

# Finite-Time Consensus on the Median Value by Discontinuous Control

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## Abstract

In this paper we propose a novel protocol to solve the consensus on the median value problem, i.e., the decentralized agreement problem for networked multi-agent systems where the quantity of interest is the median value as opposed to the average value of the agents' states. The median value is a statistical measure particularly robust to the existence of outlier agents which are a significant issue in large scale averaging networks. The proposed protocol achieves consensus on the median value in finite time by exploiting a discontinuous local interaction rule.

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## I. INTRODUCTION

A networked multi-agent system consists of a set of dynamical systems interconnected by a communication network. One of the most studied topics in distributed control of networked multi-agent systems is the consensus problem, i.e., the problem of how to force the states of dynamical agents to converge to (or "agree upon") a common value. When the control objective is to perform agreement on some average quantity related to the initial agents' state (such as position, speed or a generic measurement) the problem is denoted as consensus on the average problem. In [1], linear averaging protocols were proposed to achieve consensus in a network of single integrators. It was proved that if the graph representing the network topology is connected and balanced then consensus on the average is achieved.

In [2], [3], [4], [5], [6] distributed protocols for consensus on the average of the initial states were proposed with the idea of achieving the control objective when the graph of the multi-agent system is represented by a time-varying unbalanced directed graph, i.e., a graph in which the sum of the edge weights of incoming edges is different from that of outgoing edges. In [3] and [5] the proposed methods involve the augmentation of the state space of the agents with an additional variable to preserve the information about the average of the network states.

In [6], [7] the so called *ratio* consensus was proposed. It consists in a local interaction protocol that achieves consensus on the average by exploiting the ratio of two state variables that execute linear state updates with appropriate, distinct, initialization.

In [8] an alternative method to solve the average consensus problem based on the iterative and distributed scaling of a column stochastic matrix was proposed.

The approach presented in this paper is based on a discontinuous local interaction rule. We refer the reader to [9] and [10], [11] for an exhaustive tutorial on how to study discontinuous gradient flows and discontinuous feedback systems by means of non-smooth Lyapunov theory.

The protocol proposed in this paper converges in finite time. Protocols that achieve consensus on the average of the initial states (or on a generic arbitrary value) in finite-time can be found in [12], [13] and [14] for undirected, directed and time-varying network topologies. In [15] and [16] the case of finite time consensus for single integrators and second order systems with unknown non-linear dynamics is investigated by considering continuous local interactions.

Almost all consensus protocols that converge to the average of the initial states suffer from a significant problem: despite the large scale nature of these systems the existence of a single

outlier agent, i.e., an agent whose value of its state holds an abnormal value, may affect the whole network behavior that may ultimately fail to converge to the correct average of the agents with nominal behavior.

This issue has been investigated in [17], [18], [19] and [20] where the main idea is to identify the misbehaving agents in a decentralized way and then recover a correct network state after the outlier agent has been removed from the network. Furthermore, in [21] the fundamental limits of the consensus problem in unreliable networks are investigated while in [22] a strategy to distributively compute an arbitrary function in a network with malicious nodes is presented.

Our opinion is that, despite a clever local interaction protocol design, the average value of a set of variables is not a robust statistical measure if agents with outlier state values influence the network [23].

In this paper we propose a consensus algorithm that converges in finite-time to the median value of a set of initial network states. The median value of a sample data series is a statistical measure robust to outliers in that the existence of abnormal values is filtered out by the possibly large number of samples. Our target application is therefore a network in which some of the sensors/agents are faulty or misbehaving and thus generate abnormal values that corrupt significantly the average of the network state.

The main contributions of this work are:

- 1) A novel consensus protocol that achieves consensus on the median value of a set of initial states.
- 2) A characterization of the finite-time convergence properties of the protocol in the case of a network represented by an undirected connected graph.
- 3) Simulations to corroborate the theoretical analysis.

This paper is structured as follows. In Section II some background and the problem statement are introduced. In Section III the consensus on the median value is presented and its convergence properties are characterized. In Section IV we present some simulations to corroborate the theoretical results. Finally, in Section V concluding remarks are given.

## II. BACKGROUND AND PROBLEM STATEMENT

In a finite set of samples with values in ascending order  $S = \{z_1 \leq z_2, \dots \leq z_n\}$  the sample median is the sample value such that half of the samples is bigger and half of the samples is

smaller than such value. If the number of samples  $n$  is odd, then the  $k$ -th value with  $k = \frac{n+1}{2}$  is the sample median. If the number of samples is even we can arbitrarily choose as sample median a number between the  $k$ -th and  $k + 1$ -th value with  $k = \frac{n}{2}$ .

Let  $n \geq 2$  agents be labeled in ascending order according to their initial state value, i.e.,  $x_1(0) = z_1 \leq x_2(0) = z_2, \dots \leq x_n(0) = z_n$ . Our objective is to design a decentralized consensus protocol such the state of each agent converges in finite time toward the sample median of the initial values of the network. In particular, a protocol such that

$$\exists T : \forall t > T, \quad x_i(t) = m \quad \forall i \in V, \quad (1)$$

with

$$\begin{cases} m \in [x_k(0), x_{k+1}(0)], & k = \frac{n}{2}, \quad \text{for } n \text{ even;} \\ m = x_k(0), & k = \frac{n+1}{2}, \quad \text{for } n \text{ odd;} \end{cases}$$

We now introduce the adopted notation. Let  $G = (V, E)$  be an undirected graph where  $V = \{1, \dots, n\}$  is the set of agents and  $E \subseteq \{V \times V\}$  is the set of edges representing information flow between the agents. Let  $(i, j) \in E$  be an edge representing interaction between agent  $i$  and  $j$ . Let  $N_i = \{j \in V : (i, j) \in E\}$  be the set of neighbors of agent  $i$ , i.e., the set of edges that can send state information to agent  $i$ . A path in graph  $G$  is a sequence of consecutive edges connecting two agents. A graph is said to be *connected* if there exists a path between any pair of agents. The topology of graph  $G$  is encoded by the Laplacian matrix  $\mathcal{L}$ , an  $n \times n$  matrix the elements of which are defined as follows

$$l_{ij} = \begin{cases} |N_i|, & \text{if } i = j, \\ -1, & \text{if } (i, j) \in E \text{ and } i \neq j, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Where  $|N_i|$  denotes the cardinality of set  $N_i$ . The Laplacian matrix is a positive-semidefinite matrix with a single null eigenvalue if graph  $G$  is connected. Let  $\mathbf{1}$  and  $\mathbf{0}$  be respectively the  $n$  elements vectors of ones and zeros, then  $\mathcal{L}\mathbf{1} = \mathbf{0}$  by construction.

The dynamics of each agent are considered to be single integrators  $\dot{x}_i(t) = u_i(t)$  where  $x_i \in \mathbb{R}$  is the state of the agent representing its current estimation of the median value of the initial states  $x_i(0) = z_i$  and  $u_i \in \mathbb{R}$  is the local control input that will be specified by the consensus protocol. The proposed protocol is discontinuous and we make use of the following definition of the sign function

$$\text{sign}(y) = \begin{cases} 1, & \text{if } y > 0; \\ 0, & \text{if } y = 0; \ y \in \mathbb{R}. \\ -1, & \text{if } y < 0; \end{cases} \quad (3)$$

**Definition 2.1:** A network state  $x(t)$  is said to be at consensus if

$$|x_i(t) - x_j(t)| = 0, \quad (4)$$

with  $x_i(t) = \sup_{h \in V} x_h(t)$ ,  $x_j(t) = \inf_{h \in V} x_h(t)$ . ■

Since the proposed consensus protocol is based on a discontinuous local interaction rule, we need to recall some preliminaries on non-smooth Lyapunov theory. First, for a differential equation with discontinuous right-hand side, following [24], we understand the corresponding solution in the so-called *Filippov sense* as the solution of an appropriate differential inclusion, the existence of which is guaranteed (owing on certain properties of the associated set-valued map) and for which noticeable properties, such as absolute continuity, are in force. The reader is referred to [10] for a comprehensive tutorial of the notions of solution for discontinuous dynamical systems.

Next we define the *Clarke's Generalized Gradient* [25] of a discontinuous function that can be also found in [9], [10].

**Definition 2.2 (Clarke's Generalized Gradient):** Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be locally Lipschitz continuous and define

$$\partial V(x) \triangleq \text{co} \left\{ \lim_{i \rightarrow \infty} \nabla V(x_i) \mid x_i \rightarrow x, x_i \notin \Omega_V \cup N \right\},$$

where  $\Omega_V$  is the set of Lebesgue measure zero where  $\nabla V(x)$  does not exist and  $N$  is an arbitrary set of measure zero. ■

In Definition 2.2, *co* denotes the convex hull of a set. Set  $\Omega_V$  is a set of measure zero which contains the points in which  $V(x)$  is not differentiable. This set can be arbitrarily chosen to simplify the computation. The Clarke's generalized gradient of  $V(x)$  at  $x$  consists of all convex combinations of all of the possible limits of the gradient at neighboring points where  $V$  is differentiable [10]. The Clarke's generalized gradient corresponds to the standard gradient where the derivative of the scalar function exists. Next we recall the extended Lyapunov Theorem for discontinuous Lyapunov functions [9].

**Theorem 2.3 (Lyapunov's Theorem Generalized):** If  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $V(0) = 0$  and  $V(x) > 0 \forall x \neq 0$ , and  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $V(x(t))$  is absolutely continuous on  $[t_0, \infty]$  with  $\frac{d}{dt} [V(x(t))] < \epsilon < 0$  a.e. on  $\{t | x(t) \neq 0\}$  then  $x$  converges to 0 in finite time. ■

Many results on non-smooth Lyapunov theory involve the so called "Max functions" [9] that we define in the following

**Definition 2.4 (Max function):**  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a max function if  $V(x) = \max_{j \in Y} f_j(x)$ , where  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$  are  $C^1$  and  $Y$  is a finite index set. ■

Finally, we recall from [9] a result that simplifies the verification of conditions of Theorem 2.3.

**Proposition 2.5 (Chain rule):** Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be a max function and  $x : \mathbb{R} \rightarrow \mathbb{R}^n$  be differentiable at  $t$ . If  $\frac{d}{dt} [V(x(t))]$  exists, then

$$\frac{d}{dt} [V(x(t))] = \xi^T \dot{x}, \quad \forall \xi \in \partial V(x).$$

■

In our case, we deal with absolute value functions which are a particular case of Max functions and for which Proposition 2.5 can be exploited. This can be seen by noticing that  $|y| = \max_{y \in \mathbb{R}} \{y, -y\}$ .

### III. CONSENSUS ON THE MEDIAN VALUE

In this section we present and characterize the consensus on the median value protocol. Let the initial states of the agents be denoted as  $x_i(0) = z_i$  for  $i = 1, \dots, n$  and, without loss of generality, be ordered such that  $z_i \geq z_j$  if  $i > j$ . Each agent interacts with its neighbors according to the following protocol

$$\dot{x}_i(t) = -\alpha \text{sign}(x_i(t) - z_i) - \lambda \sum_{j \in N_i} \text{sign}(x_i(t) - x_j(t)), \quad (5)$$

where  $\alpha \in \mathbb{R}^+$  and  $\lambda \in \mathbb{R}^+$  are positive constants which are tuning parameters that govern the convergence properties of protocol (5).

**Remark 3.1:** In this paper we consider agents' labels in ascending order according to the corresponding value of their initial state  $x_1(0) = z_1 \leq x_2(0) = z_2 \leq \dots \leq x_n(0) = z_n$ . We stress the fact that agents ignore their respective labels and that this ordering is adopted only to simplify the notation in the algorithm convergence analysis.

Next we discuss how protocol (5) ensures that the network converges to a common value in finite-time.

Let

$$\delta_{ij}(t) = x_i(t) - x_j(t), \quad \text{with } (i, j) \in E, \quad (6)$$

be a set of error variables for each edge in the network. The dynamics of  $\delta_{ij}(t)$  are easily obtained by differentiating (6), and considering the closed loop dynamics of each agent

$$\dot{x}_i = -\alpha \text{sign}(x_i(t) - z_i) - \lambda \sum_{k \in V, k \neq i} r_{ik} \cdot \text{sign}(\delta_{ik}), \quad (7)$$

where  $r_{ik} = 1$  if  $(i, k) \in E$ ,  $r_{ik} = 0$  otherwise. Trivial manipulations yield

$$\begin{aligned} \dot{\delta}_{ij} = & -\alpha \text{sign}(x_i(t) - z_i) + \alpha \text{sign}(x_j(t) - z_j) \\ & - \lambda \left[ \sum_{k \in V, k \neq i} r_{ik} \cdot \text{sign}(\delta_{ik}) - \sum_{k \in V, k \neq j} r_{jk} \cdot \text{sign}(\delta_{jk}) \right]. \end{aligned} \quad (8)$$

We are now ready to prove a preliminary result that ensures that the protocol in eq. (5) achieves consensus in finite-time with respect to a value common to every agent, not necessarily the median value. Some technical insights of the next Theorem appeared in [26] where the next result was required to characterize the disturbance rejection properties of a discontinuous consensus protocol.

**Theorem 3.2:** *Consider the local interaction protocol (5) with tuning parameter selected according to*

$$\lambda \geq n\alpha + \mu^2, \quad \mu \neq 0, \quad (9)$$

*Then, if graph  $G$  is connected, there exists a  $t_r$  such that the consensus condition (4) is achieved  $\forall t \geq t_r$ , where the transient time  $t_r$  is upper bounded as follows*

$$t_r \leq \frac{1}{\mu^2} \cdot \max_{i, j \in V \times V} |x_i(0) - x_j(0)|. \quad (10)$$

Next we show that the agents' interaction protocol (5) makes the network converge to a value corresponding to the median value of the initial agents' states.

**Theorem 3.3:** *Let a network  $G$  of  $n$  agents interact according to protocol (5) with parameter  $\lambda = n\alpha + \mu^2$  and  $\mu \neq 0$ . Consider a vector of initial states or measurements  $z = [z_1, \dots, z_n]$  with  $x_1(0) = z_1 \leq \dots \leq x_n(0) = z_n$ . If graph  $G$  is connected then*

$$\exists T : \forall t > T, \quad x_i(t) = m \quad \forall i \in V, \quad (11)$$

with

$$m \in \begin{cases} [z_k, z_{k+1}(0)], & k = \frac{n}{2}, \quad \text{for } n \text{ even,} \\ z_k, & k = \frac{n+1}{2}, \quad \text{for } n \text{ odd,} \end{cases}$$

where  $m$  denotes the median value of vector  $z \in \mathbb{R}^n$ . Furthermore  $T \leq t_r + t_{r2}$ , with

$$t_r \leq \left( \frac{\max_{i,j \in V \times V} |x_i(0) - x_j(0)|}{\mu^2} \right),$$

and

$$t_{r2} \leq n \frac{\frac{\mathbf{1}^T x(0)}{n} - m}{\alpha}.$$

*Proof:*

The main ideas that guide this proof are the following:

- 1) Show that there exists a finite time  $t_r$  in which the state of all agents converge to the same value and keep such agreement for all  $t \geq t_r$ , despite the agreement value is time-varying.
- 2) Show that there exists a finite time  $t_{r2}$  in which the average of the states of the agents corresponds to the median value of  $x_i(0) = z_i$  for  $i = 1, \dots, n$  and that this value is not time-varying.
- 3) Conclude that the state value of each agent necessarily correspond after a time  $T \leq t_r + t_{r2}$  to an approximation of the median value of the initial states.

By Theorem 3.2, in a finite time less than or equal to  $t_r$  the network reaches the consensus state (4). Let us consider the average of agents states as

$$c(t) = \frac{\mathbf{1}^T x(t)}{n}. \quad (12)$$

We now study the dynamics of  $c(t)$  for  $t \geq t_r$ , i.e., after the consensus condition has been achieved. It holds

$$\dot{c}(t) = \frac{\mathbf{1}^T \dot{x}}{n} = -\alpha \frac{\sum_{i \in V} \text{sign}(x_i(t) - z_i)}{n} - \lambda \frac{\sum_{i \in V} (\sum_{j \in N_i} \text{sign}(x_i(t) - x_j(t)))}{n}, \quad (13)$$

Since graph  $G$  is undirected, it holds



$$\sum_{i \in V} \left( \sum_{j \in N_i} \text{sign}(x_i(t) - x_j(t)) \right) = 0.$$

Therefore,

$$\dot{c}(t) = \frac{\mathbf{1}^T \dot{x}}{n} = -\alpha \frac{\sum_{i \in V} \text{sign}(x_i(t) - z_i)}{n}, \quad (14)$$

We recall that the agents are labeled in ascending order with respect to their constant  $z_i$ , i.e., if  $z_i > z_j$  then  $i > j$ . Let  $m$  be the median value of vector  $x(0) = z$ , i.e.,

$$\begin{cases} m \in [z_k, z_{k+1}], & k = \frac{n}{2}, \quad \text{for } n \text{ even,} \\ m = z_k, & k = \frac{n+1}{2}, \quad \text{for } n \text{ odd.} \end{cases}$$

Let  $N_{up}(t)$  be the number of agents such that  $x_i(t) > m$  and  $x_i(t) \neq z_i$  and  $N_{down}(t)$  the number of agents such that  $x_i(t) < m$  and  $x_i(t) \neq z_i$ . Then, by construction

$$\dot{c}(t) = -\frac{\alpha}{n} (N_{up}(t) - N_{down}(t)). \quad (15)$$

Therefore,

$$\begin{cases} \dot{c}(t) \geq \frac{\alpha}{n} & \text{if } N_{up}(t) < N_{down}(t); \\ \dot{c}(t) \leq -\frac{\alpha}{n} & \text{if } N_{up}(t) > N_{down}(t); \\ \dot{c}(t) = 0 & \text{if } N_{up}(t) = N_{down}(t). \end{cases} \quad (16)$$

Let us consider the following function for  $t \geq t_r$ , i.e., after the consensus condition has been achieved

$$V(t) = |c(t) - m|. \quad (17)$$

The corresponding generalized time derivative is

$$\frac{d}{dt} [V(t)] = \text{SIGN}(c(t) - m) \dot{c}.$$

The network is by hypothesis at the approximate consensus condition, thus if  $c(t) \neq m$  then either  $N_{up}(t) < N_{down}(t)$  and  $\dot{c}(t) > 0$  or  $N_{up}(t) > N_{down}(t)$  and  $\dot{c}(t) < 0$  or  $N_{up}(t) = N_{down}(t)$  and  $\dot{c}(t) = 0$ . It follows by simple manipulations that for  $c(t) \neq m$

$$\frac{d}{dt} [V(t)] \leq -\frac{\alpha}{n} \text{SIGN}(c(t) - m) \text{sign}(c(t) - m) \leq -\frac{\alpha}{n}.$$

Therefore, according to standard non-smooth Lyapunov analysis the network average converges to  $c(t) = m$ , i.e., the median value, within a finite transient time corresponding to  $t_{r2} \leq n \frac{c(0) - m}{\alpha}$ .

□

#### IV. SIMULATIONS

In the first simulation we consider a network of 8 agents interacting by a randomly connected graph whose topology is not shown due to lack of space. We consider an initial network state such that  $[x_1(0) = z_1, \dots, x_n(0) = z_n] = [0.1, 0.2, 0.4, 0.8, 1.6, 3.2, 6.4, 12.8]$ . The initial states' average is 3.1875 while the median value, since  $n = 8$  is even, is considered to be any value between 0.8 and 1.6 which correspond to the value of either agent 4 or 5. For this simulations we choose parameters  $\alpha = 1$  and  $\mu = 1$ , thus  $\lambda = 2$ . In Figure 1 it is shown the time evolution of the network states. It is evident how after a finite transient time (of duration approximately 2 seconds, lower than the analytical upper bound of  $t_r = 12.7$ ) consensus on a common value has been achieved. At  $t = 7$  the network state converges to the value of 1.6 which corresponds to the median value of the initial states. We note that since  $n$  is even, the median value is not unique and therefore a different choice of initial conditions may lead to any legitimate median value. In Figure 2 it is shown the evolution of function  $V(t)$  defined in eq. (??) which converges to zero in finite-time.

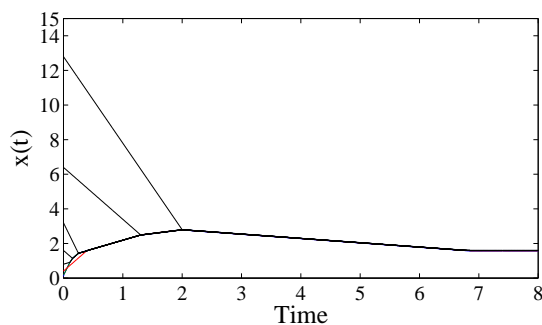


Fig. 1. Time evolution of the network state  $x(t)$ .

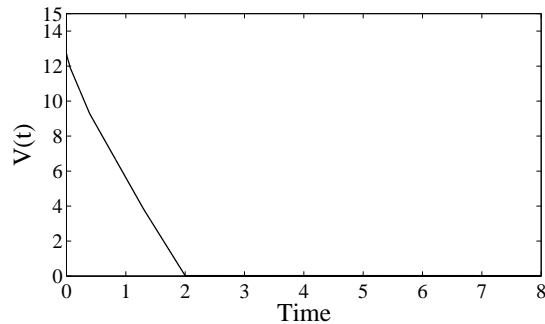


Fig. 2. Time evolution of function  $V(t)$ .

In Figure 3 it is shown the network evolution of the second simulation example that considers the case in which two agents are outliers. In the second simulation we consider the case of a network with 19 nodes, which is an odd number. We consider a random connected network topology which is not shown due to lack of space. Initial states are taken uniformly at random with  $x_i(0) = z_i \in [0, 10]$  for any  $i \in V$ . Also for this simulation we choose parameters  $\alpha = 1$ ,  $\mu = 1$  and thus  $\lambda = 2$ . Furthermore, we consider agent 1 and 2 to be outliers and artificially set their initial state values to 100 and 94 respectively. The average of the network with outliers is 14.9 while the average without considering outliers is 5.74. This sensitivity to outlier values is particularly critical in large-scale systems. On the other hand, the median value of the network with outliers is 6.8 while without considering outliers it is 5.13. Clearly, as the number of agents increases the median value becomes less and less sensitive to outlier values thus allowing a robust implementation of consensus algorithms. We point out that for many random variables with symmetric distribution, such as uniform or Gaussian random variables, the average value corresponds numerically to the median value.

Finally, in Figure 4 it is shown the evolution of the difference between the average value of the network and the median value for the second simulation. It is clear that as predicted by the analytical analysis as soon as consensus between the agents is achieved in finite time the difference between the average value and the median value of the network state converges monotonically to zero in finite time.

## V. CONCLUSIONS AND FUTURE WORK

In this paper we proposed a novel consensus protocol which achieves agreement with respect to the median value of the initial states in finite time. The proposed protocol achieves distributed agreement toward an inherently robust statistical measure with respect to outlier states due to faults or measurement errors. Future work will involve the characterization of a discrete time version of the proposed algorithm and characterization of the robustness of the protocol with respect to noisy relative state measurements.

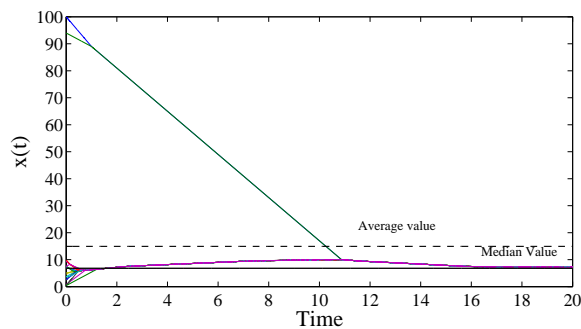


Fig. 3. Time evolution of the network state  $x(t)$  with 2 outlier agents.

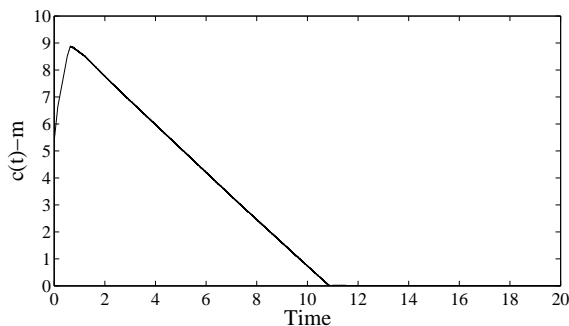


Fig. 4. Time evolution of the difference between the average value  $c(t)$  of the network state  $x(t)$  with 2 outlier agents and the median value.

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