single server semantics, i.e., we assume that each transition can fire only once at each time even its enabling degree is greater than one.

Definition 1: (Campos *et al.* [2]) *Every elementary circuit γ of a WMG is neutral, if the following condition holds.*

$$\prod_{p\in\gamma} \frac{Pre(p, p^\bullet)}{Post(p, {}^\bullet p)} = 1 \qquad \blacksquare$$

In other words, in a neutral circuit the product of the weights of all pre-arcs is equal to that of all post-arcs. This means that if the circuit initially contains enough tokens, it is possible to fire all transitions along the path returning to the same initial marking. It is well known that a WMG whose circuits are all neutral has a unique T-semiflow $\boldsymbol{x}$ and it contains all transitions in its support [1].

In this paper, we limit our study to strongly connected WMGs in which all circuits are neutral.

Proposition 1: (Benabid-Najjar *et al.* [11]) *A strongly connected WMG in which all circuits are neutral is* bounded*, i.e., there exists an integer B such that the marking of any place p is not greater than B at any reachable marking.* $\blacksquare$

Given a place p_i of a WMG, let $t_{in(p_i)}$ (resp., $t_{out(p_i)}$) be its unique input (resp., output) transition as shown in Fig. 1. We denote $w(p_i) = Post(p_i, t)$ the weight of its input arc and $v(p_i) = Pre(p_i, t)$ the weight of its output arc. Let gcd_{p_i} represent the greatest common divisor of the integers $w(p_i)$ and $v(p_i)$.

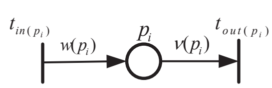

Fig. 1. A place p_i between two transitions $t_{in(p_i)}$ and $t_{out(p_i)}$.

B. Cycle time of a TWMG

The *average cycle time* $\chi(\boldsymbol{M}_0)$ of a TWMG system $\langle N, \boldsymbol{M_0}\rangle$ is the average time to fire once the T-semiflow under the earliest operational model (i.e., transitions are fired as soon as possible). Considering a net consisting only of one circuit, we define $\chi_\gamma(\boldsymbol{M}_0)$ as the average cycle time of circuit γ.

Let Γ represent the set of elementary circuits of a cyclic TWMG. It is well known that for a TMG the cycle time of the net is equal to the maximal cycle time over all circuits, i.e., $\chi(\boldsymbol{M}_0) = \max_{\gamma\in\Gamma} \chi_\gamma(\boldsymbol{M}_0)$. This result does not apply to a TWMG [7], but it holds that the average cycle time of each circuit is smaller than or equal to the one of the nets, i.e., $\chi(\boldsymbol{M}_0) \geq \max_{\gamma\in\Gamma} \chi_\gamma(\boldsymbol{M}_0)$.

The average cycle time of a TMG can be obtained by means of following LPP:

$$max \ \{\boldsymbol{y}^T \cdot \boldsymbol{Pre} \cdot \boldsymbol{\delta} | \boldsymbol{y}^T \cdot \boldsymbol{C} = 0, \boldsymbol{y}^T \cdot \boldsymbol{M_0} = 1, \boldsymbol{y} \geq 0\} \tag{2}$$

where $\boldsymbol{\delta} \in \mathbb{N}^m$ is the vector containing all firing delays of timed transitions (recall that $m = |T|$).

It is difficult to compute the average cycle time of a TWMG. From [2] a lower bound for the average cycle time of a live and bounded TWMG system can be computed by solving an LPP. Chao *et al*. [12] proposed a method to compute the cycle time of a TWMG but under restrictive conditions at initial marking.

C. Transformation of WMGs

One way to analytically compute the average cycle time of a TWMG is to convert it into an equivalent TMG. In fact, Munier [3] showed that a TWMG system $\langle N, \boldsymbol{M} \rangle$ can be transformed into an *equivalent* TMG system $\langle \hat{N}, \hat{\boldsymbol{M}} \rangle$ which describes the same precedence constraints on the firing of transitions. This implies that the average cycle time[1] of the two systems are identical, i.e., $\chi(\boldsymbol{M}) = \hat{\chi}(\hat{\boldsymbol{M}})$.

This equivalent TMG system depends on the initial marking $\boldsymbol{M}$ and the minimal T-semiflow $\boldsymbol{x}$ of the TWMG. Since it is necessary for us to use this transformation method, we present it in Algorithm 1.

Algorithm 1: Transformation from a TWMG to a TMG

Input: A TWMG system $\langle N, \boldsymbol{M} \rangle$.

Output: An equivalent TMG system $\langle \hat{N}, \hat{\boldsymbol{M}} \rangle$ such that $\chi(\boldsymbol{M}) = \hat{\chi}(\hat{\boldsymbol{M}})$.

1: Compute the minimal T-semiflow $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^T$ of net N

2: *(Transformation of transitions)*. Replace each transition $t_i \in T$ by x_i transitions, t_i^1, t_i^2, ..., $t_i^{x_i}$, with the same firing delay of t_i. These transitions are connected by an elementary circuit with all weights equal to 1. Add x_i places q_i^1, q_i^2, ..., $q_i^{x_i}$, where q_i^a, $\forall a = 1, \ldots, x_i - 1$, is a place connecting transition t_i^a to transition t_i^{a+1} and $q_i^{x_i}$ is a place connecting transition $t_i^{x_i}$ to t_i^1. Only place $q_i^{x_i}$ contains one token and the other places are empty, i.e.,

$$\begin{cases} \hat{M}(q_i^a) = 0, \ \forall i = 1, \ldots, m, \ \forall a = 1, \ldots, x_i - 1, \\ \hat{M}(q_i^{x_i}) = 1, \ \forall i = 1, \ldots, m, \end{cases} \tag{3}$$

Thus there exist m mono-marked circuits that are called *intra transition sequential systems*. They do not depend on the initial marking.

3: *(Transformation of places: case 1)*. Replace each place $p_i \in P$ such that $w(p_i) > v(p_i)$ by $n_i = x_{in(p_i)}$ places p_i^s, where for $s = 1, \ldots, n_i$:

$$\begin{cases} a_s \cdot x_{out(p_i)} + b_s = \left\lfloor \frac{M(p_i) + w(p_i) \cdot (s-1)}{v(p_i)} \right\rfloor + 1, \\ b_s \in \{1, \ldots, x_{out(p_i)}\}, \\ a_s \in \mathbb{N}, \end{cases} \tag{4}$$

[1]In the following, we will denote by $\chi(\boldsymbol{M})$ the average cycle time of a TWMG system $\langle N, \boldsymbol{M} \rangle$ and by $\hat{\chi}(\hat{\boldsymbol{M}})$ the average cycle of the equivalent TMG system $\langle \hat{N}, \hat{\boldsymbol{M}} \rangle$.

Place p_i^s connects transition $t_{in(p_i)}^s$ to transition $t_{out(p_i)}^{b_s}$ and contains a_s tokens, i.e.,

$$\hat{M}(p_i^s) = a_s. \tag{5}$$

4: *(Transformation of places: case 2).* Replace each place $p_i \in P$ such that $w(p_i) \leq v(p_i)$ by $n_i = x_{out(p_i)}$ places p_i^s, where for $s = 1, \ldots, n_i$:

$$\begin{cases} c_s \cdot x_{in(p_i)} + d_s = \left\lceil \frac{s \cdot v(p_i) - M(p_i)}{w(p_i)} \right\rceil, \\ d_s \in \{1, \ldots, x_{in(p_i)}\}, \\ c_s \in \mathbb{Z}_{\leq 0}, \end{cases} \tag{6}$$

Place p_i^s connects transition $t_{in(p_i)}^{d_s}$ to transition $t_{out(p_i)}^s$ and contains $-c_s$ tokens, i.e.,

$$\hat{M}(p_i^s) = -c_s. \tag{7}$$

Note that Eqs. (4) and (6) admit only one solution (a_s, b_s and c_s, d_s) for each value of s.

The structure of the equivalent TMG (i.e., the arcs connecting places and transitions) depends on the initial marking $\boldsymbol{M}$ of the TWMG. However, this dependence is periodic as shown in the following proposition.

Proposition 2: Consider a TWMG N with minimal T-semiflow $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^T$ and two possible initial markings $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. Let $\langle \hat{N}_1, \hat{\boldsymbol{M}}_1 \rangle$ (resp., $\langle \hat{N}_2, \hat{\boldsymbol{M}}_2 \rangle$) be the equivalent TMG obtained by Algorithm 1 with input $\langle N, \boldsymbol{M}_1 \rangle$ (resp., $\langle N, \boldsymbol{M}_2 \rangle$).

If for a place $p_i \in P$

$$M_2(p_i) = M_1(p_i) + \xi \cdot v(p_i) \cdot x_{out(p_i)} \quad \textit{with } \xi \in \mathbb{N},$$

then the structure corresponding to p_i in $\hat{N}_1$ and $\hat{N}_2$ is the same and the markings of the transformed places p_i^s corresponding to p_i in Eqs. (5) *and* (7) *satisfy*

$$\hat{M}_2(p_i^s) = \hat{M}_1(p_i^s) + \xi. \tag{8}$$

The previous result implies that the structure corresponding to place p_i in the equivalent TMG is periodic wrt $M(p_i)$ and the period ϕ_i is equal to $v(p_i) \cdot x_{out(p_i)}$ (or $w(p_i) \cdot x_{in(p_i)}$).

The size of the equivalent TMG is $O(|\boldsymbol{x}|_1)^2$. Theoretically $|\boldsymbol{x}|_1$ can grow exponentially independently with respect to the net size. However, one finds that in practical examples, this is a quite reasonable number.

We give an example in Fig. 2 and display the equivalent TMG with an initial marking $\boldsymbol{M} = (0, 0, 4)^T$ in Fig. 3. There are totally four transitions and seven places. From Proposition 2, we can compute the period of each place $\phi_1 = 2$, $\phi_2 = 2$ and $\phi_3 = 4$. For the marking $\boldsymbol{M'} = (2\xi_1, 2\xi_2, 4\xi_3)^T$, one can easily check that the structure of equivalent TMG is identical with Fig. 3 while the marking of each equivalent places are $\hat{M}'(p_1^1) = \xi_1$, $\hat{M}'(p_2^1) = \xi_2$ and $\hat{M}'(p_3^1) = \xi_3$.

[2] $|\boldsymbol{x}|_1$ denotes the 1-norm of T-semiflow $\boldsymbol{x}$.

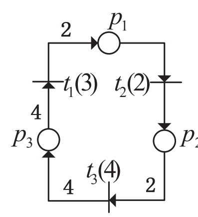

Fig. 2. TWMG of Example 1.

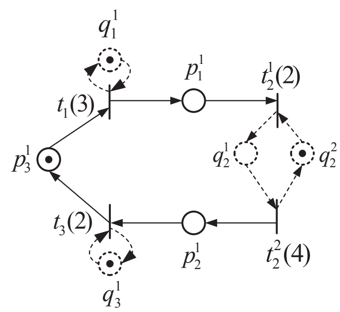

Fig. 3. TMG equivalent to the TWMG in Fig. 2 for initial marking $\boldsymbol{M}_0 = (0, 0, 4)^T$.

III. Problem statement

In this paper, the *cycle time optimization problem* of a TWMG is considered. We aim to find an initial marking $\boldsymbol{M}$ at which the weighted sum of tokens in places is less than or equal to a given value. Among all feasible solutions, we look for those that minimize the average cycle time, i.e., maximize the throughput.

In other words we look for an initial marking $\boldsymbol{M}$ that provides the optimal solution of the following problem:

$$\begin{cases} min\ \chi(\boldsymbol{M}) \\ s.t.\ \boldsymbol{y}^T \cdot \boldsymbol{M} \le f \end{cases} \tag{9}$$

where

• $\chi(\boldsymbol{M})$ is the average cycle time of the TWMG with initial marking $\boldsymbol{M}$.

• $\boldsymbol{y}^T = (y_1, \ldots, y_n)$ is a P-semiflow. In general, we choose the P-semiflow equal to the sum of all minimal P-semiflows.

• f is a given positive real number, representing the maximal available resources that can be used.

We choose $\boldsymbol{y}$ as a P-semiflow since the value of $\boldsymbol{y}^T \cdot \boldsymbol{M}$ for every reachable marking $\boldsymbol{M}' \in R(N, \boldsymbol{M})$ is an invariant. In terms of manufacturing systems, this value corresponds to the cost of the resources that remains constant as the production process proceeds.

Proposition 3: (Teruel *et al.* [1]) *A TWMG is live iff each elementary circuit is live.* ■

Teruel *et al.* [1] proposed a sufficient condition for the liveness of a weighted circuit. They defined a marking $\boldsymbol{M}_D = (v(p_1) - 1, v(p_2) - 1, \ldots, v(p_n) - 1)^T$ and a weighted function $W(\boldsymbol{M}) = \boldsymbol{y}^T \cdot \boldsymbol{M}$ of marking $\boldsymbol{M}$.

Proposition 4: ([1]) *If $W(\boldsymbol{M}) > W(\boldsymbol{M}_D)$, then the weighted circuit is live.* ■

Proposition 5: Problem (9) has a finite solution if $f \geq f^*$, where

$$f^* = min \;\; \boldsymbol{y}^T \cdot \boldsymbol{M}$$

$$s.t. \; \boldsymbol{y}_\gamma^T \cdot \boldsymbol{M} > W(\boldsymbol{M}_D^\gamma), \forall \gamma \in \Gamma \qquad ■$$

Sauer [6] proved that the lower bound of the cycle time is

$$\chi' = max\{x_i \cdot \delta_i, t_i \in T\} \tag{10}$$

where $\boldsymbol{x}$ is the minimal T-semiflow.

IV. Cycle time optimization for TWMG: a mixed ILPP solution

A. General idea

Giua *et. al*. [5] showed that for a TMG the solution[3] of problem (9) can be computed by solving the following ILPP:

$$\begin{cases} max \; \beta \\ s.t. \; \boldsymbol{C} \cdot \boldsymbol{\alpha} - \boldsymbol{Pre} \cdot \boldsymbol{\delta} \cdot \beta + \boldsymbol{M} \geq 0 \\ \qquad \boldsymbol{A} \cdot \boldsymbol{M} \leq \boldsymbol{b} \end{cases} \tag{11}$$

with variables $\boldsymbol{M} \in \mathbb{N}^n$, $\beta \in \mathbb{R}^+$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$. $\boldsymbol{A} \in \mathbb{Z}^{s \times n}$ and $\boldsymbol{b} \in \mathbb{Z}^s$ being known. It provides the optimal solution $\boldsymbol{M}$ and the corresponding maximal throughput β (i.e., the inverse of cycle time 1/$\chi(\boldsymbol{M})$), and $\boldsymbol{\alpha}$ has no physical meaning.

For TWMGs one way to find the optimal solution of the optimization problem (9) is to enumerate all possible equivalent TMGs and adopt ILPP (11) for each of them to find a marking which has the maximal throughput. However, there are two main problems.

- The number of TMG structures equivalent to a TWMG may be very large. This issue is addressed in Section IV-B.
- We have to add in Eq. (11) a series of constraints to make sure the marking $\hat{\boldsymbol{M}}$ that we find for a given net structure $\hat{N}$ is consistent with the marking $\boldsymbol{M}$. We discuss this issue in Section IV-C.

[3]The ILPP in Eq. (11) provides a solution under the assumption of infinite server semantics, but of course it can also be used under single server semantics (the assumption we consider in this paper).

B. Reduction of equivalent TMG structures

According to Proposition 2, for each place $p_i \in P$ of a TWMG system $\langle N, \boldsymbol{M} \rangle$, the structure corresponding to place p_i in the equivalent TMG is periodic with respect to $M(p_i)$ and the period is ϕ_i. Thus, we should compute the equivalent structures for initial marking $M(p_i) = 0, 1, \dots, \phi_i - 1$.

We note that the set of possible markings of place p_i can be partitioned into ϕ_i subsets such that

$$\mathbb{N} = \bigcup_{k_i=0}^{\phi_i-1} \mathcal{M}_{p_i}^{k_i},\ where\ \mathcal{M}_{p_i}^{k_i} \quad = \{k_i + \xi \cdot \phi_i | \xi \in \mathbb{N}\} \tag{12}$$

and all makings of p_i in the same partition $\mathcal{M}_{p_i}^{k_i}$ correspond to the same equivalent structure.

For each place $p_i \in P$, we define $\mathcal{N}_i = \{0, \dots, \phi_i - 1\}$. Then the set of markings of a TWMG can be partitioned into several subsets

$$\mathbb{N}^n = \bigcup_{(k_1,\dots,k_n) \in \mathcal{N}_1 \times \dots \times \mathcal{N}_n} \mathcal{M}_{p_1}^{k_1} \times \mathcal{M}_{p_2}^{k_2} \times \ldots \times \mathcal{M}_{p_n}^{k_n}. \tag{13}$$

For each vector $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathcal{N}_1 \times \dots \times \mathcal{N}_n$ corresponding to partition $\mathcal{M}_{p_1}^{k_1} \times \mathcal{M}_{p_2}^{k_2} \times \ldots \times \mathcal{M}_{p_n}^{k_n}$, the equivalent TMGs for all markings in this partition are the same. The total number of such structures (i.e., partitions) is

$$\Phi = \prod_{p_i \in P} \phi_i. \tag{14}$$

Note that the number of equivalent structures given by Eq. (14) is very large. We look for more efficient solutions that only require to consider a subset of these structures (i.e., partitions). To reach this goal, the following result is useful.

Lemma 1: (Marchetti and Munier [13]) *For a WMG, the initial marking $M(p_i)$ of any place p_i can be replaced by $M^\star(p_i) = \left\lfloor \frac{M(p_i)}{gcd_{p_i}} \right\rfloor \cdot gcd_{p_i}$ tokens without any influence on the precedence constraints induced by p_i.* ■

From Lemma 1, when looking for an optimal solution of Eq. (9), we may restrict our analysis to the markings that belong to a restricted number of partitions where the token content of each place p_i is a multiple of gcd_{p_i}. Hence the number of meaningful subsets in Eq. (12) can be reduced as follows:

$$\begin{gathered} \bigcup_{k_i=0}^{\frac{\phi_i}{gcd_{p_i}}-1} \bar{\mathcal{M}}_{p_i}^{k_i} \subseteq \mathbb{N}, \\ \bar{\mathcal{M}}_{p_i}^{k_i} = \{k_i \cdot gcd_{p_i} + \xi \cdot \phi_i | \xi \in \mathbb{N}\}. \end{gathered} \tag{15}$$

We define $\bar{\mathcal{N}_i} = \{0, \dots, \frac{\phi_i}{gcd_{p_i}} - 1\}$ and the set of markings of a TWMG in Eq. (13) can be redefined as

$$\mathcal{M}_{opt} = \bigcup_{(k_1,\dots,k_n) \in \mathcal{N}_1' \times \dots \times \mathcal{N}_n'} \bar{\mathcal{M}}_{p_1}^{k_1} \times \bar{\mathcal{M}}_{p_2}^{k_2} \times \ldots \times \bar{\mathcal{M}}_{p_n}^{k_n} \subseteq \mathbb{N}^n \tag{16}$$

where the number of partitions is reduced to

$$\bar{\Phi} = \prod_{p_i \in P} \frac{\phi_i}{gcd_{p_i}}. \tag{17}$$

In the following, for the sake of simplicity, we rename the partitions defined in Eq. (16) and write

$$\mathcal{M}_{opt} = \bigcup_{j=1}^{\bar{\Phi}} \mathcal{M}_j. \tag{18}$$

Let us see the example in Fig. 2. We have $gcd_{p_1} = 1$, $gcd_{p_2} = 1$, $gcd_{p_3} = 4$, $\phi_1 = 2$, $\phi_2 = 2$, and $\phi_3 = 4$. The number of partitions is $\Phi = 16$, while the number of meaningful partitions is $\bar{\Phi} = 4$ which is reduced significantly.

C. The mixed ILPP solution for TWMGs

We now show how it is possible to solve the optimization problem (9) assuming that the unknown initial marking $\boldsymbol{M}$ of the TWMG belongs to a generic partition

$$\mathcal{M}_j = \bar{\mathcal{M}}_{p_1}^{k_1} \times \bar{\mathcal{M}}_{p_2}^{k_2} \times \ldots \times \bar{\mathcal{M}}_{p_n}^{k_n}. \tag{19}$$

In this case, due to the special equivalent structure of a marking $\boldsymbol{M} \in \mathcal{M}_j$ in Eq. (15), problem (9) can be rewritten as

$$\begin{cases} min\ \chi(\boldsymbol{M}) \\ s.t.\ \boldsymbol{y}^T \cdot \boldsymbol{M} \leq f, \\ \qquad M(p_i) = k_i \cdot gcd_{p_i} + \xi(p_i) \cdot \phi_i,\ \forall p_i \in P, \\ \qquad \xi(p_i) \in \mathbb{N}, \end{cases} \tag{20}$$

For each place p_i with an initial marking

$$M(p_i) = k_i \cdot gcd_{p_i},\ k_i = 0, \ldots, \frac{\phi_i}{gcd_{p_i}} - 1, \tag{21}$$

we compute

- The equivalent structure of place p_i, i.e., places $p_i^1, \ldots, p_i^{n_i}$.
- The initial markings correspond to Eq. (21), i.e., $\hat{M}(p_i^1) = \mu_{k_i}(p_i^1), \ldots, \hat{M}(p_i^{n_i}) = \mu_{k_i}(p_i^{n_i})$.

Thus for each partition $\mathcal{M}_j$ given in Eq. (19), we can compute the equivalent net structure $\hat{N}_j$, incidence matrix $\hat{\boldsymbol{C}}_{\boldsymbol{j}}$ and pre-incidence $\boldsymbol{P}\hat{\boldsymbol{r}}\boldsymbol{e}_{\boldsymbol{j}}$.

Proposition 6: For each partition $\mathcal{M}_j$ in Eq. (19), we consider the following ILPP

$$\begin{cases} max\ \beta_j \\ s.t.\ \hat{\boldsymbol{C}}_j \cdot \hat{\boldsymbol{\alpha}}_j - \boldsymbol{P}\hat{\boldsymbol{r}}\boldsymbol{e}_j \cdot \hat{\boldsymbol{\delta}}_j \cdot \beta_j + \hat{\boldsymbol{M}}_j \geq 0, & (a) \\ \boldsymbol{y}^T \cdot \boldsymbol{M}_j \leq f, & (b) \\ M_j(p_i) = k_i \cdot gcd_{p_i} + \xi_j(p_i) \cdot \phi_i,\ \forall p_i \in P, & (c) \\ \hat{M}_j(p_i^s) = \mu_{k_i}(p_i^s) + \xi_j(p_i), s = 1, \ldots, n_i, & (d) \\ \hat{M}_j(q_i^a) = 0,\ \forall i = 1, \ldots, m,\ \forall a = 1, \ldots, x_i - 1, & (e) \\ \hat{M}_j(q_i^{x_i}) = 1,\ \forall i = 1, \ldots, m, & (f) \\ \xi_j(p_i) \in \mathbb{N}, & (g) \end{cases} \tag{22}$$

and let ($\beta_j^*, \boldsymbol{M}_j^*, \hat{\boldsymbol{M}}_j^*, \hat{\boldsymbol{\alpha}}_j^*, \boldsymbol{\xi}_j^*$) be the optimal solution. Thus $\boldsymbol{M}_j^*$ is also the optimal solution of problem (9) restricted to partition $\mathcal{M}_j$. ■

It is obvious that if we compute Eq. (22) for all $\bar{\Phi}$ partitions, we can find the maximal throughput $\tilde{\beta} = \max_{j=1,\ldots,\bar{\Phi}} \beta_j^*$ and the corresponding marking $\tilde{\boldsymbol{M}}$. The optimal solution of problem (9) are $\boldsymbol{M} = \tilde{\boldsymbol{M}}$ and $\chi(\boldsymbol{M}) = 1/\tilde{\beta}$.

As we can see Example 1 in Fig. 2, the markings of the TWMG can be partitioned into four subsets: $\mathcal{M}_1 = (2\xi_1(p_1), 2\xi_1(p_2), 4\xi_1(p_3))^T$, $\mathcal{M}_2 = (1+2\xi_2(p_1), 2\xi_2(p_2), 4\xi_2(p_3))^T$, $\mathcal{M}_3 = (2\xi_3(p_1), 1+2\xi_3(p_2), 4\xi_3(p_3))^T$, and $\mathcal{M}_4 = (1+2\xi_4(p_1), 1+2\xi_4(p_2), 4\xi_4(p_3))^T$.

For TMG system $\langle \hat{N}_1, \hat{\boldsymbol{M}}_1 \rangle$ equivalent to TWMG system $\langle N, \boldsymbol{M}_1 \rangle$, we adopt Eq. (22) and obtain the following equation

$$\begin{cases} max\ \beta_1 \\ s.t.\ \hat{\boldsymbol{C}}_1 \cdot \hat{\boldsymbol{\alpha}}_1 - \hat{\boldsymbol{Pre}}_1 \cdot \hat{\boldsymbol{\delta}}_1 \cdot \beta_1 + \hat{\boldsymbol{M}}_1 \geq 0, \\ \boldsymbol{y}^T \cdot \boldsymbol{M}_1 \leq f, \\ \boldsymbol{M}_1 = (2\xi_1(p_1), 2\xi_1(p_2), 4\xi_1(p_3))^T, \\ \hat{M}_1(p_1^1) = \xi_1(p_1),\ \hat{M}_1(p_2^1) = \xi_1(p_2),\ \hat{M}_1(p_3^1) = \xi_1(p_3), \\ \hat{M}_1(q_1^1) = 1,\ \hat{M}_1(q_2^1) = 0,\ \hat{M}_1(q_2^2) = 1,\ \hat{M}_1(q_2^2) = 1. \end{cases} \tag{23}$$

Proposition 7: Any marking $\boldsymbol{M}$ that produces a cycle time $\chi(\boldsymbol{M}) = \chi'$ as defined in Eq. (10) *and satisfies $\boldsymbol{y}^T \cdot \boldsymbol{M} \leq f$ is an optimal solution.* ■

From theoretical point of view, we should compute the markings for all $\bar{\Phi}$ partitions. However, in practical if we find a marking $\boldsymbol{M}$ whose average cycle time converges to the lower bound, there is no need to do more computations. According to Proposition 7, we can conclude that marking $\boldsymbol{M}$ is one of the optimal solutions.

V. NUMERICAL RESULTS

We consider the first example in Fig. 2. Let f=6 be the available resources and all the optimal solutions for each equivalent TMG system $\langle \hat{N}_j, \hat{\boldsymbol{M}}_j \rangle$ ($j = 1, \ldots, 4$) are computed by Eq. (22), i.e.,

$$\begin{cases} \boldsymbol{M}_1 = (2,0,0)^T, \chi(\boldsymbol{M}_1) = 11 \\ \boldsymbol{M}_2 = (3,0,0)^T, \chi(\boldsymbol{M}_2) = 9 \\ \boldsymbol{M}_3 = (2,1,0)^T, \chi(\boldsymbol{M}_3) = 9 \\ \boldsymbol{M}_4 = (1,1,0)^T, \chi(\boldsymbol{M}_4) = 11 \end{cases} \tag{24}$$

Then the optimal solution is $\boldsymbol{M} = (3,0,0)^T$ or $\boldsymbol{M} = (2,1,0)^T$ and the minimal average cycle time of the TWMG system is $\chi(\boldsymbol{M}) = 9$.

Let us consider another example in Fig. 4. The minimal T-semiflows is $\boldsymbol{x} = (1,3,1,1)^T$, while the minimal P-semiflows are $\boldsymbol{y}_1 = (1,0,6,1,0)^T$ and $\boldsymbol{y}_2 = (0,2,2,0,1)^T$. Therefore, we choose the P-semiflow $\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 = (1,2,8,1,1)^T$. We have $\phi_1 = 6$, $\phi_2 = 1$, $\phi_3 = 1$, $\phi_4 = 6$, $\phi_5 = 2$, $gcd_{p_1} = 2$, $gcd_{p_2} = 1$, $gcd_{p_3} = 1$, $gcd_{p_4} = 2$, and $gcd_{p_5} = 2$. The markings of the TWMG are partitioned into $\bar{\Phi} = 9$ subsets.

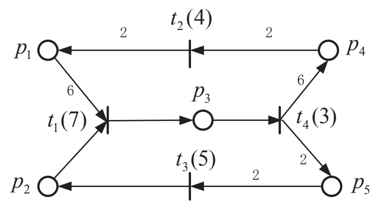

Fig. 4. TWMG of Example 2.

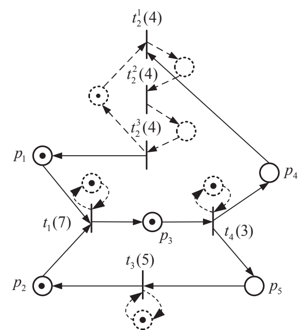

Fig. 5. Equivalent TMG for initial marking $\boldsymbol{M} = (6, 1, 1, 0, 0)^T$.

The equivalent TMG for $\boldsymbol{M} = (6, 1, 1, 0, 0)^T$ is depicted in Fig. 5. We can observe from Table I that the cycle time $\chi(\boldsymbol{M})$ is equal to the lower bound.

TABLE I

THE SOLUTION OF EXAMPLE 1.

$\boldsymbol{M}$	β	$\chi(\boldsymbol{M})$	χ'	$\boldsymbol{y}^T \cdot \boldsymbol{M}$	f
(6, 1, 1, 0, 0)	0.083	12	12	16	20

VI. CONCLUSION

This paper deals with the cycle time optimization problem of deterministic TWMGs. The problem consists in finding an initial marking to minimize the average cycle time while the weighted sum of tokens in places is less than or equal to a given value. We transform a TWMG into several equivalent TMGs and adopt a mixed ILPP solution from the study in [5] to compute a proper initial marking. The conversion of the obtained marking for the equivalent TMG to a marking associated with the TWMG is presented. We show that the proposed method can always find an optimal solution.

Future work will pertain to the cycle time optimization problem of TWMGs under infinite server semantics which is a more general case.

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