Generalized Mutual Exclusion Constraints and Monitors for Colored Petri Nets*

Maria Pia Fanti
Dip. di Elettrotecnica ed Elettronica
Politecnico di Bari, Italy
fanti@deemail.poliba.it

Alessandro Giua, Carla Seatzu
Dip. di Ing. Elettrica ed Elettronica
Università di Cagliari, Italy
{giua,seatzu}@diee.unica.it

Abstract – *A Generalized Mutual Exclusion Constraint (GMEC) is a linear constraint that limits the weighted sum of tokens in a subset of places of a place/transition net system. The corresponding controller takes the simple form of a monitor place that can be added to the net to obtain the closed-loop system. In this paper we extend this approach to the case of colored Petri nets, showing that a colored GMEC can express a set of linear constraints and can be enforced by a colored monitor place. We also develop a matrix representation of multisets that is useful for the design of the monitor place.*

Keywords: Petri nets, colored Petri nets, generalized mutual exclusion constraints, monitor places.

1 Introduction

A problem that has been widely investigated in the Petri net (PN) literature [3, 8, 9, 10] is that of designing a supervisory controller that restricts the open-loop reachability set $R(N, \boldsymbol{M}_0)$ of a plant $\langle N, \boldsymbol{M}_0 \rangle$, to a closed-loop reachability set $\mathcal{L} \cap R(N, \boldsymbol{M}_0)$, where $\mathcal{L} \subseteq \mathbb{N}^m$ is a given set of *legal markings*. Of particular interest in many applications are those control problems where the set of legal markings $\mathcal{L}$ is expressed by a set of n_c linear inequality constraints called *Generalized Mutual Exclusion Constraints* (GMECs) [2]. Each GMEC is a couple $(\boldsymbol{w}, k)$ where $\boldsymbol{w} : P \to \mathbb{Z}$ is a $m \times 1$ weight vector and $k \in \mathbb{Z}$, and defines a set of legal markings: $\mathcal{L} = \mathcal{M}(\boldsymbol{w}, k) = \{\boldsymbol{M} \in \mathbb{N}^m \mid \boldsymbol{w}^T \boldsymbol{M} \leq k\}$.

A controlling agent, called supervisor, must ensure that only legal markings are reached. If all transitions are controllable, the maximally permissive supervisor for a GMEC takes the form of a single *monitor place* p_c. If $\boldsymbol{C}$ is the incidence matrix of the open-loop plant and $\boldsymbol{M}_0$ is its initial marking, the monitor that enforces $(\boldsymbol{w}, k)$ has incidence matrix $\boldsymbol{C}_c = -\boldsymbol{w}^T \boldsymbol{C}$ and initial marking $m_{c,0} = k - \boldsymbol{w}^T \boldsymbol{M}_0$. A monitor solution can also be found when some of the transitions are not controllable, although in this case the monitor may not be maximally permissive [6, 7].

In this paper we show how the results presented in [2] for place/transition nets can be extended to the more general case of colored Petri nets [4]. A colored GMEC may represent in a compact way several constraints, and can be unfolded into a set of uncolored GMECs. We show that a GMEC can still be enforced by adding a monitor place p_c, and provide a systematic procedure to compute the incidence matrix defining such a monitor place, as well as its initial marking. We also formally prove that, under the assumption that all transitions are controllable and observable, the monitor place minimally restricts the behavior of the closed-loop system, in the sense that it prevents only those transition firings that yield forbidden markings.

In a companion paper [1] we apply these theoretical results to the design of control logic for a railway system modelled by colored Petri nets.

When not all transitions are controllable and observable, it may well be possible that a monitor designed for a given GMEC is not admissible, i.e., either it disables an uncontrollable transition or observes an unobservable one. It may be possible to construct a less permissive, but admissible monitor, extending to the case of colored nets the parametrization proposed by Moody and Antsaklis [6]. This problem is not considered in this paper.

2 Multisets

In this section we recall some notation that will be useful in the following.

Definition 2.1. *Let D be a set. A* multiset *(resp.,* non negative multiset*) $\boldsymbol{\alpha}$ over D is defined by a mapping $\alpha : D \to \mathbb{Z}$ ($\alpha : D \to \mathbb{N}$) and may be represented as*

$$\boldsymbol{\alpha} = \sum_{d \in D} \alpha(d) \otimes d$$

where the sum is limited to the elements such that $\alpha(d) \neq 0$.

Let $\mathcal{Z}(D)$ (resp., $\mathcal{N}(D)$) denote the set of all multisets (resp., non negative multisets) over D.

The multiset $\boldsymbol{\varepsilon}$ is the empty multiset such that for all $d \in D$, $\varepsilon(d) = 0$. ■

Proposition 2.2. *Given two multisets $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{Z}(D)$ and a number $a \in \mathbb{Z}$:*

- *The sum of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is denoted as $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ and is defined as $\forall d \in D : \gamma(d) = \alpha(d) + \beta(d)$.*

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- *The difference of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *is denoted as* $\boldsymbol{\gamma} = \boldsymbol{\alpha} - \boldsymbol{\beta}$ *and is defined as* $\forall d \in D : \gamma(d) = \alpha(d) - \beta(d)$. *Note that the difference of two non negative multisets may be negative.*
- *The product of* $\boldsymbol{\alpha}$ *and* a *is denoted as* $\boldsymbol{\gamma} = a\ \boldsymbol{\alpha}$ *and is defined as* $\forall d \in D : \gamma(d) = a\ \alpha(d)$.
- *We write* $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ *iff* $\forall d \in D : \alpha(d) \leq \beta(d)$. ■

Now, given two sets D and D', let $\boldsymbol{F} : D \to \mathcal{Z}(D')$ be a function that associates to each element $d \in D$ a multiset on D':

$$\boldsymbol{F}(d) = \sum_{d' \in D'} F(d, d') \otimes d' \in \mathcal{Z}(D').$$

We can naturally extend this application to a function $\boldsymbol{F} : \mathcal{Z}(D) \to \mathcal{Z}(D')$ as follows.

Definition 2.3. *Given two sets* D *and* D', *a function* $\boldsymbol{F} : D \to \mathcal{Z}(D')$, *and a multiset* $\boldsymbol{\alpha} \in \mathcal{Z}(D)$, *we define*

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{\alpha}) \triangleq\ & \boldsymbol{F} \circ \boldsymbol{\alpha} \triangleq \textstyle\sum_{d \in D} \alpha(d) \boldsymbol{F}(d) = \\ & \textstyle\sum_{d \in D} \sum_{d' \in D'} \alpha(d) F(d, d') \otimes d' \in \mathcal{Z}(D') \end{aligned}$$

i.e., $\boldsymbol{F}(\boldsymbol{\alpha})$ *is the linear combination with coefficients* $\alpha(d)$ *of the multisets* $\boldsymbol{F}(d)$ *over* D'. ■

Example 2.4. Let us consider the two sets $D = \{c_1, c_2\}$ and $D' = \{z_1, z_2, z_3\}$, and the multiset $\boldsymbol{\alpha}$ over D, where $\boldsymbol{\alpha} = 2 \otimes c_1 + 3 \otimes c_2$. Let $\boldsymbol{F}(c_1) = 4 \otimes z_1 + 5 \otimes z_2 + 2 \otimes z_3$ and $\boldsymbol{F}(c_2) = 3 \otimes z_1 + 2 \otimes z_2 + 2 \otimes z_3$ be two multisets over D'. Then, by definition,

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{\alpha}) &= \boldsymbol{F} \circ \boldsymbol{\alpha} = \textstyle\sum_{d \in \{c_1, c_2\}} \alpha(d) \boldsymbol{F}(d) \\ &= 2\boldsymbol{F}(c_1) + 3\boldsymbol{F}(c_2) \\ &= (2 \cdot 4 + 3 \cdot 3) \otimes z_1 + (2 \cdot 5 + 3 \cdot 2) \otimes z_2 + \\ &\qquad + (2 \cdot 2 + 3 \cdot 2) \otimes z_3 \\ &= 17 \otimes z_1 + 16 \otimes z_2 + 10 \otimes z_3 \in \mathcal{Z}(D') \end{aligned}$$

■

We finally observe that it is possible to give a matrix representation of multisets and of functions over multisets.

Remark 2.5. *Given two sets* D *and* D', *let us arbitrary order their elements as follows:* $D = \{d_1, \ldots, d_k\}$ *and* $D' = \{d'_1, \ldots, d'_{k'}\}$.

A multiset $\boldsymbol{\alpha} \in \mathcal{Z}(D)$ *can be represented by a vector:*

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha(d_1) \\ \alpha(d_2) \\ \vdots \\ \alpha(d_k) \end{bmatrix} \in \mathbb{Z}^k.$$

Thus, given a function $\boldsymbol{F} : D \to \mathcal{Z}(D')$ *for all* $d \in D$ *we can write*

$$\boldsymbol{F}(d) = \begin{bmatrix} F(d, d'_1) \\ F(d, d'_2) \\ \vdots \\ F(d, d'_{k'}) \end{bmatrix} \in \mathbb{Z}^{k'}.$$

while its extension $\boldsymbol{F} : \mathcal{Z}(D) \to \mathcal{Z}(D')$ *can be represented by the matrix*

$$\boldsymbol{F} = [\ \boldsymbol{F}(d_1) \quad \boldsymbol{F}(d_2) \quad \ldots \quad \boldsymbol{F}(d_k)\ ] \in \mathbb{Z}^{k' \times k}$$

and finally the multiset $\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{F} \circ \boldsymbol{\alpha}$ *can be computed with the usual matrix-vector product denoted by* $\cdot$, *i.e.,*

$$\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{F} \circ \boldsymbol{\alpha} = \boldsymbol{F} \cdot \boldsymbol{\alpha} = \begin{bmatrix} \sum_{i=1}^{k} \alpha(d_i) F(d_i, d'_1) \\ \sum_{i=1}^{k} \alpha(d_i) F(d_i, d'_2) \\ \vdots \\ \sum_{i=1}^{k} \alpha(d_i) F(d_i, d'_{k'}) \end{bmatrix} \in \mathbb{Z}^{k'}.$$

■

Example 2.6. Let us go back to the Example 2.4. We can write

$$\boldsymbol{F} = [\ \boldsymbol{F}(c_1) \quad \boldsymbol{F}(c_2)\ ] = \begin{array}{c} \begin{matrix} c_1 & c_2 \end{matrix} \\ \begin{bmatrix} 4 & 3 \\ 5 & 2 \\ 2 & 2 \end{bmatrix} \end{array} \begin{matrix} z_1 \\ z_2 \\ z_3 \end{matrix}$$

and thus

$$\boldsymbol{F} \cdot \boldsymbol{\alpha} = \begin{bmatrix} 4 & 3 \\ 5 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 10 \end{bmatrix}.$$

■

3 Colored Petri nets

A *Colored Petri Net* (CPN) is a bipartite directed graph represented by a quintuple $N = (P, T, Co, \boldsymbol{Pre}, \boldsymbol{Post})$ where P is the set of places, T is the set of transitions, $Co : P \cup T \to \mathcal{Cl}$ is a color function that associates to each element in $P \cup T$ a non empty ordered set of colors in the set of possible colors $\mathcal{Cl}$.

Therefore, for all $p_i \in P$, $Co(p_i) = \{a_{i,1}, a_{i,2}, \cdots, a_{i,u_i}\} \subseteq \mathcal{Cl}$ is the ordered set of possible colors of tokens in p_i, and u_i is the number of possible colors of tokens in p_i. Analogously, for all $t_j \in T$, $Co(t_j) = \{b_{j,1}, b_{j,2}, \cdots, b_{j,v_j}\} \subseteq \mathcal{Cl}$ is the ordered set of possible occurrence colors of t_j, and v_j is the number of possible occurrence colors in t_j.

In the following we assume that $m = |P|$ and $n = |T|$.

Matrices $\boldsymbol{Pre}$ and $\boldsymbol{Post}$ are the pre-incidence and the post-incidence $m \times n$ dimensional matrices respectively. In particular, each element $\boldsymbol{Pre}(p_i, t_j)$ is a mapping from the set of occurrence colors of t_j to a non negative multiset over the set of colors of p_i, namely, $\boldsymbol{Pre}(p_i, t_j) : Co(t_j) \to \mathcal{N}(Co(p_i))$, for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. In the following we denote $\boldsymbol{Pre}(p_i, t_j)$ as a matrix of $u_i \times v_j$ non negative integers, whose generic element $Pre(p_i, t_j)(h, k)$ is equal to the weight of the arc from place p_i with respect to (wrt) color $a_{i,h}$ to transition t_j wrt color $b_{j,k}$.

Analogously, $\boldsymbol{Post}(p_i, t_j) : Co(t_j) \to \mathcal{N}(Co(p_i))$, for $i = 1, \cdots, m$ and $j = 1, \cdots, n$, and we denote $\boldsymbol{Post}(p_i, t_j)$ as a matrix of $u_i \times v_j$ non negative integers. The generic element $Post(p_i, t_j)(h, k)$ is equal to the weight of the arc from transition t_j wrt color $b_{j,k}$ to place p_i wrt color $a_{i,h}$.

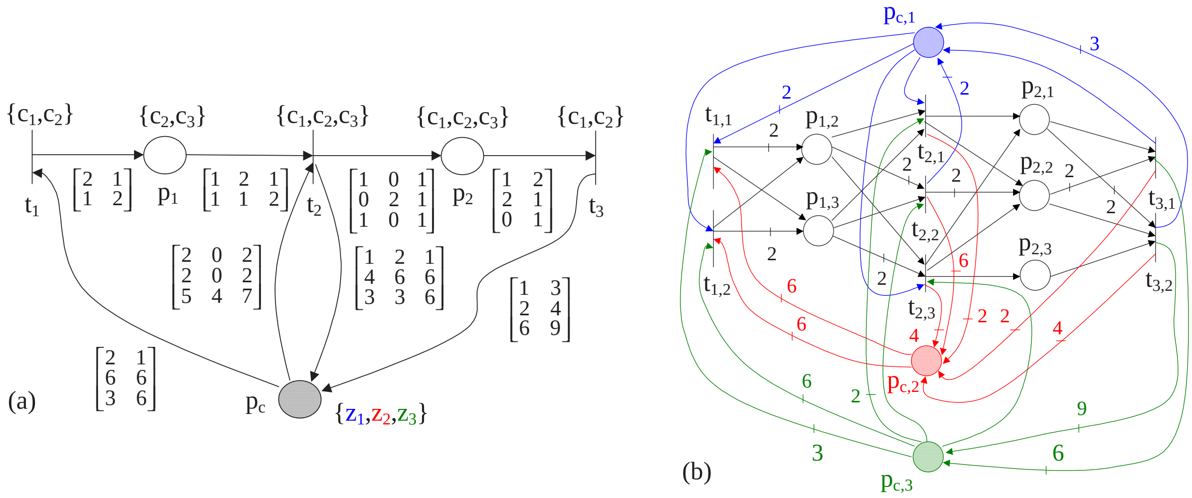

Figure 1: The closed-loop colored Petri net of Example 1 and the unfolded closed-loop net.

The incidence matrix $\boldsymbol{C}$ is an $m \times n$ matrix, whose generic element $\boldsymbol{C}(p_i, t_j) : Co(t_j) \to \mathcal{Z}(Co(p_i))$, for $i = 1, \cdots, m$ and $j = 1, \cdots n$. In particular $\boldsymbol{C}(p_i, t_j) = \boldsymbol{Post}(p_i, t_j) - \boldsymbol{Pre}(p_i, t_j)$.

For each place $p_i \in P$, we define the *marking* $\boldsymbol{m}_i$ of p_i as a *non negative multiset* over $Co(p_i)$. The mapping $m_i : Co(p_i) \to \mathbb{N}$ associates to each possible token color in p_i a non negative integer representing the number of tokens of that color that is contained in place p_i, and

$$\boldsymbol{m}_i = \sum_{d \in Co(p_i)} m_i(d) \otimes d.$$

In the following we denote $\boldsymbol{m}_i$ as a column vector of u_i non negative integers, whose h-th component $m_i(h)$ is equal to the number of tokens of color $a_{i,h}$ that are contained in p_i.

Finally, the marking $\boldsymbol{M}$ of a CPN is an m-dimensional column vector of multisets, i.e.,

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 \\ \vdots \\ \boldsymbol{m}_m \end{bmatrix}.$$

A colored Petri net system $\langle N, \boldsymbol{M}_0 \rangle$ is a colored Petri net N with initial marking $\boldsymbol{M}_0$.

A transition $t_j \in T$ is *enabled* wrt color $b_{j,k}$ at a marking $\boldsymbol{M}$ if and only if for each place $p_i \in P$ and for all $h = 1, \cdots, u_i$, we have $m_i(h) \geq Pre(p_i, t_j)(h, k)$.

If an enabled transition t_j fires at $\boldsymbol{M}$ wrt color $b_{j,k}$, then we get a new marking $\boldsymbol{M}'$ where, for all $p_i \in P$ and for all $h = 1, \cdots, u_i$, $m_i'(h) = m_i(h) + Post(p_i, t_j)(h, k) - Pre(p_i, t_j)(h, k)$.

We will write $\boldsymbol{M}[t_j(k)\rangle \boldsymbol{M}'$ to denote that t fires at $\boldsymbol{M}$ wrt color $b_{j,k}$ yielding $\boldsymbol{M}'$.

A *firing sequence* from $\boldsymbol{M}_0$ is a (possibly empty) sequence of transitions, each one firing wrt a given color,

$$\sigma = t_{j_1}(k_{j_1}) t_{j_2}(k_{j_2}) \ldots t_{j_r}(k_{j_r})$$

such that

$$\boldsymbol{M}_0[t_{j_1}(k_{j_1})\rangle \boldsymbol{M}_1[t_{j_2}(k_{j_2})\rangle \boldsymbol{M}_2 \ldots t_{j_r}(k_{j_r})\rangle \boldsymbol{M}_r.$$

A marking $\boldsymbol{M}$ is *reachable* in $\langle N, \boldsymbol{M}_0 \rangle$ iff there exists a firing sequence σ such that $\boldsymbol{M}_0[\sigma\rangle \boldsymbol{M}$.

Given a system $\langle N, \boldsymbol{M}_0 \rangle$, the set of firing sequences (also called *language* of the net) is denoted $L(N, \boldsymbol{M}_0)$ and the set of reachable markings (also called the *reachability set* of the colored net) is denoted $R(N, \boldsymbol{M}_0)$.

If the marking $\boldsymbol{M}$ is reachable in $\langle N, \boldsymbol{M}_0 \rangle$ by firing a sequence σ, then the following *state equation* is satisfied:

$$\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{C} \circ \boldsymbol{\Sigma}$$

$$\boldsymbol{\Sigma} = [\ \boldsymbol{\sigma}_1 \quad \cdots \quad \boldsymbol{\sigma}_n\ ]^T$$

is a vector of non negative multisets, and $\boldsymbol{\sigma}_j \in \mathcal{N}(Co(t_j))$, for $j = 1, \ldots, n$ is a multiset that specifies how many times transition t_j has fired wrt each of its colors. The vector $\boldsymbol{\Sigma}$ is called the *firing count vector* of the firing sequence σ.

Finally, let $\boldsymbol{x}$ be an m-dimensional vector of multisets where for all $i = 1, \ldots, m$, $\boldsymbol{x}_i \in \mathcal{N}(Co(p_i))$. Let $P' \subseteq P$. The projection of $\boldsymbol{x}$ on P' is the restriction of $\boldsymbol{x}$ to P' and will be denoted $\boldsymbol{x} \uparrow_{P'}$. This definition is extended in the usual way to the projection of a set of vectors $\mathcal{X}$, i.e., $\mathcal{X} \uparrow_{P'} = \{\boldsymbol{x} \uparrow_{P'} \mid \boldsymbol{x} \in \mathcal{X}\}$.

Example 3.1. Let us consider the CPN in Figure 1.a apart from place p_c and all connected arcs. The set of colors is $\mathcal{Cl} = \{c_1, c_2, c_3\}$. Place p_1 may only contain tokens of colors c_2 and c_3, while place p_2 may contain tokens of any color in $\mathcal{Cl}$. Finally, transitions t_1 and t_3 may only fire wrt to colors c_1 and c_2, while transition t_2 may fire wrt any color in $\mathcal{Cl}$.

Given the structure of the net, the only non null matrices $\boldsymbol{Pre}$ and $\boldsymbol{Post}$ are those reported Figure 1.a using the matrix notation, or equivalently,

$$\begin{aligned} \boldsymbol{Post}(p_1, t_1)(c_1) &= 2 \otimes c_2 + 1 \otimes c_3, \\ \boldsymbol{Post}(p_1, t_1)(c_2) &= 1 \otimes c_2 + 2 \otimes c_3, \\ \boldsymbol{Pre}(p_1, t_2)(c_1) &= 1 \otimes c_2 + 1 \otimes c_3, \\ \boldsymbol{Pre}(p_1, t_2)(c_2) &= 2 \otimes c_2 + 1 \otimes c_3, \end{aligned}$$

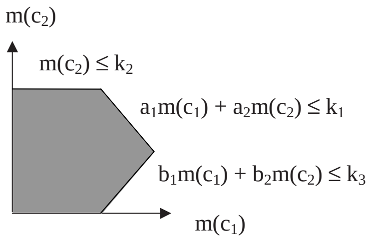

Figure 2: The graphical representation of the $\mathcal{M}(\boldsymbol{w},\boldsymbol{k})$ in the case of a generic GMEC involving a single place.

and so on.

Assuming that no token is initially contained in the net, i.e.,

$$\boldsymbol{M}_0 = [\ \boldsymbol{\varepsilon} \quad \boldsymbol{\varepsilon}\ ]^T,$$

if t_1 fires wrt to c_1 then we reach a new marking

$$\boldsymbol{M}_1 = [\ 2\otimes c_2 + 1\otimes c_3 \quad \boldsymbol{\varepsilon}\ ]^T.$$

Now, if t_2 fires wrt c_2, then we reach a new marking

$$\boldsymbol{M}_2 = [\ \boldsymbol{\varepsilon} \quad 2\otimes c_2\ ]^T.$$

The firing vector associated to the whole sequence $\sigma = t_1(c_1)t_2(c_2)$ is

$$\boldsymbol{\Sigma} = [\ 1\otimes c_1 \quad 1\otimes c_2 \quad \boldsymbol{\varepsilon}\ ]^T.$$

■

4 GMECs in colored Petri nets

In this section we extend the notion of GMEC to the case of colored Petri nets. Then, in the next session, we provide a systematic procedure to design the controller that enforces such a constraint.

Definition 4.1. *A GMEC is a couple $(\boldsymbol{W},\boldsymbol{k})$ where*

$$\boldsymbol{W} = [\ \boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_m\ ], \qquad \boldsymbol{k}\in\mathcal{Z}(D), \tag{1}$$

for all i, $\boldsymbol{w}_i : Co(p_i) \to \mathcal{Z}(D)$, and D is a set of colors different from $Co(p_i)$, $i = 1,\cdots,m$. Thus $\boldsymbol{W}$ can also be represented by a matrix with $|D|$ rows and $\sum_{i=1}^m |Co(p_i)|$ columns.

The set of legal markings defined by $(\boldsymbol{W},\boldsymbol{k})$ can be written as

$$\mathcal{M}(\boldsymbol{W},\boldsymbol{k}) = \left\{ \boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 \\ \vdots \\ \boldsymbol{m}_m \end{bmatrix} \middle| \ \boldsymbol{m}_i \in \mathcal{N}(Co(p_i)), \right. \\ \left. \boldsymbol{W}\circ\boldsymbol{M} \triangleq \textstyle\sum_{i=1}^m \boldsymbol{w}_i\circ\boldsymbol{m}_i \le \boldsymbol{k} \right\}. \tag{2}$$

■

Note that here we are extending the $\circ$ operator to the case of scalar product of vectors of multisets.

Now, to better clarify the above definition, let us assume that a single place p is involved in the constraint, i.e., the GMEC is a couple $(\boldsymbol{w},\boldsymbol{k})$ where $\boldsymbol{w} : Co(p) \to \mathcal{Z}(D)$ and $\boldsymbol{k}\in\mathcal{Z}(D)$. Assume we want the set of legal markings have the generic structure shown in Figure 2.

A set of legal markings of this kind can be written in the form (2) provided that $D = \{z_1, z_2, z_3\}$. In fact, in such a case

$$\begin{aligned} &\boldsymbol{w}(c_1) = w_1(z_1)\otimes z_1 + w_1(z_3)\otimes z_3,\\ &\boldsymbol{w}(c_2) = w_2(z_1)\otimes z_1 + 1\otimes z_2 + w_2(z_3)\otimes z_3,\\ &\boldsymbol{k} = k(z_1)\otimes z_1 + k(z_2)\otimes z_2 + k(z_3)\otimes z_3, \end{aligned}$$

$$\begin{aligned} \boldsymbol{w}\circ\boldsymbol{M} &= w(c_1)m(c_1) + w(c_2)m(c_2)\\ &= [w_1(z_1)m(c_1) + w_2(z_1)m(c_2)]\otimes z_1 +\\ &\quad m(c_2)\otimes z_2 +\\ &\quad [w_1(z_3)m(c_1) + w_2(z_3)m(c_2)]\otimes z_3 \end{aligned}$$

that implies that the set of legal markings is

$$\begin{aligned} \mathcal{M}(\boldsymbol{w},\boldsymbol{k}) = \ &\{\boldsymbol{M}\in\mathcal{N}(Co(p)) \mid\\ &\ w_1(z_1)m(c_1) + w_2(z_1)m(c_2) \le k(z_1),\\ &\ m(c_2) \le k(z_2),\\ &\ w_1(z_3)m(c_1) + w_2(z_3)m(c_2) \le k(z_3)\}. \end{aligned}$$

The above set coincides with the dark area in Figure 2 provided that $k_i = k(z_i)$, $i = 1,2,3$, $a_1 = w_1(z_1)$, $a_2 = w_2(z_2)$, $b_1 = w_1(z_3)$ and $b_2 = w_2(z_3)$.

Remark 4.2. *Note that the same reasoning can be trivially extended to the case of an arbitrary large number N of linear constraints by simply defining the set D as a set of cardinality N.* ■

Example 4.3. Let us consider again the CPN in Figure 1.a apart from place p_c and all connected arcs. Assume $D = \{z_1, z_2, z_3\}$. Moreover, let

$$\begin{aligned} &\boldsymbol{w}_1 = [\ \boldsymbol{w}_1(c_2) \quad \boldsymbol{w}_1(c_3)\ ]\\ &\boldsymbol{w}_1(c_1) = 1\otimes z_1 + 2\otimes z_2,\\ &\boldsymbol{w}_1(c_3) = 2\otimes z_2 + 3\otimes z_3, \end{aligned}$$

$$\begin{aligned} &\boldsymbol{w}_2 = [\ \boldsymbol{w}_2(c_1) \quad \boldsymbol{w}_2(c_2) \quad \boldsymbol{w}_2(c_3)\ ]\\ &\boldsymbol{w}_2(c_1) = 1\otimes z_1 + 2\otimes z_2 + 2\otimes z_3,\\ &\boldsymbol{w}_2(c_2) = 2\otimes z_3,\\ &\boldsymbol{w}_2(c_3) = 1\otimes z_1 + 4\otimes z_3. \end{aligned}$$

$$\boldsymbol{k} = 3\otimes z_1 + 5\otimes z_2 + 6\otimes z_3.$$

Using the matrix notation we can write:

$$\boldsymbol{w}_1 = \overset{\begin{matrix} c_2 & c_3 \end{matrix}}{\begin{bmatrix} 1 & 0\\ 2 & 2\\ 0 & 3 \end{bmatrix}} \begin{matrix} z_1\\ z_2\\ z_3 \end{matrix} \qquad \boldsymbol{w}_2 = \overset{\begin{matrix} c_1 & c_2 & c_3 \end{matrix}}{\begin{bmatrix} 1 & 0 & 1\\ 2 & 0 & 0\\ 2 & 2 & 3 \end{bmatrix}} \begin{matrix} z_1\\ z_2\\ z_3 \end{matrix}$$

and

$$\boldsymbol{k} = [\ 3 \quad 5 \quad 6\ ]^T.$$

Therefore,

$$\boldsymbol{W}\circ\boldsymbol{M} \triangleq \sum_{i=1}^m \boldsymbol{w}_i\circ\boldsymbol{m}_i$$

$$= \begin{bmatrix} 1 & 0\\ 2 & 2\\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} m_1(c_2)\\ m_1(c_3) \end{bmatrix} +$$

$$+ \begin{bmatrix} 1 & 0 & 1\\ 2 & 0 & 0\\ 2 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} m_2(c_1)\\ m_2(c_2)\\ m_2(c_3) \end{bmatrix} \le \begin{bmatrix} 2\\ 1\\ 3 \end{bmatrix}$$

and

$$\mathcal{M}(\boldsymbol{W},\boldsymbol{k}) = \{ \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{bmatrix} \mid \boldsymbol{m}_i \in \mathcal{N}(Co(p_i)),$$
$$m_1(c_2) + m_2(c_1) + m_2(c_3) \leq 3,$$
$$2m_1(c_2) + 2m_1(c_3) + 2m_2(c_1) \leq 5$$
$$3m_1(c_3) + 2m_2(c_1) + 2m_2(c_2) + 3m_2(c_3) \leq 6\}\,.$$

■

5 Monitors for colored Petri nets

In this section we show how the results presented in [2] for place/transition nets can be extended to the more general case of colored Petri nets. In particular, we show that a GMEC can still be enforced by adding a monitor place p_c, and we provide a systematic procedure to compute the incidence matrix defining such a monitor place, as well as its initial marking.

Definition 5.1. *Given a colored Petri net system $\langle N_p, \boldsymbol{M}_{p,0}\rangle$, with $N_p = (P, T, Co, \boldsymbol{Pre}_p, \boldsymbol{Post}_p)$, and a GMEC $(\boldsymbol{W},\boldsymbol{k})$ with $\boldsymbol{k} \in \mathcal{Z}(D)$, the* monitor *that enforces this constraint is a new place p_c with $Co(p_c) = D$, to be added to N_p. The resulting system is denoted $\langle N, \boldsymbol{M}_0\rangle$, with $N = (P \cup \{p_c\}, T, Co, \boldsymbol{Pre}, \boldsymbol{Post})$. Then N will have incidence matrix*

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_p \\ \boldsymbol{C}_c \end{bmatrix}, \quad \text{where } \boldsymbol{C}_c = -\boldsymbol{W} \circ \boldsymbol{C}_p. \qquad (3)$$

We are assuming that there are no selfloops containing p_c in N, hence $\boldsymbol{Pre}$ and $\boldsymbol{Post}$ may be uniquely determined by $\boldsymbol{C}$. The initial marking of $\langle N, \boldsymbol{M}_0\rangle$ is

$$\boldsymbol{M}_0 = \begin{bmatrix} \boldsymbol{M}_{p,0} \\ \boldsymbol{m}_{c,0} \end{bmatrix}, \quad \text{where } \boldsymbol{m}_{c,0} = \boldsymbol{k} - \boldsymbol{W} \circ \boldsymbol{M}_{p,0}. \quad (4)$$

We assume that the initial marking $\boldsymbol{M}_{p,0}$ of the system satisfies the constraint $(\boldsymbol{W},\boldsymbol{k})$. ■

In the case of controllable and observable transitions we can prove the following result.

Theorem 5.2. *Let $\langle N_p, \boldsymbol{M}_{p,0}\rangle$ be a CPN system, and $(\boldsymbol{W},\boldsymbol{k})$ a colored GMEC. Let $\langle N, \boldsymbol{M}_0\rangle$ be the system with the addition of the monitor place p_c.*

(1) The monitor place p_c enforces the GMEC $(\boldsymbol{W},\boldsymbol{k})$ when included in the closed-loop system $\langle N, \boldsymbol{M}_0\rangle$.

(2) The monitor place p_c minimally restricts the behavior of the closed-loop system $\langle N, \boldsymbol{M}_0\rangle$, in the sense that it prevents only transition firings that yield forbidden markings.

Proof. Let $\boldsymbol{m}_c$ denote the generic marking of the monitor place p_c.

(1) The first statement is proved if we demonstrate that the projection on P of the reachability set of $\langle N, \boldsymbol{M}_0\rangle$ is contained in the set of legal reachable markings of $\langle N_p, \boldsymbol{M}_{p,0}\rangle$, i.e., $R(N, \boldsymbol{M}_0) \uparrow_P \subseteq R(N_p, \boldsymbol{M}_{p,0}) \cap \mathcal{M}(\boldsymbol{W},\boldsymbol{k})$.

Clearly, $R(N, \boldsymbol{M}_0) \uparrow_P \subseteq R(N_p, \boldsymbol{M}_{p,0})$, since the addition of a place can only further constrain the behavior of a system. To prove $R(N, \boldsymbol{M}_0) \uparrow_P \subseteq \mathcal{M}(\boldsymbol{W},\boldsymbol{k})$, let $\boldsymbol{M}' \in R(N, \boldsymbol{M}_0)$ and $\boldsymbol{M}'_p = \boldsymbol{M}' \uparrow_P$. Then, there exists a firing vector $\boldsymbol{\Sigma}$ such that $\boldsymbol{M}' = \boldsymbol{M}_0 + \boldsymbol{C} \circ \boldsymbol{\Sigma}$, or equivalently, $\boldsymbol{M}'_p = \boldsymbol{M}_{p,0} + \boldsymbol{C}_p \circ \boldsymbol{\Sigma}$ and $\boldsymbol{m'}_c = \boldsymbol{m}_{c,0} - \boldsymbol{W} \circ \boldsymbol{C}_p \circ \ \boldsymbol{\Sigma} = \boldsymbol{k} - \boldsymbol{W} \circ (\boldsymbol{M}_{p,0} + \boldsymbol{C}_p \circ \boldsymbol{\Sigma}) \geq \boldsymbol{\varepsilon}$. Hence $\boldsymbol{W} \circ \boldsymbol{M}'_p = \boldsymbol{W} \circ (\boldsymbol{M}_{p,0} + \boldsymbol{C}_p \circ \boldsymbol{\Sigma}) \leq \boldsymbol{k}$, i.e., $\boldsymbol{M}'_p \in \mathcal{M}(\boldsymbol{W},\boldsymbol{k})$.

(2) Let $\sigma t(k) \in L(N_p, \boldsymbol{M}_{p,0})$ be such that $\boldsymbol{M}_{p,0}[\sigma\rangle \boldsymbol{M}'_p[t(k)\rangle \boldsymbol{M}''_p$ and $\sigma \in L(N, \boldsymbol{M}_0)$ be such that $\boldsymbol{M}_0[\sigma\rangle \boldsymbol{M}'$. We need to prove that $\sigma t(k) \notin L(N, \boldsymbol{M}_0) \Rightarrow \boldsymbol{W} \circ \boldsymbol{M}''_p > \boldsymbol{k}$.

Let $\boldsymbol{C}_p(\cdot,t)(\cdot,k)$ be the column of $\boldsymbol{C}_p$ corresponding to transition t firing wrt color k. Then $\boldsymbol{Pre}(p_c,t)(\cdot,k) - \boldsymbol{Post}(p_c,t)(\cdot,k) = -\boldsymbol{C}(p_c,t)(\cdot,k) = \boldsymbol{W} \circ \boldsymbol{C}_p(\cdot,t)(\cdot,k)$. Since t is not enabled wrt color k at marking $\boldsymbol{M}'$ and since there are no selfloops containing p_c, it follows that $\boldsymbol{\varepsilon} \leq \boldsymbol{m'}_c < \boldsymbol{Pre}(p_c,t)(\cdot,k) \ \Rightarrow \ \boldsymbol{Post}(p_c,t)(\cdot,k) = \boldsymbol{\varepsilon}$, i.e., $\boldsymbol{Pre}(p_c,t)(\cdot,k) = \boldsymbol{W} \circ \boldsymbol{C}_p(\cdot,t)(\cdot,k)$. Then $\boldsymbol{k} - \boldsymbol{W} \circ \boldsymbol{M}'_p = \boldsymbol{m'}_c < \boldsymbol{Pre}(p_c,t)(\cdot,k) = \boldsymbol{W} \circ \boldsymbol{C}_p(\cdot,t)(\cdot,k)$, from which follows $\boldsymbol{W} \circ \boldsymbol{M}''_p = \boldsymbol{W} \circ [\boldsymbol{M}'_p + \boldsymbol{C}(\cdot,t)(\cdot,k)] > \boldsymbol{k}$. □

Example 5.3. Let us consider again the CPN in Figure 1.a apart from place p_c and all connected arcs. The incidence matrix is

$$\boldsymbol{C}_p = \begin{bmatrix} \boldsymbol{Post}(p_1,t_1) & -\boldsymbol{Pre}(p_1,t_2) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Post}(p_2,t_2) & -\boldsymbol{Pre}(p_2,t_3) \end{bmatrix}$$

where $\boldsymbol{Post}(p_1,t_1)$, $\boldsymbol{Pre}(p_1,t_2)$, $\boldsymbol{Post}(p_2,t_2)$ and $\boldsymbol{Pre}(p_2,t_3)$ are shown in Figure 1.a using the matrix notation introduced in Section 3.

Assume that we want to enforce the GMEC $(\boldsymbol{W},\boldsymbol{k})$ considered in the previous Example 4.3.

This constraint can be enforced by adding a monitor place p_c whose incidence matrix $\boldsymbol{C}_c$ is

$$\begin{aligned}
\boldsymbol{C}_c &= [\ \boldsymbol{C}_c(p_c,t_1) \quad \boldsymbol{C}_c(p_c,t_2) \quad \boldsymbol{C}_c(p_c,t_3)\ ] \\
&= -\boldsymbol{W} \circ \boldsymbol{C}_p \\
&= -[\ \boldsymbol{w}_1 \quad \boldsymbol{w}_2\ ] \circ \\
&\quad \begin{bmatrix} \boldsymbol{Post}(p_1,t_1) & -\boldsymbol{Pre}(p_1,t_2) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Post}(p_2,t_2) & -\boldsymbol{Pre}(p_2,t_3) \end{bmatrix} \\
&= \begin{bmatrix} -(\boldsymbol{w}_1 \circ \boldsymbol{Post}(p_1,t_1)\ )^T \\ (\ \boldsymbol{w}_1 \circ \boldsymbol{Pre}(p_1,t_2) - \boldsymbol{w}_2 \circ \boldsymbol{Post}(p_2,t_2)\ )^T \\ (\boldsymbol{w}_2 \circ \boldsymbol{Pre}(p_2,t_3)\ )^T \end{bmatrix}^T \\
&= \begin{array}{c} \begin{array}{cc|ccc|cc} c_1 & c_2 & c_1 & c_2 & c_3 & c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|ccc|cc} -2 & 11 & -1 & 2 & -1 & 1 & 3 \\ -6 & -6 & 2 & 6 & 4 & 2 & 4 \\ -3 & -6 & -2 & -1 & -1 & 6 & 9 \end{array}\right] \end{array} \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}
\end{aligned}$$

The resulting closed-loop net is reported in Figure 1.a. For completeness in the same figure we have also reported the unfolding of the closed-loop net $\overline{N}$. Note that an uncolored Petri net is a CPN where for all $p \in P$ and for all $t \in T$, $Co(p) = Co(t) = \{\bullet\}$ where $\bullet$ is the usual uncolored token. In particular, we used the following notation. Different colors have been used to denote the open-loop net and the monitor places. More precisely, black has been used to represent the open-loop net, while blue, red and green denote the monitor places ($p_{c,1}$, $p_{c,2}$, and $p_{c,3}$), and all arcs connected to them, relative to the constraints associated to colors z_1, z_2 and z_3, respectively. Moreover, the marking of the generic place $p_{i,j}$ denotes the number of tokens of color c_j that are contained in the place p_i of the original CPN. Finally, the firing of transition $t_{i,j}$ corresponds to the firing of transition t_i wrt color c_j. If we order the set of places and transitions of the unfolded open-loop net so that its marking $\overline{\boldsymbol{M}}$ is equal to

$$\overline{\boldsymbol{M}} = [\ m_{1,2} \quad m_{1,3} \quad m_{2,1} \quad m_{2,2} \quad m_{2,3}\ ]^T$$

and the set of transitions is

$$\overline{T} = \{t_{1,1}, t_{1,2}, t_{2,1}, t_{2,2}, t_{2,3}, t_{3,1}, t_{3,2}\},$$

the incidence matrix of the unfolded open-loop net is equal to

$$\overline{\boldsymbol{C}} = \begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 0 & 0 \\ 1 & 2 & -1 & -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 2 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

By looking at the definition of the set of consistent markings, it is easy to write the constraint matrix $\overline{\boldsymbol{W}}$, i.e.,

$$\overline{\boldsymbol{W}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}.$$

Therefore the incidence matrix of the controller of the unfolded net is

$$\overline{\boldsymbol{C}}_c = -\overline{\boldsymbol{W}}^T \cdot \overline{\boldsymbol{C}} = \begin{bmatrix} -2 & -1 & -1 & 2 & -1 & 1 & 3 \\ -6 & -6 & 2 & 6 & 4 & 2 & 4 \\ -3 & -6 & -2 & -1 & -1 & 6 & 9 \end{bmatrix}$$

in accordance with the results obtained using the colored Petri net.

Now, assume that the initial marking of the open-loop colored net is

$$\boldsymbol{M}_{p,0} = \begin{bmatrix} 1 \otimes c_1 \\ 1 \otimes c_1 + 1 \otimes c_3 \end{bmatrix}$$

that satisfies the GMEC. In such a case the initial marking of the monitor place should be taken equal to

$$\begin{aligned} \boldsymbol{m}_{c,0} = \quad & \boldsymbol{k} - \boldsymbol{W} \circ \boldsymbol{M}_{p,0} = \boldsymbol{k} - \sum_{i=1}^{3} \boldsymbol{w}_i \circ \boldsymbol{m}_{p,0,i} \\ & = 1 \otimes z_1 + 3 \otimes z_2 + 1 \otimes z_3. \end{aligned}$$

Analogously, if we consider the unfolded net, using the well known theory of the GMECs, we find out that the initial marking of the monitor places is $m_{c,1} = m_{c,3} = 1$ and $m_{c,2} = 3$. ■

6 Conclusions

We have extended the classic Petri net control approach based on GMECs and monitor places to the case of colored Petri nets. A colored GMEC can express a set of linear constraints and can be enforced by a colored monitor place. We have also developed a matrix representation of multisets that is useful for the design of the monitor place.

We have assumed that all transitions are controllable and observable. If such is not the case, a colored monitor designed for a given colored GMEC may not be admissible, (it may disable an uncontrollable transition or observe an unobservable one). We plan to extend this work to show how it may be possible to construct a less permissive, but admissible monitor, extending to the case of colored nets the parametrization proposed by Moody and Antsaklis [6].

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