IPA for Continuous Petri Nets

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Abstract

Recently there has been a considerable interest in the application of Infinitesimal Perturbation Analysis (IPA) to continuous queues, where its sample derivatives (gradients) were shown to be unbiased for a large class of systems. This paper extends the investigation to a class of hybrid Petri nets, where the special algebraic structure of continuous transitions yields simple algorithms for the IPA derivatives. We derive such algorithms for the performance functions of throughput and average workload, and show them to be model-free and easily computable from the sample paths.

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I. Introduction

Continuous queueing networks have been investigated in the past few years as a suitable setting for the Infinitesimal Perturbation Analysis (IPA) sensitivity-analysis technique. Their advantage over the traditional, discrete queueing models is due to the fact that they yield statistically-unbiased IPA derivatives in a far-larger class of systems and performance functions. Furthermore, the formulas for the IPA derivatives often are model-free and easily computable from the sample paths (see, e.g., Cassandras (2006) and references therein). This makes it possible to compute them not only from a simulation output but also from observation of an actual system, thereby suggesting their possible use in real-time parameter control and performance optimization. Examples in various application areas including telecommunications, manufacturing, traffic networks, and supply chains, have been presented in Miyoshi (2008); Panayiotou et al. (2004); Yu and Cassandras (2005); Panayiotou and Cassandras (2006); Xie (2002); Chen et al. (2004); Zhao and Melamed (2007). More recently, a systematic approach to queueing networks has been developed in Wardi et al. (2010); Cassandras et al. (2009), which yields iterative distributed algorithms for the IPA derivatives. Our purpose is to extend these results to a class of event graphs, a kind of Petri nets having a wide scope in manufacturing applications.

The network-models mentioned above lack the notion of concurrency and synchronization. These are key concepts in Petri nets, where they are represented by the element of the transition. The inclusion of fluid transitions alters the structure of the IPA algorithms significantly, and as this paper will show, yields a different suite of algorithms than those published in the past. Thus, the focal point of this paper is an investigation of the fluid transition from the standpoint of sensitivity analysis via IPA. The main results concern simple, iterative algorithms that lend themselves to distributed, synchronous implementation in a natural way. Two kinds of performance measures will be considered: throughput-related measures and workload-related measures, and they will be viewed as functions of structural and distributional parameters such as maximum firing rate or the duration of “down” periods at a network’s transitions.

Reference Xie (2002) addressed the application of IPA to a general class of continuous event graphs, and derived algorithms for the IPA derivatives. This paper is different from it in the following three ways: (1). Xie (2002) assumes piecewise-constant maximum firing speeds while this paper allows for general functions. (2). Xie (2002) assumes no direct impact of the continuous-time dynamics on the discrete-event dynamics, while we make no such assumption. (3). The variational parameter in Xie (2002) is either the max firing speeds or the initial marking, while we allow for a general parameter, including a variable of the probability law underlying the max firing times. As a result, the system discussed in Xie (2002) has evolution equations in whose terms the IPA derivatives are derived, while this paper does not have such equations and its IPA derivatives are expressed by other means.

Hybrid and continuous Petri nets have been extensively analyzed in recent years (see, e.g., Alla and David (1998); Balduzzi et al. (2000); Silva and Recalde (2004)). They constitute an abstraction of the traditional (discrete) timed-Petri net model, where token-firing by a transition is described by a flow-rate process. This can be viewed as tokens consisting of fluid "molecules", and their firing by a transition amounts to their continuous elimination from the input places of the transition at a certain given rate, and concurrent regeneration, at the same rate, at the transition's output places. Furthermore, it is possible to dispense with the token-concept entirely, and characterize the network's dynamics by fluid flow rates on its arcs.

Continuous Petri nets often are used to evaluate a network's performance measure such as throughput or average workload. When the network is stochastic, the performance of interest typically is an expected-value quantity. In the setting of perturbation analysis, the performance measure is a function of a variable $\theta \in R^n$ such as the maximum rate of a transition or a control parameter regulating fluid flow rates into the network. In this case the performance metric becomes a performance function of θ. Lacking closed-form expression, such a performance function may have to be estimated by a sample-path technique like simulation. The role of IPA is to compute the gradient of the sample performance function with respect to θ, which possibly can be used in stochastic optimization of the performance function. To be practical, the IPA gradients have to be statistically unbiased and easily computable from the sample path. This paper shows that indeed they have these two properties for the systems and performance functions under study.

II. Modeling and Problem Formulation

A continuous Petri net is a directed graph having two kinds of nodes, transitions and places, whose arcs can connect a place to a transition or a transition to a place (see, e.g., Xie (2002) and references therein). Fluid flows on the network's arcs are characterized by flow rates, which we assume to be piecewise continuous functions of time t. The places can store fluid and are assumed to have unlimited storage capacities. The *firing rate* of a transition is defined as the fluid flow rate through it, and generally it is a function of time. The *maximum firing rate* of a transition is a given (possibly time-dependent) upper bound on its firing rate. The transitions will be denoted by the upper-case letters $T,\ U$, and W, and the places will be denoted by the lower-case letters p and q. We assume that the network's dynamics evolve throughout an interval $[0, \mathcal{T}]$ for a given $\mathcal{T} > 0$. We next describe the dynamics of flow in such a Petri net.

Consider a transition T having k input places, denoted by $p_1, \ldots, p_k$, and m output places, denoted by $q_1, \ldots, q_m$. Denote the firing rate of the transition at time t by $v_T(t)$. This is the rate at which fluid flows from each one of its input places towards the transition, as well as the outflow rate from the transition towards each one of its output places. The transition does not store any fluid, and we note that its aggregate inflow rate from its input places generally is not equal to its aggregate outflow rate to its output places unless $k = m$. The maximum firing rate of the transition (as a function of time) is denoted by $V_T(t)$, and it is given as a part of the transition's characterization.

In this paper we consider *event graphs* (Murata, 1989) namely Petri nets each of whose places has a single input transition and a single output transition. A place p can store fluid, and the amount of fluid stored in it at time t, called the *workload*, is denoted by $m_p(t)$. For a transition T and a place p we further define the following notation: $in(T)$ - set of input places to T; $in(p)$ - the input transition to p; $pre(T)$ - the set of transitions immediately upstream from T, i.e., $pre(T) = \{U = in(p) : p \in in(T)\}$; $out(T)$ - set of output places of T; $out(p)$ - the output transition of p; and $post(T)$ - the set of transitions immediately downstream from T. For example, Figure 1 depicts a path in a network where $p \in in(T)$ and $q \in out(T)$, while $U = in(p) \in pre(T)$ and $W = out(q) \in post(T)$. The figure does not include all of the input places and output places of the various transitions. A transition is called a *source* if $in(T) = \emptyset$, and a *sink* if $out(T) = \emptyset$. We define an empty period of the place p to be a maximal time-interval during which $m_p(t) = 0$, and contiguous complements of empty periods are called nonempty periods. For a given time t, we denote by $\varepsilon_T(t)$ the set of input places of transition T that are empty at time t, namely, $\varepsilon_T(t) \ := \ \{p \in in(T) \ : \ m_p(t) = 0\}$.

The maximum firing rates at the transitions, $V_T(t)$, are assumed to be given and to have the

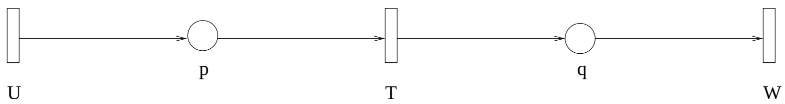

Fig. 1. Typical network path

following properties:

***Assumption** 2.1:* For every transition T, the function $V_T(\cdot)$ is piecewise continuous and piecewise continuously differentiable on the interval $[0, \mathcal{T}]$.

The following two equations, satisfied for every transition T and place p, define the flow-dynamics in the network:

$$v_T(t) = \begin{cases} V_T(t), & \text{if } \varepsilon_T(t) = \emptyset \\ \min_{p \in \varepsilon_T(t)} \{ v_U(t) \mid U = in(p) \} \cup \{ V_T(t) \}, & \text{if } \varepsilon_T(t) \neq \emptyset, \end{cases} \tag{1}$$

and

$$\dot{m}_p(t) = v_{in(p)}(t) - v_{out(p)}(t), \tag{2}$$

where "dot" denotes derivative with respect to time. Note that these equations do not define the transition rates $v_T(t)$ uniquely even if the initial condition $m_p(0)$ is specified. For example, consider a cyclic network consisting of a single place p and a single transition T, with $V(t) := V$ for a constant $V > 0$. If $m_p(0) = 0$ then (31) and (2) are satisfied whenever $v_T(t) < V$ $\forall t \in [0, \mathcal{T}]$. However, if $m_p(0) > 0$ then these equations have the unique solution $v_T(t) = V$ and $m_p(t) = m_p(0)$, $\forall t \in [0, \mathcal{T}]$. More generally, define an elementary circuit as a closed path $< t_1, p_1, t_2, p_2, \ldots, t_n, p_n, t_1 >$ such that for every $i \in \{1, \ldots n\}$, and for every $j \in \{1, \ldots, n\} \setminus \{i\}$, $t_i \neq t_j$ and $p_i \neq p_j$ (see Xie (2002)). The following result ascertains that Eqs. (31) and (2) jointly have a unique solutions:

***Proposition** 2.2:* Suppose that Assumption 2.1 is in force. 1). The sum of the workloads in all places of every elementary circuit in an event graph is a constant function of time t. 2). Suppose that at time $t = 0$ every elementary circuit of an event graph contains at least one place p such that $m_p(0) > 0$. Then Eqs. (31), and (2) together with the initial conditions $m_p(0)$, have a unique solution for every transition T, place p, $t \in [0, \mathcal{T}]$.

Proof. See Xie (2002). □

The role of IPA. IPA can be applied to stochastic hybrid systems by computing the gradients (derivatives) of sample performance functions defined on them, with respect to structural or distributional parameters (see Cassandras and Lafortune (1999)). For the class of Petri nets considered in this paper, some of the maximum transition firing rates are assumed to be random functions of time t and a variable parameter $\theta \in R^n$, and hence are denoted by $V_T(\theta, t)$. Consequently the transitions rates and places' workloads also are random functions of t and θ, and hence are denoted by $v_T(\theta, t)$ and $m_p(\theta, t)$, respectively. The performance functions of interest to us are the cumulative firing rate at a particular transition and the cumulative workload at a particular place, over a given time-interval $[0, \mathcal{T}]$. Respectively denoted by $J_T(\theta)$ and $J_p(\theta)$, these performance functions are defined as follows,

$$J_T(\theta) \ := \ \int_0^{\mathcal{T}} v_T(\theta, t)dt, \tag{3}$$

and

$$J_p(\theta) \ := \ \int_0^{\mathcal{T}} m_p(\theta, t)dt, \tag{4}$$

and we note that they are related to the average firing rate $\frac{1}{\mathcal{T}}J_T(\theta)$ and average workload $\frac{1}{\mathcal{T}}J_p(\theta)$, respectively. We point out that $J_T(\theta)$ and $J_p(\theta)$ are random functions whose realizations depend on a drawn sample path. Their respective expected values, denoted by $\ell_T(\theta)$ and $\ell_p(\theta)$, are closely related to performance functions of interest in manufacturing applications. We comment that all of these functions depend on the final time $\mathcal{T}$, but we suppress this notational dependence since we assume that $\mathcal{T}$ is given and fixed. These functions also depend on the initial conditions for Eq. (2), $m_p(0)$ for every place p, which we assume to be given.

The purpose of IPA is to compute the sample gradients $\nabla J_T(\theta)$ and $\nabla J_p(\theta)$, which then can be used in sensitivity analysis and stochastic optimization of performance functions involving $\ell_T(\theta)$ and $\ell_p(\theta)$. Section 3 derives a simple algorithm for computing the sample gradients (called the IPA gradients, or IPA derivatives), and Section 4 demonstrates their use in optimization.

III. ALGORITHMS FOR THE IPA DERIVATIVES

Throughout this section we assume, without loss of generality, that the controlled parameter θ is one-dimensional, and that it is a variable of the maximum firing rates at one or more transitions. Let us denote by Σ the set of transitions where θ is a parameter of their maximum firing rate, and we denote by S a typical transition in Σ. Thus, we denote the maximum transition rate-process of $S \in \Sigma$ by $V_S(\theta, t)$, and for $T \notin \Sigma$, we use the notation $V_T(t)$ established earlier. We further assume that $\theta \in \Theta$ for a closed, bounded interval $\Theta \subset R$. In this section we develop an iterative algorithm for computing the IPA derivatives $\frac{dJ_T}{d\theta}(\theta)$ and $\frac{dJ_p}{d\theta}(\theta)$ for every transition T and place p, and at a fixed $\theta \in \Theta$.

We assume that for every $S \in \Sigma$, $V_S(\theta, t)$ depends θ in the following way, extending Assumption 2.1.

Assumption *3.1:* For every $S \in \Sigma$, w.p.1, there exists a finite sequence of real-valued functions of θ, $\{z_i(\theta)\}_{i=1}^{K_S}$, such that (i) $z_i(\theta)$ is C^1 in θ, $i = 1, \ldots, K_S$; (ii) for every $\theta \in \Theta$, $0 \leq z_1(\theta) \leq \ldots \leq z_{K_S}(\theta) \leq \mathcal{T}$; (iii) for every $\theta \in \Theta$ and $t \in \big(0, \mathcal{T}\big) \setminus \{z_1(\theta), \ldots, z_{K_S}(\theta)\}$, there exists an open neighborhood of (θ, t) where the function $V_S(\cdot, \cdot)$ is C^1 and bounded.

In other words, the function $V_S(\theta, t)$ is C^1 in θ, piecewise C^1 in t, and the time-points where it is not C^1 (where it may be discontinuous) are C^1 functions of θ.

The next assumption is stated in terms of the sample-path functions, and can be verified fairly easily for a given particular problem.

Assumption *3.2:* For every $\theta \in \Theta$, w.p.1, the following statements hold. 1). For every transition T the functions $v_T(\theta, t)$ is C^1 in θ, piecewise C^1 in t, and the time points where it is not C^1 are C^1 functions of θ. 2). (For every place p, the function $m_p(\theta, t)$ is continuous in (θ, t), piecewise C^1 in θ for a given t, and piecewise C^1 in t for a given θ. 3). No empty period consists of a single point.

Fix $\theta \in \Theta$, and consider first the IPA derivatives $\frac{dJ_T}{d\theta}$ for a given transition T; $\frac{dJ_p}{d\theta}$ will be discussed later. Let $z_{i,T}(\theta)$, $i = 1, \ldots, M_T$, denote the jump-times of $v_T(\theta, \cdot)$ in increasing order. By Eq. (3),

$$\begin{aligned} \frac{dJ_T}{d\theta}(\theta) &= \int_0^{\mathcal{T}} \frac{\partial v_T}{\partial \theta}(\theta, t) dt \\ &+ \sum_{i=1}^{M_T} \big(v_T(\theta, z_{i,T}(\theta)^-) - v_T(\theta, z_{i,T}(\theta)^+) \big) \frac{dz_{i,T}}{d\theta}(\theta). \end{aligned} \tag{5}$$

Let us consider the two main additive terms in the Right-Hand Side (RHS) of (5) separately. Regarding the first term, we have the following immediate result.

Proposition *3.3:* Fix a transition T, and consider $\theta \in \Theta$ and $t \in [0, \mathcal{T}]$.

- Case 1: $\varepsilon_T(\theta,t)=\emptyset$. Then,

$$\frac{\partial v_T}{\partial\theta}(\theta,t) \;=\; \begin{cases} \frac{\partial V_T}{\partial\theta}(\theta,t), & \text{if } T\in\Sigma \\ 0, & \text{if } T\notin\Sigma. \end{cases} \tag{6}$$

- Case 2: $\varepsilon_T(\theta,t)\neq\emptyset$, and t lies in the interior of an empty period at every place $p\in in(T)$. Then for every $p\in\varepsilon_T(\theta,t)$, with $U=in(p)$,

$$\frac{\partial v_T}{\partial\theta}(\theta,t) \;=\; \frac{\partial v_U}{\partial\theta}(\theta,t). \tag{7}$$

Proof. In case 1 either T is a source or all of the input places of T are nonempty, and hence, $v_T(\theta,t)=V_T(\theta,t)$; this implies (6) by the definition of Σ. In case 2 $v_T(\theta,t)=v_U(\theta,t)$ for every $U\in pre(T)$, hence (7). ☐

Consider next the second main additive term in the RHS of (5). For every transition T and time t, define

$$\Delta v_T(\theta,t) \;:=\; v_T(\theta,t^-)-v_T(\theta,t^+). \tag{8}$$

Obviously, $\Delta v_T(\theta,t)=0$ if $v_T(\theta,\cdot)$ is continuous at t. But in the last additive term of (5), the functions $v_T(\theta,\cdot)$ are not continuous at $t=z_{i,T}(\theta)$, and Eq. (5) becomes

$$\frac{dJ_T}{d\theta}(\theta) \;=\; \int_0^{\mathcal{T}}\frac{\partial v_T}{\partial\theta}(\theta,t)dt+\sum_{i=1}^{M_T}\Delta v_T(\theta,z_{i,T}(\theta))\frac{dz_{i,T}}{d\theta}(\theta). \tag{9}$$

To analyze the last term in (9) we use the notion of discrete events. The events we define are manifested by, and co-occur with discontinuities (jumps) in any one of the functions $v_T(\theta,\cdot)$, and therefore, each event is associated with a particular transition T and the time it occurs. We are concerned with the timing of events, and especially with the terms $\Delta v_T(\theta,z_{i,T}(\theta))\frac{dz_{i,T}}{d\theta}(\theta)$ that arise in Eq. (9). We point out that it may not be possible to compute the terms $\frac{dz_{i,T}}{d\theta}(\theta)$, but we will show that it is possible to compute the terms $\Delta v_T(\theta,z_{i,T}(\theta))\frac{dz_{i,T}}{d\theta}(\theta)$. In fact, the main result of our analysis is a an algorithm, recursive in time as well as in the network's flow, for computing these terms. To this end we classify events as *exogenous*, *endogenous*, or *induced*. Exogenous events are either jumps in the functions $V_S(\theta,\cdot)$ under certain circumstances, or jumps in the functions $v_T(\theta,\cdot)$ whose timing is independent of θ in a sense to be defined below. Endogenous events are the end of nonempty periods at the various places, which generally result in jumps in the functions $v_T(\theta,\cdot)$ at their output transitions. Induced events are those events that are caused directly by other events. We call the event that causes an induced event the *triggering event*, and as we shall see, the triggering event and its associated induced event occur at the same time. We next define formally the various types of events; further explanation will follow the definition. As a matter of notation, we will use the generic term $z_T(\theta)$ to denote the occurrence time of an event at transition T.

Definition 3.4:

1) A type-I exogenous event at transition T is an event at T whose occurrence time satisfies the condition $\frac{dz_T}{d\theta}(\theta) = 0$.
2) A type-II exogenous event at a transition $S \in \Sigma$ is a jump in the function $V_S(\theta, \cdot)$ at a time $t = z_S(\theta)$ such that $\varepsilon_S(\theta, z_S(\theta)^+) = \emptyset$.
3) An endogenous event at transition T is the end of a nonempty period at one or more places $p \in in(T)$.
4) An induced event at transition T is an event that occurs at the same time as an event at a transition $U \in pre(T)$ at a time the place $p \in in(T) \cap out(U)$ is empty. In this case we say that the event at U is the triggering event of the induced event at T.

A few remarks are due.

1) The aforementioned events' categories are not mutually exclusive since, for instance, an event can be both type-I exogenous, and endogenous or induced.
2) As earlier mentioned, the main purpose of the analysis is to develop a recursive algorithm for computing the quantities $\Delta V_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta)$ throughout the network. It is evident that if $z_T(\theta)$ is the timing of a type-I exogenous event at transition T then $\Delta V_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) = 0$, and therefore, type-I exogenous events need not be considered by the algorithm.
3) By Eq. (31), for $S \in \Sigma$, a jump in $V_S(\theta, \cdot)$ at time t causes a jump in $v_S(\theta, \cdot)$ at the same time if no place $p \in in(S)$ is empty at time t^+. Thus, type-II exogenous events are such jumps in $V_S(\theta, \cdot)$ that result in a jump in $v_S(\theta, \cdot)$ at the same time.
4) A triggering event and its associated induced event occur at the same time, but the induced event occurs at the transition that is immediately downstream from that where the triggering event occurs.
5) Induced events can form chains. Specifically, we say that a finite sequence of events, $e_1, \ldots, e_n$, is an *induced chain* of events if e_1 is a triggering but not an induced event; for all $i = 1, \ldots, n-1$, e_i triggers e_{i+1}; and e_n is not a triggering event.

The following assumption implies that induced chains are loop-free.

***Assumption** 3.5:* For every $\theta \in \Theta$, at time $t = 0$, every elementary circuit in the network contains a place p such that $m_p(\theta, 0) > 0$.

***Proposition** 3.6:* For every $\theta \in \Theta$, the sequence of transitions associated with an induced chain at any time $t \in [0, \mathcal{T}]$ does not contain loops.

Proof. Immediate from Assumption 3.2 and Proposition 2.1. □

The following assumption practically excludes the co-occurrence of multiple events except for induced chains. Variants thereof are routinely made in the literature on IPA.

***Assumption** 3.7:* For every $\theta \in \Theta$, w.p.1, the following statements hold. 1). No two or more events can occur at the same time unless they belong to the same induced chain. 2). No induced event can be type-II exogenous or endogenous.

The recursive nature of the algorithm gives it the structure of a discrete event dynamical system whose inputs are generated by the type-II exogenous events and whose state trajectory consists of the sequence of variables $\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta)$, computed at the times of endogenous and induced events at the network's transitions. Thus, we assume that when a type-II exogenous event occurs at a transition $S \in \Sigma$, it is possible to compute the term $\Delta V_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta) := \big(V_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)\big)\frac{dz_S}{d\theta}(\theta)$. By definition of type-II exogenous events, none of the input places of S is empty at time $z_S(\theta)^+$, and this means that either none of the input places of S is empty at time $t = z_S(\theta)$, or an empty periods ends at some of the input places of S. In the first case $\varepsilon_S(\theta, z_S(\theta)) = \emptyset$, and in the latter case $\varepsilon_S(\theta, z_S(\theta)) \neq \emptyset$. Accordingly, the term $\Delta v_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta)$ is related to $\Delta V_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta)$ in the following ways.

***Proposition** 3.8:* Consider a type-II exogenous event at a transition $S \in \Sigma$ occurring at time $z = z_S(\theta)$. If $\varepsilon_S(\theta, z_S(\theta)) = \emptyset$ then

$$\Delta v_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta) = \ \Delta V_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta). \tag{10}$$

On the other hand, if $\varepsilon_S(\theta, z_S(\theta)) \neq \emptyset$ then

$$\Delta v_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta) = \frac{v_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)}{V_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)}\Delta V_S(\theta, z_S(\theta))\frac{dz_S}{d\theta}(\theta) \tag{11}$$

and the ratio $\frac{v_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)}{V_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)}$ satisfies the inequalities

$$0 \ \leq \ \frac{v_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)}{V_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)} \ \leq \ 1. \tag{12}$$

Proof. Consider first the case where $\varepsilon_S(\theta, z_S(\theta)) = \emptyset$. Then (10) follows immediately from (8) and (31). Next, suppose that $\varepsilon_S(\theta, z_S(\theta)) \neq \emptyset$. By definition $\varepsilon_S(\theta, z_S(\theta)^+) = \emptyset$ and hence, by (31), $v_S(\theta, z_S(\theta)^+) = V_S(\theta, z_S(\theta)^+)$. Consequently $\Delta v_S(\theta, z_S(\theta)) = v_S(\theta, z_S(\theta)^-) - V_S(\theta, z_S(\theta)^+)$, and by dividing and multiplying this term by $\Delta V_S(\theta, z_S(\theta))$, Eq. (11) follows. Furthermore, by Assumption 3.4 there is no event at any transition $U \in pre(S)$ at time $t = z_S(\theta)$, and hence, the only way $z_S(\theta)$ is the end-time of an empty period at some $p \in in(S)$ is if $v_S(\theta, z_S(\theta)^-) > v_S(\theta, z_S(\theta)^+)$. But by assumption $\varepsilon_S(\theta, z_S(\theta)^+) = \emptyset$ and hence $v_S(\theta, z_S(\theta)^+) = V_S(\theta, z_S(\theta)^+)$, while $v_S(\theta, z_S(\theta)^-) \leq V_S(\theta, z_S(\theta)^-)$ by definition of the maximum transition-flow rate. All of this implies (12). □

Remark: The fraction term in Eq. (11) requires exact flow rates at certain times, which may be either available from the underlying model or approximated via moving averages from the sample path. Similar terms also arise in other quantities later in the sequel, and more generally, in networks of fluid queues. In practical situations it may be desired to avoid them, and this can be done by replacing them by a number (random or deterministic) $\zeta \in [0, 1]$. This was done in Wardi et al. (2010) with little impact on the convergence of an algorithm for optimizing performance of a queueing network.

Next, consider an induced event at transition T occurring at a time $z = z_T(\theta)$. Let $U \in pre(T)$ be the transition where the triggering event occurs, and let $p \in in(T) \cap out(U)$ be the place connecting U to T. Observe that $z_T(\theta)$ is also the time of the triggering event at U, and we note this fact by the notation $z_U(\theta) := z_T(\theta)$. By definition of induced events, $m_p(\theta, z_T(\theta)) = 0$, namely $z_T(\theta)$ lies in an empty period at p. Now there are two possibilities: either $z_T(\theta)$ lies in the interior of an empty period, or it is an end-point of an empty period. In the latter case, by Assumption 3.4 precluding the co-occurrence of events, $z_T(\theta)$ must be the end-time of an empty period at p. The following result relates $\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta)$ to $\Delta v_U(\theta, z_U(\theta))\frac{dz_U}{d\theta}(\theta)$ according to these two situations.

***Proposition** 3.9:* Consider an induced event at transition T occurring at time $z_T(\theta)$, and let $U \in pre(T)$ be the transition where the triggering event occurs. Let $p \in in(T) \cap out(U)$ be the place connecting U to T. If $z_T(\theta)$ lies in the interior of an empty period at p then

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \Delta v_U(\theta, z_U(\theta))\frac{dz_U}{d\theta}(\theta). \tag{13}$$

On the other hand, if $z_T(\theta)$ is the end-time of an empty period at p then

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) = \frac{v_T(\theta, z_T(\theta)^+) - v_U(\theta, z_U(\theta)^-)}{v_U(\theta, z_U(\theta)^+) - v_U(\theta, z_U(\theta)^-)}\Delta v_U(\theta, z_U(\theta))\frac{dz_U}{d\theta}(\theta) \tag{14}$$

and the term $\frac{v_T(\theta,z_T(\theta)^+)-v_U(\theta,z_U(\theta)^-)}{v_U(\theta,z_U(\theta)^+)-v_U(\theta,z_U(\theta)^-)}$ satisfies the following inequalities,

$$0 \leq \frac{v_T(\theta, z_T(\theta)^+) - v_U(\theta, z_U(\theta)^-)}{v_U(\theta, z_U(\theta)^+) - v_U(\theta, z_U(\theta)^-)} \leq 1. \tag{15}$$

Proof. If $z_T(\theta)$ lies in the interior of an empty period of p then (13) follows directly from (31). Suppose that $z_T(\theta)$ is the end-point of an empty period at p. Then $v_T(\theta, z_T(\theta)^-) = v_U(\theta, z_T(\theta)^-)$, and recalling that $z_T(\theta) = z_U(\theta)$, (14) follows from (31). Regarding (8), since $z_T(\theta)$ is the end-point of an empty period at p, it follows that (i) the triggering event at U raises $v_U(\theta), \cdot)$ at $t = z_T(\theta)$, which causes $v_T(\theta, \cdot)$ to rise there as well, implying that $v_T(\theta, z_T(\theta)^-) \leq v_T(\theta, z_T(\theta)^+)$; and (ii) $v_U(\theta, z_U(\theta)^+) - v_T(\theta, z_T(\theta)^+) \geq 0$. Recalling that $v_T(\theta, z_T(\theta)^-) = v_U(\theta, z_U(\theta)^-)$, this implies, after some algebra, that the inequalities in (15) are satisfied. □

Next, consider an endogenous event at transition T occurring at time $z_T(\theta)$. By definition of endogenous events, there exists $p \in in(T)$ such that an nonempty period ends at p at time $z_T(\theta)$. Let us denote by $\xi(\theta)$ the starting time of that nonempty period, which therefore is comprised of the interval $\big(\xi(\theta), z_T(\theta)\big)$. Furthermore, let $\eta_{k,T}(\theta)$, $k = 1, \ldots, K$ denote the times of all events except for type-I exogenous events that occurred at T during the nonempty period $\big(\xi(\theta), z_T(\theta)\big)$ in increasing order. Let $U := in(p)$, and denote by $\eta_{j,U}(\theta)$, $j = 1, \ldots, M$, the times of all events except for type-I exogenous events that occurred during the nonempty period $\big(\xi(\theta), z_T(\theta)\big)$ in increasing order.

***Proposition** 3.10:* The term $\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta)$ has the following form,

$$\begin{aligned}\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;&= \\ \int_{\xi(\theta)}^{z_T(\theta)} \Big(\frac{\partial v_U}{\partial\theta}(\theta,t) - \frac{\partial v_T}{\partial\theta}(\theta,t)\Big)dt & \\ +\sum_{j=1}^{M}\Delta v_U(\theta,\eta_{j,U}(\theta))\frac{d\eta_{j,U}}{d\theta}(\theta) \;&- \\ \sum_{k=1}^{K}\Delta v_T(\theta,\eta_{k,T}(\theta))\frac{d\eta_{k,T}}{d\theta}(\theta) \;&+ \\ \frac{v_U(\theta,\xi(\theta)^+) - v_T(\theta,\xi(\theta)^+)}{v_U(\theta,\xi(\theta)^+) - v_U(\theta,\xi(\theta)^-)}\Delta v_U(\theta,\xi(\theta))\frac{d\xi(\theta)}{d\theta}(\theta), &\end{aligned} \tag{16}$$

where the term $\frac{v_U(\theta,\xi(\theta)^+) - v_T(\theta,\xi(\theta)^+)}{v_U(\theta,\xi(\theta)^+) - v_U(\theta,\xi(\theta)^-)}$ satisfies the following inequalities,

$$0 \;\le\; \frac{v_U(\theta,\xi(\theta)^+) - v_T(\theta,\xi(\theta)^+)}{v_U(\theta,\xi(\theta)^+) - v_U(\theta,\xi(\theta)^-)} \;\le\; 1.$$

Proof. Since $(\xi(\theta), z_T(\theta))$ is a nonempty period at p, we have that $\int_{\xi(\theta)}^{z_T(\theta)} \dot{m}_p(\theta,t)dt \;=\; m(\theta, z_T(\theta)) - m(\theta,\xi(\theta)) \;=\; 0$. By (2), $\int_{\xi(\theta)}^{z_T(\theta)} \big(v_U(\theta,t) - v_T(\theta,t)\big)dt = 0$. Taking derivatives with respect to θ we obtain,

$$\begin{aligned}\big(v_U(\theta,z_T(\theta)^-) - v_T(\theta,z_T(\theta)^-)\big)\frac{dz_T}{d\theta}(\theta) \;+& \\ \int_{\xi(\theta)}^{z_T(\theta)} \Big(\frac{\partial v_U}{\partial\theta}(\theta,t) - \frac{\partial v_T}{\partial\theta}(\theta,t)\Big)dt & \\ +\sum_{j=1}^{M}\Delta v_U(\theta,\eta_{j,U}(\theta))\frac{d\eta_{j,U}}{d\theta}(\theta) & \\ -\;\sum_{k=1}^{K}\Delta v_T(\theta,\eta_{k,T}(\theta))\frac{d\eta_{k,T}}{d\theta}(\theta) & \\ -\big(v_U(\theta,\xi(\theta)^+) - v_T(\theta,\xi(\theta)^+)\big)\frac{d\xi}{d\theta}(\theta) \;&=\; 0.\end{aligned} \tag{17}$$

By Assumption 3.4 there is no event at U at time $z_T(\theta)$ and hence $v_U(z_T(\theta)^-) = v_U(z_T(\theta)^+)$, and since $z_T(\theta)$ is the starting time of an empty period at p, $v_U(\theta, z_T(\theta)^+) = v_T(\theta, z_T(\theta)^+)$; consequently, $\Delta v_T(\theta, z_T(\theta)) = v_T(\theta, z_T(\theta)^-) - v_U(\theta, z_T(\theta)^-)$. Plug this in (18) to obtain (16). Next, Eq. (17) is provable by similar arguments to those used in proving (12) and (15). □

We next describe a procedure for computing the IPA derivative $\frac{dJ}{d\theta}(\theta)$ as given in Eq. (9). Fix $\theta \in \Theta$. The integrant term $\frac{\partial v_T}{\partial\theta}$ can be computed iteratively by Proposition 3.1 as follows. For every transition T, (i) if every place $p \in in(T)$ is nonempty then $\frac{\partial v_T}{\partial \theta} = \frac{\partial V_T}{\partial \theta}$; and if there exists $p \in in(T)$ which is non-empty at time t, then, with $U = in(p)$, $\frac{\partial v_T}{\partial \theta} = \frac{\partial v_U}{\partial \theta}$.

The more-complicated term in the RHS of (9) is the second term, namely the sum-term $\sum_{i=1}^{M_T} \Delta v_T(\theta, z_{i,T}(\theta)) \frac{dz_{i,T}}{d\theta}(\theta)$, and we describe its computation by a formal, recursive algorithm. The algorithm follows Propositions 3.8 - 3.10 according to the events that occur, and in doing so it uses transition-related and place-related accumulators, denoted by D_T, B_p, and A_p, whose role will become evident from the sequel. The algorithm is assumed to be run synchronously for all of the network's transitions according to the order of the events occurring in them. In its description we define a *null event* at a transition T to be the start of an empty period at a place $p \in in(T)$ that is not caused by a type-II exogenous event, an endogenous event, or an induced event occurring at the same time at T. Strictly speaking this is not an event according to Definition 3.1, but the algorithm has to record the timing of these null events in order to compute the first term in the RHS of (16). To this end, we use a running variable, ξ_p, for every place p. Furthermore, the algorithm uses the term "unprocessed event" to mean an event at any transition T for which the corresponding term $\Delta v_T(\theta, z_T(\theta)) \frac{dz_T}{d\theta}(\theta)$ has not yet been computed. To reflect the order of induced events in an induced chain, the algorithm defines an *identified event* as the next event in the chain, as will be made clear in its Step 7. Figure 1 and its notation can be used as a visual aid in the forthcoming discussion.

Algorithm 1:

Initialize: For every transition T, set $D_T = 0$; for every place p, set $A_p = B_p = 0$ and set $\xi_p = 0$; and set $t = 0$. No induced event is labeled as identified.

Step 1: If there are no unprocessed events in the time-interval $[t, \mathcal{T}]$, then stop and exit. Otherwise, define t_{next} to be the next time $\tau \geq t$ when an unprocessed event occurs at some transition. Let T be the transition where this event occurs; in case of induced events, T must be the transition where the event occurring at it is identified. Set $z_T(\theta) := t_{next}$ to denote the timing of that event at T. Depending on whether this event is type-II exogenous (in which case $T \in \Sigma$), endogenous, identified induced, or a null event, go to Step 2, Step 3, Step 4, or Step 5, respectively.

Step 2: type-II exogenous event. Compute

$\Delta V_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta)$ from the sample path. If $\varepsilon_T(\theta) = \emptyset$ then set

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \Delta V_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta). \tag{18}$$

Otherwise, if $\varepsilon_T(\theta) \neq \emptyset$, then set

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \frac{v_T(\theta, z_T(\theta)^-) - V_T(\theta, z_T(\theta)^+)}{V_T(\theta, z_T(\theta)^-) - V_T(\theta, z_T(\theta)^+)}\Delta V_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta). \tag{19}$$

Step 3: Endogenous event. Let $p \in in(T)$ be the place where an empty period starts at time $z_T(\theta)$, and let $U := in(p)$. Set

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \int_{\xi_p}^{z_T(\theta)} \Big(\frac{\partial v_U}{\partial\theta}(\theta, t) - \frac{\partial v_T}{\partial\theta}(\theta, t)\Big)dt + B_p - A_p. \tag{20}$$

Step 4: Identified induced event. Let U be the transition where the inducing event occurs, and let p be the place connecting U to T. Then $z_T(\theta) = z_U(\theta)$. If $z_T(\theta)$ lies in the interior of an empty period at p, then set

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \Delta v_U(\theta, z_U(\theta))\frac{dz_U}{d\theta}(\theta). \tag{21}$$

On the other hand, if $z_T(\theta)$ is the end-time of an empty period at p, then set

$$\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) \;=\; \frac{v_T(\theta, z_T(\theta)^+) - v_U(\theta, z_T(\theta)^-)}{v_U(\theta, z_T(\theta)^+) - v_U(\theta, z_T(\theta)^-)}\Delta v_U(\theta, z_U(\theta))\frac{dz_U}{d\theta}(\theta). \tag{22}$$

Step 5: Null event. Set $\Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta) = 0$. With $p \in in(T)$ the place where the end of an empty period is the null event, set $\xi_p = z_T(\theta)$.

Step 6: Updating the variables A_p, B_p, and ξ_p. Do substeps (1) and (2), below.

1) For every $p \in in(T)$: If $z_T(\theta)$ lies in a nonempty period at p, then set

$$A_p \;=\; A_p + \Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta). \tag{23}$$

Alternatively, if $z_T(\theta)$ is the end-point of an empty period at p, then set $\xi_p = z_T(\theta)$. Furthermore, set $B_p = 0$ unless the event at T is induced and the triggering event is at

$U := in(p)$, in which case set

$$B_p \;=\; \Delta v_U(\theta, z_U(\theta))\frac{dz_U(\theta)}{d\theta}(\theta) - \Delta v_T(\theta, z_T(\theta))\frac{dz_T(\theta)}{d\theta}(\theta). \tag{24}$$

2) For every $q \in out(T)$: If $z_T(\theta)$ lies in a nonempty period at q, then set

$$B_q \;=\; B_q + \Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta). \tag{25}$$

Step 7: Check if the event is triggering. For every $q \in out(T)$, if $z_T(\theta)$ either lies in an empty period at q (including its starting point) then label the corresponding induced event at transition $W := out(q)$ as identified.

Step 8. Set

$$D_T \;=\; D_T + \Delta v_T(\theta, z_T(\theta))\frac{dz_T}{d\theta}(\theta), \tag{26}$$

set $t = z_T(\theta)$, and go to Step 1.

Tracing through the steps of this algorithm it can be seen that indeed it yields the sum-term in the RHS of Eq. (9).

Consider next the IPA derivative $\frac{dJ_p}{d\theta}$ for a given place p. Let us denote by $B_m := (\xi_m(\theta), \eta_m(\theta))$ the mth nonempty period at p in the interval $[0, \mathcal{T}]$, $m = 1, \ldots, M$. Let $U = in(p)$ and $T = out(p)$, as in Figure 1. For every $t \in B_m$, let $\eta_{m,j,U}(\theta)$, $j = 1, \ldots, J_m(t)$ be the times of events at U, in increasing order, in the interval $(\xi_m(\theta), t)$ (excluding type-I exogenous events), and let $\zeta_{m,k,U}(\theta)$, $k = 1, \ldots, K_m(t)$ be the times of events at T, in increasing order, in the interval $(\xi_m(\theta), t)$ (excluding type-I exogenous. The following result follows from Eq. (4) and some algebra, and its proof is omitted since it uses the same arguments as for the proofs of Propositions 3.3-3.5.

***Proposition** 3.11:* Fix a place p and $\theta \in \Theta$.

1). The IPA derivative $\frac{dJ_p}{d\theta}(\theta)$ has the following form.

$$\frac{dJ_p}{d\theta}(\theta) \;=\; \int_0^{\mathcal{T}} \frac{\partial m_p}{\partial \theta}(\theta, t)dt. \tag{27}$$

2). If t lies in the interior of an empty period at p. Then $\frac{\partial m_p}{\partial \theta}(\theta, t) = 0$.

3). For every $m = 1, \ldots, M$, and for every $t \in B_m$,

$$\begin{aligned}\frac{\partial m_p}{\partial \theta}(\theta, t) \;=\; & \int_{\xi_m(\theta)}^{t} \Big(\frac{\partial v_U}{\partial \theta}(\theta, \tau) - \frac{\partial v_T}{\partial \theta}(\theta, \tau)\Big) d\tau + \\ & \sum_{j=1}^{J_m(t)} \Delta v_U(\theta, \eta_{m,j,U}(\theta)) \frac{d\eta_{m,j,U}}{d\theta}(\theta) - \\ & \sum_{k=1}^{K_m(t)} \Delta v_T(\theta, \zeta_{m,k,T}(\theta)) \frac{d\zeta_{m,k,T}}{d\theta}(\theta) + \\ & \frac{v_U(\theta, \xi_m(\theta)^+) - v_T(\theta, \xi_m(\theta)^+)}{v_U(\theta, \xi_m(\theta)^+) - v_U(\theta, \xi_m(\theta)^-)} \Delta v_U(\theta, \xi_m(\theta)),\end{aligned} \tag{28}$$

and the last fraction-term satisfies the following inequalities,

$$0 \;\leq\; \frac{v_U(\theta, \xi_m(\theta)^+) - v_T(\theta, \xi_m(\theta)^+)}{v_U(\theta, \xi_m(\theta)^+) - v_U(\theta, \xi_m(\theta)^-)} \;\leq 1. \tag{29}$$

Note that all the computation of all the terms in the RHS of (28) has been discussed earlier, and in particular, the Δ_u and Δ_T terms are computable via Algorithm 1.

IV. Numerical example

Let us consider the Petri net system in Fig. 2. Here the firing of transition t_1 models the arrival of orders, while the firing of t_2 represents the arrival of the supplies. Finally, t_3 represents a production facility.

The instantaneous firing speed of t_1, $v_1(t)$, that coincides with its maximum firing speed $V_1(t)$, is a random process that alternates between values of $V_1 > 0$ that are random, and 0. We assume that the 0 periods are long and random, while the V_1 periods are short and the mean value $E[V_1]$ is large.

The instantaneous firing speed of t_2, $v_2(t)$, that coincides with its maximum firing speed $V_2(\theta, t)$, is a function of the time t and of a parameter θ. In particular, we assume that it is deterministic and cyclical. Let C be the duration of a cycle. Within a cycle, $V_2(\theta, t) = V_2$ (for a given $V_2 > 0$) during the first θ time units of the cycle, and $V_2(\theta, t) = 0$ for the last $C - \theta$ time units of the cycle.

Finally, we assume $V_3 < V_2 << E[V_1]$.

Note that all kind of events may occur in this system. In particular, we have exogenous events of type I when v_1 switches from V_1 to 0 and viceversa. We have exogenous events of type I

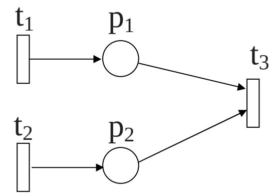

Fig. 2. The Petri net system considered in Section IV.

when v_2 switches from 0 to V_2, while the switch of v_2 from V_2 to 0 is an exogenous event of type II being it dependent on θ. Places p_1 and p_2 becoming empty correspond to endogenous events. Finally, v_3 switching from 0 to V_3 corresponds to an induced event.

Our goal is that of minimizing the integral, over a given time horizon $\mathcal{T}$, of $m_1(\theta,t)$ and $m_2(\theta,t)$, i.e., we assume that our performance index to be minimized is

$$J(\theta) = \int_0^{\mathcal{T}} (C_1 m_1(\theta,t) + C_2 m_2(\theta,t))dt. \tag{30}$$

In words, we want to minimize, possibly with different weights, the amount of orders that are waiting to be satisfied and the inventory of supplies.

We now show how to use the IPA formulas introduced in the previous section to compute the derivative $dJ(\theta)/d\theta$. To this aim we first observe that, given a generic time instant t belonging to a busy period of p_i, $i = 1, 2$, starting at time ξ_i, it is

$$m_i(\theta,t) = \int_{\xi_i}^{t} (V_i(\theta,\tau) - v_3(\theta,\tau))d\tau,$$

therefore

$$\begin{aligned} \frac{\partial m_i(\theta,t)}{\partial\theta} &= \int_{\xi_i}^{t} \left(\frac{\partial V_i(\theta,\tau)}{\partial\theta} - \frac{\partial v_3(\theta,\tau)}{\partial\theta} \right) d\tau \\ &\quad -(V_i(\theta,\xi_i^+) - v_3(\theta,\xi_i^+))\frac{d\xi_i}{d\theta} \\ &\quad + \sum_{k=1}^{N_i} \Delta V_i(\theta, z_{i,k})\frac{dz_{i,k}}{d\theta} \\ &\quad - \sum_{k=1}^{N_3} \Delta v_3(\theta, z_{3,k})\frac{dz_{3,k}}{d\theta} \end{aligned} \tag{31}$$

where $z_{i,k}$ is the k-th jump-time of v_i and N_i is the number of jumps of v_i during the time period $[\xi_i, t]$, for all $i = 1, 2, 3$.

Now, it is easy to verify that

$$\int_{\xi_i}^{t} \left(\frac{\partial V_i(\theta,\tau)}{\partial\theta} - \frac{\partial v_3(\theta,\tau)}{\partial\theta} \right) d\tau = 0, \qquad \frac{d\xi_i}{d\theta} = 0,$$

$$\Delta V_i(\theta, z_{i,k}) = \pm V_i, \quad \Delta v_3(\theta, z_{3,k}) = V_3,$$

The computation of the terms $dz_{3,k}/d\theta$ should be done recursively as explained in Section III.

Now, to show the effectiveness of the above IPA formulas, we present the results of a series of numerical simulations carried out on the system at hand. In particular, we assumed $E[V_1] = 10$, $V_2 = 1.5$, $V_3 = 1$, $m_1(0) = m_2(0) = 0$, $C = 4$, $C_1 = C_2 = 1$, $\mathcal{T} = 40$, and θ varying from 0.5 to 3.5. Table I enables us to compare the values of $dJ(\theta)/d\theta|_{sim}$ computed numerically, as an incremental ratio, and the values of $dJ(\theta)/d\theta_{IPA}$ using the IPA formulas. Note that in all cases the values of $dJ(\theta)/d\theta_{sim}$ are determined using an increment of θ equal to $\Delta\theta = 0.4$. Moreover, in all cases the same evolution of $v_1(t)$ has been considered. In the last column of Table I $err\%$ is equal to the absolute value of the percentage error in the computation of the derivatives using IPA, namely

$$err\% = \left| \frac{dJ(\theta)/d\theta|_{sim} - dJ(\theta)/d\theta_{IPA}}{dJ(\theta)/d\theta|_{sim}} \right| \cdot 100.$$

As it can be easily verified the derivative computed using IPA are absolutely satisfactory, since they differ at most of 10.81% wrt those computed numerically.

We conclude that the optimal value of θ is equal to 2.25 that leads to a performance index equal to $J^* = 262.876$.

V. Conclusions

This paper presented a systematic approach for computing the IPA derivatives of sample performance functions defined on event graphs. The approach consists of recursive algorithms that propagate the perturbations in the system's state. It is expressed in abstract and general terms and hence its description appears complicated, but its applications to specific examples can be quite simpler depending on their special characteristics. Future research will be pursued in two directions: applications of the approach to classes of applications, and its extension to Petri nets that are not event graphs.

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θ	$J(\theta)$	$\left.\frac{dJ(\theta)}{d\theta}\right\|_{sim}$	$\left.\frac{dJ(\theta)}{d\theta}\right\|_{IPA}$	$err\%$
0.5	679.208	-309.640	-306.332	1.07
0.75	603.560	-298.184	-298.832	0.22
1	530.916	-288.592	-285.740	0.99
1.25	460.696	-278.660	-275.992	0.96
1.5	392.916	-268.792	-266.240	0.95
1.75	328.204	-242.472	-240.400	0.85
2	269.884	-220.660	-226.148	2.49
2.25	262.876	138.868	123.856	10.81
2.5	323.996	294.800	292.500	0.78
2.75	396.656	290.948	288.752	0.75
3	468.376	287.156	285.000	0.75
3.25	539.160	283.328	281.252	0.73
3.5	609.004	279.564	277.580	0.71

TABLE I

A COMPARISON AMONG DIFFERENT VALUES OF $dJ(\theta)/d\theta$ COMPUTED NUMERICALLY AND USING THE IPA FORMULAS.

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