Robust reconstruction of the discrete state for a class of nonlinear uncertain switched systems

N. Orani, A. Pisano, M. Franceschelli, A. Giua and E. Usai

Abstract

This paper presents an approach to the robust state reconstruction for a class of nonlinear switched systems affected by model uncertainties. Under the assumption that the continuous state is available for measurement, an approach is presented based on concepts and methodologies derived from the sliding mode control theory. The time needed for reconstructing the discrete state after a transition can be made arbitrarily small by sufficiently increasing a certain observer tuning parameter.

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I. INTRODUCTION

In the last decade the control community has devoted a great deal of attention to the study of switched systems. A switched system has a discrete dynamics represented by a finite state machine that evolves according to the occurrence of discrete events. To each discrete state (or "mode") a continuous dynamics is associated [16], [6], [5], [7], [17], [1].

A problem of great interest is the reconstruction of the discrete-state through the observation of measurable system outputs. The techniques developed in this framework can be applied to several problems where the discrete events are not observable. In the framework of discrete event systems, several approaches have been proposed to estimate the discrete state [10], [11]. In a more general hybrid context, the discrete state estimation has been discussed in [13], [14].

In a recent work [2], the problem of invertibility of nonlinear systems [12] was addressed for the class of switched nonlinear systems affine in the control variables. In general the problem of invertibility for switched systems, especially linear switched systems, has received considerable attention [4], [3], [15], [9]. In this paper we investigate the problem of the discrete state reconstruction for switched systems building on the idea that a general class of switched systems can be modeled by nonlinear systems with an affine boolean input representing the system discrete state.

Objective of the present work is to reconstruct such a boolean input despite bounded uncertainties affecting the system dynamics.

We propose a sliding mode based technique by relying on the remarkable properties of robustness against uncertainties and disturbances featured by such an approach [26]. We refer to the theoretical framework of the observers for systems with unknown inputs. Sliding mode observers offer the opportunity of reconstructing the unmeasurable quantities affecting the system dynamics, which can be exploited to solve the problem addressed in this work.

The organization of the paper is as follows. Section II describes the considered class of uncertain switched systems. It is pointed out, in particular, that the considered class, that embeds both analog and boolean terms, can capture switched dynamics of a certain degree of generality. Section III presents the proposed discrete state reconstruction scheme which is based on the second-order sliding mode (2-SM) approach. Section IV introduces a case study of a physical example (an hydraulic three-tank system) that falls into the considered class of switched systems. Section V deals with the simulation results and the final Section VI summarizes the attained results and draws possible lines for future improvements of the presented results.

II. PROBLEM FORMULATION

In this paper we examine the class of switched systems that can be represented in the form:

$$\dot{\mathbf{x}}(t) = \mathbf{G}(\mathbf{x},\mathbf{u},t) + \mathbf{D}(\mathbf{x},\mathbf{u},t)\delta(\sigma(t)) + \varepsilon(\mathbf{x},t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the continuous state, $\mathbf{u}(t) \in \mathbb{R}^p$ is the input to the system, $\mathbf{G}(\mathbf{x},\mathbf{u},t) \in \mathbb{R}^n$ and $\mathbf{D}(\mathbf{x},\mathbf{u},t) \in \mathbb{R}^{n\times L}$ are known vector fields, and $\varepsilon(\mathbf{x},t) \in \mathbb{R}^n$ is an **uncertain** term that represents possible sources of uncertainties such as modelling errors and/or external disturbances.

The piecewise-constant integer function $\sigma(t) \in \{0,\ldots,k-1\}$ represents the **discrete state** of system (1). Vector $\delta(\sigma(t)) \in \{0,1\}^L$ contains **boolean** elements. It maps the discrete state $\sigma(t)$ into an L-dimensional boolean vector which "encodes" the actual discrete state.

Model (1) can represent switched systems with, at most, $k \leq 2^L$ different dynamics. The problem tackled in this paper is the reconstruction of the discrete state $\sigma(t)$.

A. Assumptions

We now specify the assumptions which are met about the considered class of systems (1).

The continuous state $\mathbf{x}(t)$ is supposed to be fully measurable, and $\mathbf{G}(\mathbf{x},\mathbf{u},t)$, $\mathbf{D}(\mathbf{x},\mathbf{u},t)$ are supposed to be known.

The dimension L of vector $\delta(t)$ must not exceed the dimension of the continuous state:

$$L \leq n \tag{2}$$

The boolean vector $\delta(t)$ is not available due to the uncertainty in the discrete state. The discrete state $\sigma(t)$ can be uniquely recovered from the boolean vector $\delta(t)$, and viceversa.

Let the time evolutions of the continuous state $\mathbf{x}$ and exogenous input $\mathbf{u}$ variables be a-priori confined in the compact domains $\mathbb{X}$ and $\mathbb{U}$.

As for the matrix field $\mathbf{D}(\mathbf{x},\mathbf{u},t)$, let it be norm-bounded and smooth in the domain $\mathbb{X}\times\mathbb{U}$ uniformly in time so that two constants D_0 and D_1 exist such that

$$\|\mathbf{D}(\mathbf{x},\mathbf{u},t)\| \leq D_0 \qquad \|\dot{\mathbf{D}}(\mathbf{x},\mathbf{u},t)\| \leq D_1 \tag{3}$$

$\mathbf{D}(\mathbf{x},\mathbf{u},t)$ is also assumed to be full rank which implies that a constant D_2 exists such that

$$\|[\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\mathbf{D}(\mathbf{x},\mathbf{u},t)]^{-1}\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\| \geq D_2 \tag{4}$$

As for the unmeasurable state-dependent uncertainty/disturbance term $\varepsilon(\mathbf{x},t)$, let it be uniformly norm-bounded and smooth in the domain $\mathbb{X}$ so that two constants $\bar{\varepsilon}_0$ and $\bar{\varepsilon}_1$ exist such that

$$\|\varepsilon(\mathbf{x},t)\| \leq \bar{\varepsilon}_0 \qquad \|\dot{\varepsilon}(\mathbf{x},t)\| \leq \bar{\varepsilon}_1 \tag{5}$$

Note that no boundedness or smoothing requirements are met on the vector field $\mathbf{G}(\mathbf{x},\mathbf{u},t)$.

B. Comments on the considered class of systems

We now show that the considered class of switched systems (1) is general enough to represent the following continuous-time switched nonlinear dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{G}_{\sigma(t)}(\mathbf{x},\mathbf{u},t) + \varepsilon(\mathbf{x},t) \tag{6}$$

where $\sigma(t) \in \{0,\ldots,k-1\}$ is the discrete state. A simple systematic procedure to put system (6) into the form (1) is now given.

Define the matrices $\mathbf{D}(\mathbf{x},\mathbf{u},t)$ and $\mathbf{G}(\mathbf{x},\mathbf{u},t)$ as follows (omitting the dependence of its entries from x,u and t)

$$\begin{array}{l}\mathbf{D}(\mathbf{x},\mathbf{u},t) = \left[\begin{array}{cccc}\mathbf{G}_1-\mathbf{G}_0 & \mathbf{G}_2-\mathbf{G}_0 & \cdots & \mathbf{G}_{k-1}-\mathbf{G}_0\end{array}\right] \\ \mathbf{G}(\mathbf{x},\mathbf{u},t) = \mathbf{G}_0(\mathbf{x},\mathbf{u},t)\end{array} \tag{7}$$

and let

$$\delta(\sigma(t)) = [\delta_1(\sigma(t)), \delta_2(\sigma(t)), ..., \delta_L(\sigma(t))]^T \tag{8}$$

where

$$\delta_j(\sigma(t)) = \begin{cases} 1 & \text{if} \quad j = \sigma(t) \quad j = 1, 2, ..., L \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

One can readily verify that system (6) is equivalent to (1),(7)-(9), whose main characteristics is that of being affine in the boolean vector $\delta(\sigma(t))$ defining the current mode of operation. Thus the considered class (1) is capable of representing nonlinear switched systems with significant degree of generality.

III. DISCRETE STATE OBSERVER DESIGN

The proposed discrete state estimator (which assumes the knowledge of the continuous system state) takes the following form:

$$\dot{\mathbf{z}} \;=\; \mathbf{G}(\mathbf{x}, \mathbf{u}, t) + \mathbf{w}(t) \tag{10}$$

where $\mathbf{z}$ represents the observer state and $\mathbf{w}$ is an observer input to be designed. Let $\mathbf{e} = \mathbf{x} - \mathbf{z}$ be the observer error variable. Then, from (1) and (10), the observer error dynamics is

$$\dot{\mathbf{e}} \;=\; \mathbf{D}(\mathbf{x}, \mathbf{u}, t)\delta(t) + \varepsilon(\mathbf{x}, t) - \mathbf{w}(t) \tag{11}$$

A. Observer input design

Our objective is to design an observer control vector $\mathbf{w}$ guaranteeing the finite-time convergence to zero of $\mathbf{e}$ and $\dot{\mathbf{e}}$.

In [24], within a distinct framework related to a fault detection and insulation problem, an approach to unknown input reconstruction was suggested based on standard first-order sliding mode control technique [26]. Such an approach could be used to reconstruct the discrete state of the switched systems using the equivalent control principle through **low pass filtering**. However, it is well known that the methods based on low pass filtering are intrinsically *approximate* methods [26] which can guarantee, at best, the asymptotic reconstruction of the discrete system state [23]-[24].

Here we propose a different approach based on second-order sliding modes that enables us to reconstruct the discrete state **without any filtering**, therefore leading to a solution converging in finite time and theoretically exact.

Consider the second time derivative of the error variable $\mathbf{e}$

$$\ddot{\mathbf{e}} = \dot{\mathbf{D}}(\mathbf{x}, \mathbf{u}, t)\delta(t) + \mathbf{D}\frac{d}{dt}\delta(t) - \dot{\varepsilon}(\mathbf{x}, t) - \dot{\mathbf{w}}(t) \tag{12}$$

which can be rewritten in compact form as follows:

$$\ddot{\mathbf{e}} = \varphi(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{w}}(t) \tag{13}$$

The uncertain "drift term" $\varphi(\cdot) = [\varphi_1(\cdot), \varphi_2(\cdot), ..., \varphi_n(\cdot)]^T$ takes the following form

$$\varphi(\mathbf{x},\mathbf{u},t) = \dot{\mathbf{D}}(\mathbf{x},\mathbf{u},t)\delta(t) + \mathbf{D}\frac{d}{dt}\delta(t) - \dot{\varepsilon}(\mathbf{x},t) \tag{14}$$

which is affected by the unmeasurable vector of boolean elements $\delta(t)$ that must be reconstructed. Note that the time derivative of the observer input vector $\mathbf{w}(t)$ appears in (13). Denote $\mathbf{v}(t) = [v_1, v_2, ..., v_n]^T \equiv -\dot{\mathbf{w}}(t)$ and

$$\begin{aligned} y_{i,1} &= e_i \\ y_{i,2} &= \dot{e}_i \end{aligned} \tag{15}$$

where e_i and $\dot{e}_i$ represent the i-th entry of vectors $\mathbf{e}$ and $\dot{\mathbf{e}}$. Then it is possible to rewrite system (13) in terms of n de-coupled single input subsystems having the following form

$$\begin{cases} y_{i,1} = y_{i,2}, \quad i = 1,2,..,n \\ \dot{y}_{i,2} = \varphi_i(\mathbf{x},\mathbf{u},t) + v_i \end{cases} \tag{16}$$

The problem is to find a set of control inputs v_i stabilizing the uncertain SISO systems (16) in finite time. To solve this problem, the second-order sliding mode control approach [25] appears to be particularly appropriate because of systems (16) have relative degree two with respect to the inputs v_i's which are treated as auxiliary control variables. The control task is complicated by the two issues: (a) variables $y_{i,2}$ $(i = 1,2,...,n)$ are not measurable, and (b) the drift terms $\varphi_i(\mathbf{x},\mathbf{u},t)$ are uncertain.

The boundedness properties of the drift terms $\varphi_i(\mathbf{x},\mathbf{u},t)$ play a crucial role and critically affect the solution to the control problem. In [19] the problem was solved under the condition that a constant upperbound Φ_i to the drift term magnitude can be computed

$$|\varphi_i(\mathbf{x},\mathbf{u},t)| \leq \Phi_i \qquad \forall t \tag{17}$$

Denote as t_i $(i = 1,2,...)$ switching instants at which the active dynamics is commuting. The discrete state $\sigma(t)$, and then also the vector $\delta(t)$, are piecewise constant during the time intervals $T_i = (t_i, t_{i+1})$ between two adjacent mode switchings. This means that the time derivative $\frac{d}{dt}\delta(t)$ is identically zero during the time intervals T_i between two adjacent mode switching, and feature an impulsive behaviour at the switching instants t_i.

The effect of the impulsive $\frac{d}{dt}\delta(t)$ term is a **jump** in the $\mathbf{e}-\dot{\mathbf{e}}$ state trajectories of system (13). More precisely, as it is clear from (11), the $\dot{\mathbf{e}}$ trajectories will be discontinuous at the switching instants, while the $\mathbf{e}$ trajectories will be continuous.

Let the mode switching sequence of the hybrid dynamics have a dwell time t_d. This means that $t_{i+1} - t_i \geq t_d$, for $i \geq 0$.

The **main idea** is to use the discontinuous controller in [19]. Under the condition (17), such controller is able to stabilize the uncertain SISO systems (16) in a finite time $t^* << t_d$ starting from any initial conditions $\mathbf{e}(0) = \mathbf{e}_0$, $\dot{\mathbf{e}}(0) = \dot{\mathbf{e}}_0$, with $\|\mathbf{e}_0\|$ and $\|\dot{\mathbf{e}}_0\|$ upper bounded by arbitrarily large constants. Thus, at any time $t \in [t^*, t_1)$ the conditions $\mathbf{e}(t) = \dot{\mathbf{e}}(t) = 0$ will be satisfied.

The state jumps occurring at the switching instants t_i $(i = 1,2,...)$ where the vector $\delta(t)$ is undergoing a discontinuity, will have the main effect of setting new nonzero initial conditions for $\dot{\mathbf{e}}(t_i^+)$.

So, at the first switching instants t_1 the system (16) will leave the origin according to $\dot{\mathbf{e}}(t_1^+) = \dot{\mathbf{e}}_{t_1}$ with $\|\dot{\mathbf{e}}_{t_1}\| \leq \|\mathbf{D}(\mathbf{x},\mathbf{u},t)\|\|\delta(t)\| \leq D_0\sqrt{n}$ (the norm of the boolean vector δ will never exceed the value $\sqrt{n}$).

After a new transient whose duration can be made less than t^* the system will be steered back to the origin. Thus, at any time $t \in [t_1 + t^*, t_2)$ the conditions $\mathbf{e}(t) = \dot{\mathbf{e}}(t) = 0$ will be satisfied. The reasoning is repeated over all the successive switching intervals. The key point is the capability of the robust controller presented in [19], that will be specified in the sequel, of steering to zero the SISO systems (16) **arbitrarily fast** during the time interval between two adjacent mode switchings.

Along any interval $T_i \equiv (t_{i-1}, t_i)$, $(i = 1, 2, \ldots)$, $t_0 \equiv 0$, the drift term $\varphi(\mathbf{x},\mathbf{u},t)$ is

$$\varphi(\mathbf{x},\mathbf{u},t) = \dot{\mathbf{D}}(\mathbf{x},\mathbf{u},t)\delta(t) - \dot{\varepsilon}(\mathbf{x},t) \quad t \in (t_{i-1}, t_i) \tag{18}$$

which can be upper bounded in the form

$$\|\varphi(\mathbf{x},\mathbf{u},t)\| \leq \bar{\Phi} \equiv D_1\sqrt{n} + \bar{\varepsilon}_1, \quad t \in (t_{i-1}, t_i) \tag{19}$$

The existence of such a constant upper bound allows for the possibility to design a robust controller featuring the desired finite time convergence properties. Next theorem outlines the main result by introducing the so-called "Suboptimal" second-order sliding mode control algorithm [19], [25], [22], together with the tuning rules that allow to obtain an **arbitrarily fast convergence**, which is a basic requirement of the present problem.

Theorem 1: *Consider system* (16) *and the control law*

$$v_i(t) = -V_M \operatorname{sign}\left(y_{i,1}(t) - \tfrac{1}{2}y_{i,1}(\xi_{i,j})\right) \quad \begin{array}{r} \xi_{i,j} \leq t < \xi_{i,j+1} \\ j = 1, 2, \ldots \end{array} \tag{20}$$

where $\xi_{i,0} = t_0 \equiv 0$, $\xi_{i,j}$ is the sequence of time instants at which $y_{i,2}(t) = 0$, and

$$V_M = \Gamma\bar{\Phi} \tag{21}$$

Denote the sequence of the switching instants as t_h, $h = 1, 2, \ldots$. Then there is Γ^* such that, for any $\Gamma \geq \Gamma^*$, the following conditions are provided.

$$y_{i,1}(t) = y_{i,2}(t) = 0, \quad t \in (t_h + t^*, t_{h+1}). \tag{22}$$

where $i = 1, 2, \ldots, n$ and with t^* being arbitrarily small.

Proof of Theorem 1 The proof exploits the basic convergence properties of the suboptimal second-order sliding mode control algorithm [25], [22]. During the first switching interval $[0, t_1)$ the suggested algorithm provides for the attainment of condition (22) with a transient time t^* fulfilling the next inequality [19], [25]

$$t^* \leq \frac{|y_{i,1}(0)| + |y_{i,2}(0)|}{g(\Gamma; \bar{\Phi})} \tag{23}$$

where $g(\Gamma; \bar{\Phi})$ is a function such that

$$lim_{\Gamma\to\infty} g(\Gamma; \bar{\Phi}) = \infty \quad \forall\bar{\Phi} \tag{24}$$

On the basis of the boundedness assumptions outlined in the Section II.A, there exist constant bound to $\|\mathbf{e}(0)\|$ and $\|\dot{\mathbf{e}}(0)\|$. Then, by choosing Γ sufficiently large one can guarantee that $t^* << t_d$.

At $t = t_1$, at which the $y_{i,1}$ and $y_{i,2}$ variables are all zero, vector $\dot{\mathbf{e}}$ undergoes a jump such that

$$\|\dot{\mathbf{e}}(t_1^+)\| \leq \|\mathbf{D}(\mathbf{x},\mathbf{u},t)\|\|\delta(t)\| \leq D_0\sqrt{n} \tag{25}$$

It then starts a new transient with the new initial conditions $y_{i,1}(t_1^+) = 0$, and $y_{i,2}(t_1^+)$ such that $|y_{i,2}(t_1^+)| \leq D_0$ (the square root of n does not appear when a specific value of i is considered). The time needed for steering the system back to the origin is then bounded by a relationship similar to (23) but with different error initial conditions as follows

$$t^* \leq \frac{D_0}{g(\Gamma;\bar{\Phi})} \tag{26}$$

Considering (26) it is possible to select Γ sufficiently large to make $t^* << t_d$, which means that the following condition will be achieved during the second switching interval $t \in (t_1, t_2)$

$$y_{i,1}(t) = y_{i,2}(t) = 0, \quad t \in (t_1 + t^*, t_2). \tag{27}$$

By iterating the same reasoning it is proven that the proposed controller is able to guarantee the attainment of conditions (22) with the transient time $t^* << t_d$ that can be made arbitrarily small by taking the observer gain parameter Γ sufficiently large, which concludes the proof. □

B. Discrete state reconstruction

It was shown in the previous Section that there is t^* such that, in every "inter-switching" interval $T_i \equiv (t_{i-1}, t_i)$ the next conditions hold

$$\mathbf{e} = \dot{\mathbf{e}} = 0, \quad t \in [t_{i-1} + t^*, t_i) \tag{28}$$

From the definition (11) of $\dot{\mathbf{e}}$, its zeroing implies that

$$\mathbf{D}(\mathbf{x},\mathbf{u},t)\delta(t) + \varepsilon(\mathbf{x},t) - \mathbf{w}(t) = 0 \tag{29}$$

Notice that the observer input $\mathbf{w}(t)$ is obtained integrating the discontinuous signal $\mathbf{v}(t)$, whose sign switches at very high (theoretically infinite) frequency (Zeno behaviour), then $\mathbf{w}(t)$ is a **continuous** signal.

By neglecting the uncertainty $\varepsilon(\mathbf{x},t)$ in (29) it yields naturally the following reconstruction formula that defines a **"non-thresholded" estimate** of the boolean vector δ.

$$\tilde{\delta}(t) = [\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\mathbf{D}(\mathbf{x},\mathbf{u},t)]^{-1}\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\mathbf{w}(t) \tag{30}$$

The non-thresholded estimate is not robust against the uncertainty $\varepsilon(\mathbf{x},t)$. By (29), the estimation error $\tilde{\delta}(t) - \delta(t)$ will be such that

$$\|\tilde{\delta}(t) - \delta(t)\| \leq \|[\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\mathbf{D}(\mathbf{x},\mathbf{u},t)]^{-1}\mathbf{D}^T(\mathbf{x},\mathbf{u},t)\|\|\varepsilon\| \tag{31}$$

It can be fruitfully exploited the boolean nature of the vector $\delta(t)$ by introducing a **thresholding** that rounds the value of $\tilde{\delta}(t)$ to the closest integer value between 0 and 1. It yields the "**thresholded" estimate** $\hat{\delta}(t)$ defined according to

$$\hat{\delta}_i(t) = \begin{cases} 1 & \tilde{\delta}_i(t) > 0.5 \\ 0 & \tilde{\delta}_i(t) \leq 0.5 \end{cases} \tag{32}$$

where $\tilde{\delta}_i(t)$ and $\hat{\delta}_i(t)$ are the i-th entries of vectors $\tilde{\delta}(t)$ and $\hat{\delta}(t)$ respectively.

The thresholded estimate results to be robust against any error $(\tilde{\delta}_i(t) - \delta_i)$ less, in magnitude, than 0.5. Thus it can be explicitly given a bound to the maximal tolerated magnitude for the uncertainty term.

From the requirement that $\|\tilde{\delta}(t) - \delta(t)\| \leq 0.5$ it yields by (31), (4), (5) the following maximal acceptable bound for the norm of the uncertainty term

$$\|\varepsilon(\mathbf{x},t)\| \leq \bar{\varepsilon}_0 \leq \frac{0.5}{D_2} \tag{33}$$

The fulfillment of (33) guarantees the **insensitivity** of the estimate $\hat{\delta}$ against the uncertainty, namely the condition

$$\hat{\delta}(t) = \delta(t), \quad t \in [t_{i-1} + t^*, t_i), \qquad i = 1, 2, ... \tag{34}$$

Lemma 1 Under the condition that the norm of the uncertain term $\varepsilon(\mathbf{x},t)$ fulfills the restriction (33), the proposed estimation procedure given by (30), (32) provides the **exact** reconstruction of the boolean vector δ in the time intervals $t \in [t_{i-1} + t^*, t_i)$, according to (34)

Remark 1 The requirement of providing the observer convergence within the arbitrarily small transient time $t^* << t_d$ would correspond, in the linear context, to locate the eigenvalues of the error dynamics far away from the origin. Generally, this strongly deteriorate the robustness against the measurement noise of the resulting linear "high gain" observer. It can be argued, due to the analysis made in [20], [21], that the magnification of the noise in the considered 2-SMC observer could be less severe than in the linear observer counterpart. This topic will be addressed in more detail in next research activities.

IV. The Three-tank system case study

The three-tank water process is regarded as a valuable setup for investigating nonlinear multivariable control as well as fault diagnosis schemes [27]. Let us show that it can be modelled as a switched affine system according to the general formulation (1).

The vertical multi-tank system that we shall consider is composed of three tanks of different shape interconnected as shown in Fig. 1, a water inflow $q(t)$ that supplies the upper tank n. 1 and three on-off valves $V1_{sw}, V2_{sw}, V3_{sw}$ that determine whether an outflow from each Tank exists or not. The on-off state of the three valves define the 8 possible operating mode of the considered system.

Refer to the schematic representation in Fig. 1. Signal $q(t)$, the inflow to the upper tank, represent a measurable input to the system, the binary signals U_1, U_2 and U_3 are the unknown states of the on-off valves, and H_1, H_2 and H_3 are the water levels which represent the continuous state of the three-tank system. It is the objective of the present work to present a scheme for reconstructing the states of the three on-off valves by assuming the knowledge of the water levels and of the input inflow $q(t)$ to the upper tank.

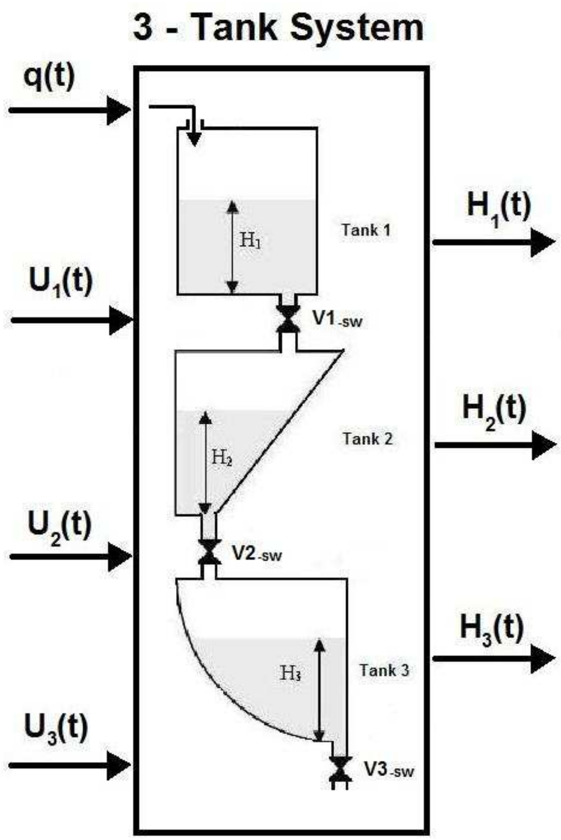

Fig. 1. System inputs and outputs

Let us model the three tank system. The flow balance equations lead to

$$\dot{V}_1 = q(t) - C_1(t)\sqrt{H_1} \tag{35}$$

$$\dot{V}_2 = C_1(t)\sqrt{H_1} - C_2(t)\sqrt{H_2} \tag{36}$$

$$\dot{V}_3 = C_2(t)\sqrt{H_2} - C_3(t)\sqrt{H_3} \tag{37}$$

where V_1, V_2, V_3 correspond to the actual volume of water in the three tanks and $C_1(t), C_2(t), C_3(t)$ are nonnegative coefficients. Due to the on-off nature of the valves, the $C_i(t)$ coefficients ($i = 1,2,3$) can assume two values only according to

$$C_i(t) = \begin{cases} 0 & \texttt{when valve } Vi_{sw} \texttt{ is OFF} \\ C_i^* & \texttt{when valve } Vi_{sw} \texttt{ is ON} \end{cases} \tag{38}$$

Let us define three boolean variables $U_1(t), U_2(t), U_3(t)$ representing the status of the three valves, so that the $C_i(t)$ terms can be rewritten as

$$C_i(t) = C_i^* U_i(t) \tag{39}$$

Further, the following simple relationships holds

$$\dot{V}_i = \beta_i(H_i)\dot{H}_i, \quad i = 1,2,3 \tag{40}$$

where $\beta_i(H_i)$ $(i = 1,2,3)$ represents the cross sectional area of the $i-th$ tank i at the level height H_i.

It yields the simple model

$$\dot{H}_1 = \frac{1}{\beta_1(H_1)}\left[q(t) - C_1^*\sqrt{H_1}U_1(t)\right] \tag{41}$$

$$\dot{H}_2 = \frac{1}{\beta_2(H_2)}\left[C_1^*\sqrt{H_1}U_1(t) - C_2^*\sqrt{H_2}U_2(t)\right] \tag{42}$$

$$\dot{H}_3 = \frac{1}{\beta_3(H_3)}\left[C_2^*\sqrt{H_2}U_2(t) - C_3^*\sqrt{H_3}U_3(t)\right] \tag{43}$$

Collecting into a boolean vector the discrete states of the on-off valves as follows

$$\delta(t) = [U_1(t), U_2(t), U_3(t)]^T \in \{0,1\}^3 \tag{44}$$

it is straightforward to rewrite the model (41)-(43) in the form (1) with $\mathbf{x} = [H_1, H_2, H_3]^T$, $\mathbf{u} = q(t)$ and

$$\mathbf{G}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} \frac{q(t)}{\beta_1(H_1)} \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

$$\mathbf{D}(\mathbf{x},\mathbf{u},t) = \begin{bmatrix} -\frac{C_1{}^*\sqrt{H_1}}{\beta_1(H_1)} & 0 & 0 \\ \frac{C_1{}^*\sqrt{H_1}}{\beta_2(H_2)} & -\frac{C_2{}^*\sqrt{H_2}}{\beta_2(H_2)} & 0 \\ 0 & \frac{C_2{}^*\sqrt{H_2}}{\beta_3(H_3)} & -\frac{C_3{}^*\sqrt{H_3}}{\beta_3(H_3)} \end{bmatrix} \tag{46}$$

According to the notation introduced in section II, it must be highlighted that in our case a particular instance for (44) represent one of the possible $k = 8$ discrete states $\sigma(t)$ in witch the three-tank system could be found.

In the derived three tank system the dimension L of vector $\delta(t)$ is $L = 3$ which does not exceed the dimension $n = 3$ of the continuous state, as required in assumption (2).

The assumptions (3) on the matrix $\mathbf{D}(\mathbf{x},\mathbf{u},t)$ are trivially satisfied if the water levels $H_1(t), H_2(t), H_3(t)$ remains strictly positive during the observation process. Further the assumption (4) requires that the square matrix $\mathbf{D}(\mathbf{x},\mathbf{u},t)$ is **nonsingular**. Since

$$\det\mathbf{D}(\mathbf{x},\mathbf{u},t) = -\frac{C_1{}^*C_2{}^*C_3{}^*}{\beta_1(H_1)\beta_2(H_2)\beta_3(H_3)}\sqrt{H_1H_2H_3} \tag{47}$$

again the assumption (4) is fulfilled if none of the water levels become zero during the observation process. Assuming that an appropriate closed-loop supervisory system has been designed, capable of guaranteeing that $H_i(t) \geq H_i^* > 0,\ i = 1,2,3$, the proposed observer can provide for the reconstruction of the binary signal vector $\delta(t)$.

An additive error term $\varepsilon(\mathbf{x},t)$ may take into account possible discrepancies between the actual and nominal system model as well as possible external disturbances. It is stated in the Lemma 1 that the discrete state can be still reconstructed exactly provided that the norm of $\varepsilon(\mathbf{x},t)$ is sufficiently small.

It is worth noting that the discrete state $\sigma(t) \in \{0,1,\cdots,7\}$ can be reconstructed from the **thresholded** estimates $\hat{\delta}_1(t), \hat{\delta}_2(t), \hat{\delta}_3(t)$ by means of the following expression

$$\hat{\sigma}(t) = \hat{\delta}_1(t) \cdot 2^2 + \hat{\delta}_2(t) \cdot 2^1 + \hat{\delta}_3(t) \cdot 2^0 \tag{48}$$

V. Simulation results

The effectiveness of the suggested discrete state observer is now studied by means of some simulative analysis conducted on the three tank model (41)−(43). The inflow input $q(t)$ and the binary state $\delta(t)$ have been selected in such a way that the tanks never become empties, that would cause the loss of observability for the system.

The cross-sectional area functions $\beta_1(H_1), \beta_2(H_2), \beta_3(H_3)$ have the following analytical expressions which depends on the particular, different, shapes for the considered three tank system represented in Figure 1:

$$\begin{aligned} \beta_1(H_1) &= aw \\ \beta_2(H_2) &= cw + bwH_2 \\ \beta_3(H_3) &= w\sqrt{R^2 - (R - H_3)^2} \end{aligned}$$

where a,b,c,w,R are appropriate constant geometric parameters. The parameter values used in the simulations are reported in the Table 1. Euler integration method with the fixed sampling time $T_s = 0.001s$ has been used.

TABLE I

Table 1. Parameters of the Tree Tank System

Parameter	Value	Unit
$C_1{}^*$	$4.702 * 10^{-5}$	$\frac{m^3}{h}$
$C_2{}^*$	$4.49 * 10^{-5}$	$\frac{m^3}{h}$
$C_3{}^*$	$4.51 * 10^{-5}$	$\frac{m^3}{h}$
α_1	0.28	
a	0.035	m
w	0.035	m
b	0.348	m
c	0.1	m
R	0.365	m

A disturbance vector with elements of the form

$$\varepsilon_i(\mathbf{x},t) = 0.1(|H_1(t)| + |H_2(t)| + |H_3(t)|)sin(\omega t), \quad i = 1,2,3 \tag{49}$$

is considered, and a band-limited additive white noise is added to the level measurements H_1, H_2, H_3. The binary signal inputs $U_1(t), U_2(t), U_3(t)$ defining the discrete state of the system have been selected as shown in the plot of the next figure 2 (the same profile for all the simulation tests has been used). It can be noted that a dwell time of $0.5s$ has been used.

In the first TEST 1, the disturbance vector $\varepsilon(\mathbf{x},t)$ and the measurement noise are **not** included. The plots in the figure 2 show the actual $\delta_i(t)$ values together with the non-thresholded reconstructed ones $\tilde{\delta}_i(t)$. Figure 3 makes the same comparison by considering the thresholded reconstructed values $\hat{\delta}_i(t)$. Figure 4 shows the actual and reconstructed discrete states $\sigma(t)$ and $\hat{\sigma}(t)$. It can be seen that the suggested method provides a prompt identification of the active mode.

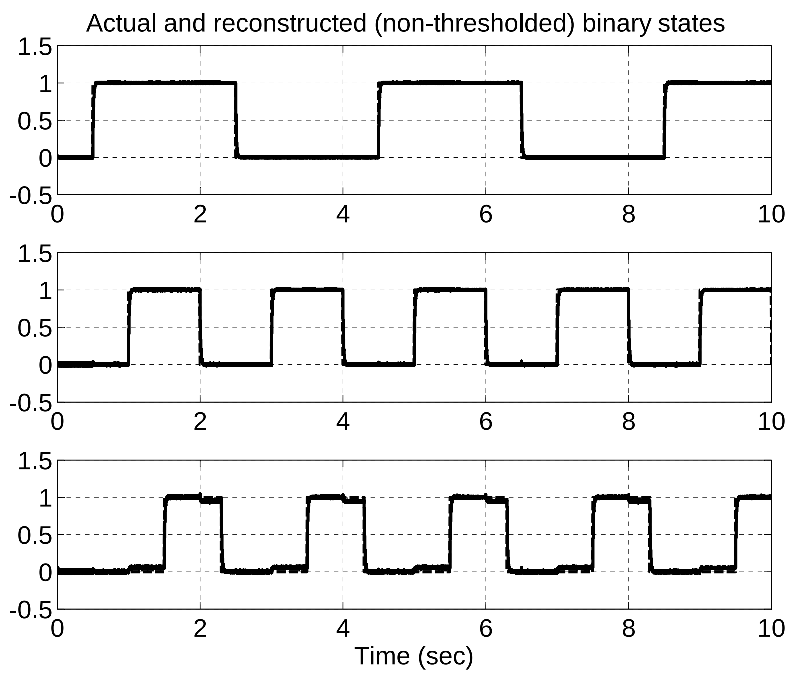

Fig. 2. TEST 1. $\delta_i(t)$ vs. the corresponding non-thresholded estimations $\tilde{\delta}_i(t)$. From topo to bottom: $i = 1,2,3$.

In TEST 2 it is shown that by increasing the V_M observer parameter it can be achieved an arbitrarily fast identification of the current mode after the mode switchings. To this end three three different values of V_M have been considered, and a zoom across some switching instant is made in the Fig. 5. The differences in the transient duration confirm the expected performance.

In the last TEST 3, disturbances and measurement noise are considered. Figure 6 shows that signals $\delta_{1i}(t)$ are corrupted by the noise as compared with the TEST 1. But, since the resulting errors are less than 0.5, the successive thresholding removes the errors and recovers the correct discrete state estimates according to Lemma 1 (see figure 7)

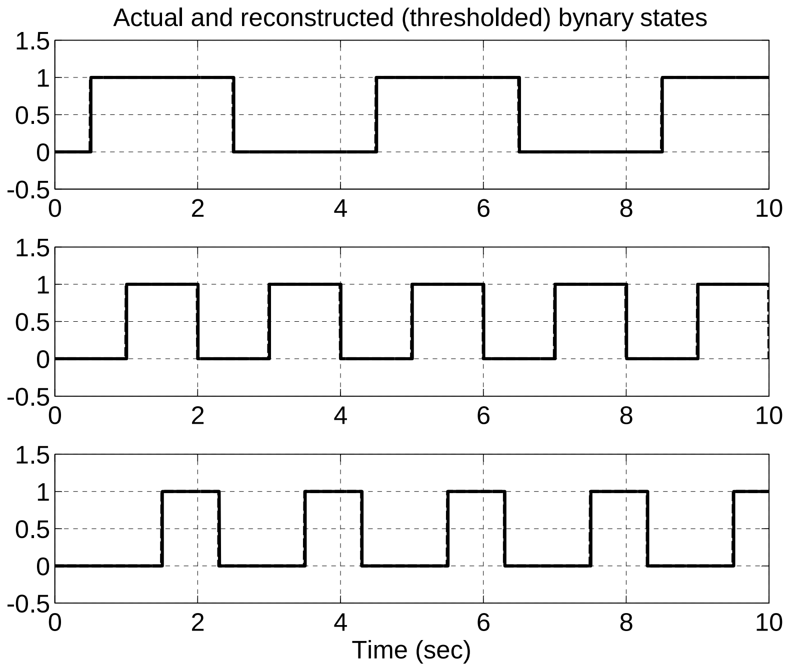

Fig. 3. TEST 1. $\delta_i(t)$ vs. the corresponding thresholded estimations $\hat{\delta}_i(t)$. From topo to bottom: $i = 1, 2, 3$.

VI. CONCLUSIONS

A scheme for the reconstruction of the discrete state in a class of nonlinear uncertain switched systems has been proposed. Key ingredients of the proposed approach are the use of a second order sliding mode observer approach followed by a thresholding procedure. Robustness against significant classes of disturbances is guaranteed by the proposed procedure. Next activities could be devoted to relax the requirement of knowing the continuous state by providing the simultaneous reconstruction of continuous and discrete state by output measurements, and to enlarge the observed system class by including, for instance, switching dynamics which are not-affine on the boolean vector $\delta(t)$.

VII. ACKNOWLEDGMENTS

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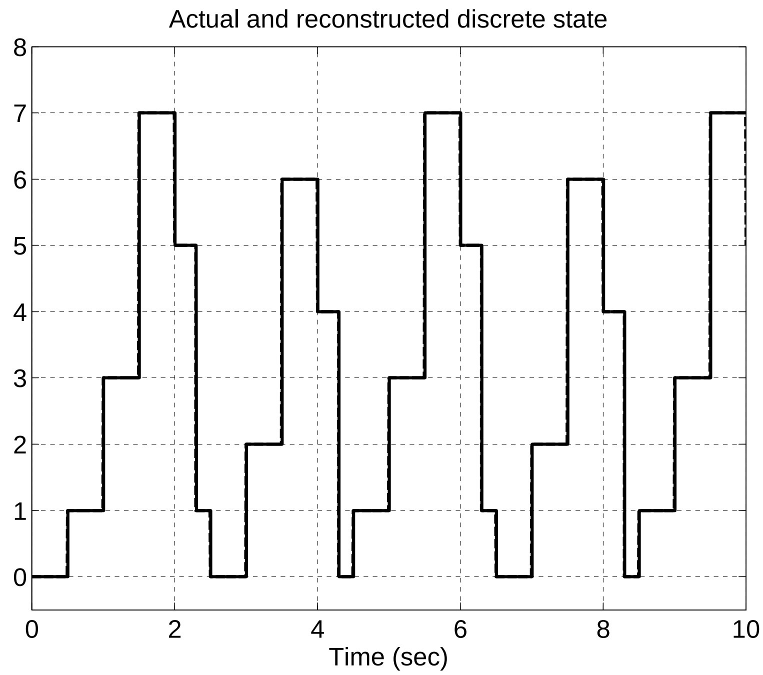

Fig. 4. TEST 1. Actual $\sigma(t)$ and reconstructed $\hat{\sigma}(t)$

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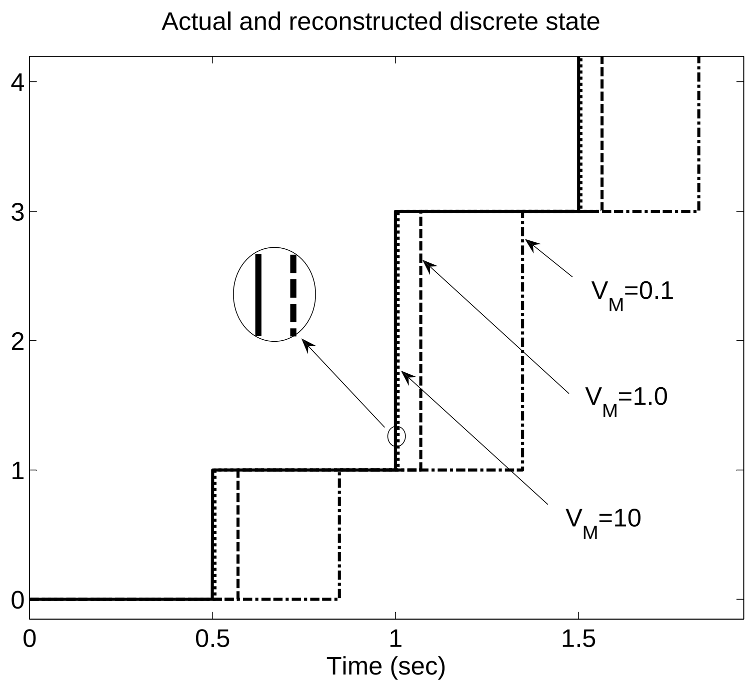

Fig. 5. TEST 2. $\sigma(t)$ (solid line) and $\hat{\sigma}(t)$ (dashed lines) using different values for the V_M observer gain

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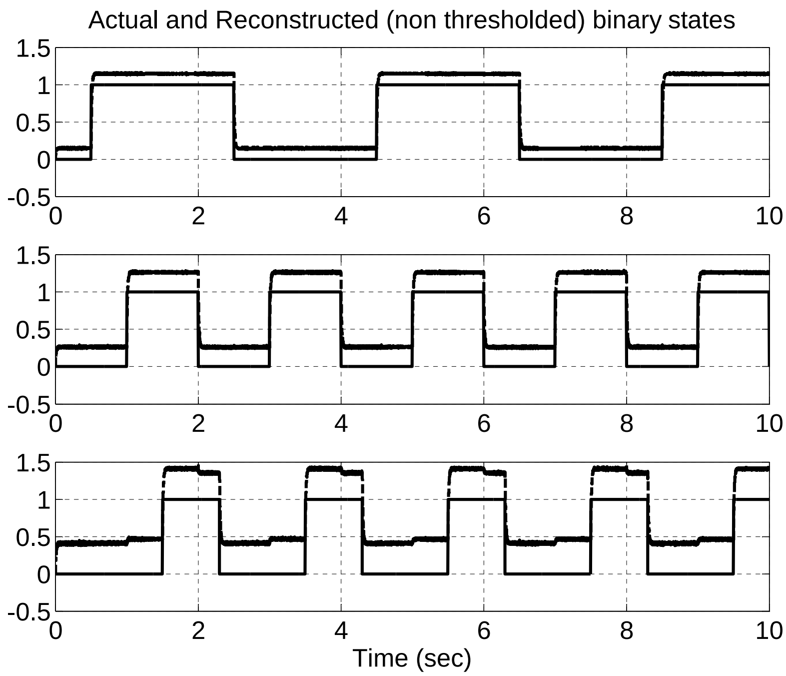

Fig. 6. TEST 3. Actual $\delta_i(t)$ and non-thresholded reconstructed $\tilde{\delta}_i(t)$ discrete inputs

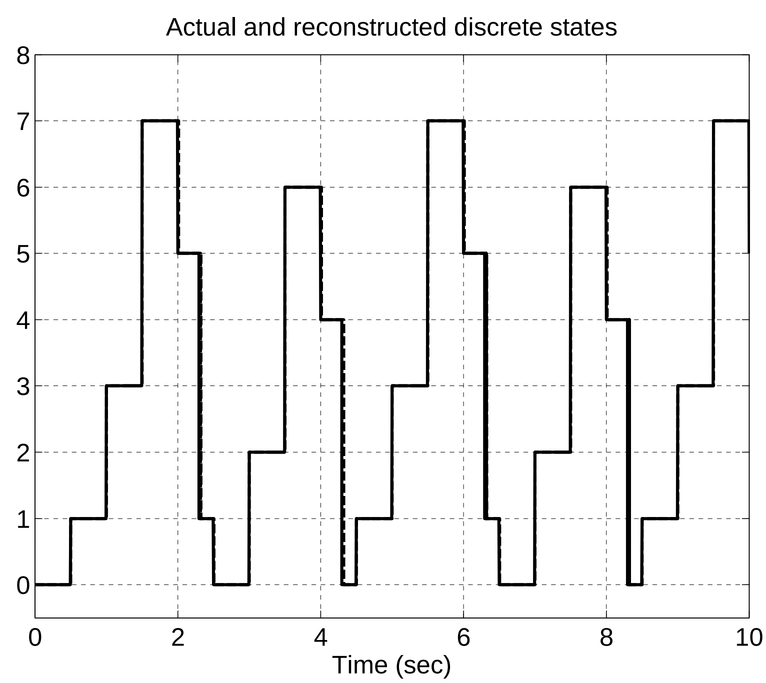

Fig. 7. TEST 3. Actual $\sigma(t)$ and reconstructed $\hat{\sigma}(t)$ discrete states